Optimal Entanglement Swapping in Quantum Repeaters

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We formulate the problem of finding the optimal entanglement swapping scheme in a quantum repeater chain as a Markov decision process and present its solution for different repeaters' sizes. Based on this, we are able to demonstrate that the commonly used "doubling" scheme for performing probabilistic entanglement swapping of probabilistically distributed entangled qubit pairs in quantum repeaters does not always produce the best possible raw rate. Focusing on this figure of merit, without considering additional probabilistic elements for error suppression such as entanglement distillation on higher "nesting levels," our approach reveals that a power-of-two number of segments has no privileged position in quantum repeater theory; the best scheme can be constructed for any number of segments. Moreover, classical communication can be included into our scheme, and we show how this influences the raw waiting time for different numbers of segments, confirming again the optimality of "nondoubling" in some relevant parameter regimes. Thus, our approach provides the minimal possible waiting time of quantum repeaters in a fairly general physical setting.

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Introduction.—A long-standing problem in the theory of quantum repeaters involves determining the optimal entanglement distribution time as a function of a repeater's characteristics like the distribution success probability p for a single segment and the entanglement swapping probability a between two segments [1]. It is commonly assumed that the number of segments n is a power of two, $n = 2^d$, and the only scheme considered is "doubling," where the segments are divided into two equal halves, which are then treated as independent smaller repeaters. When both halves have finally distributed an entangled state the last swapping is attempted. Such a doubling scheme can be useful in a "nested quantum repeater" allowing for a systematic inclusion of entanglement distillation [2] or, exploiting the repeater's "self-similarity," for a recursive and approximate calculation of repeater rates in certain regimes [3]. However, it is unknown whether "doubling" gives the optimal rates, i.e., the shortest repeater waiting times. Furthermore, one may ask: is there an optimal scheme if $n \neq 2^d$?

Rate calculations for quantum networks so far have focused either on the ultimate, information-theoretical limits independent of experimental constraints such as nondeterministic gate operations [4] or on more realistic systems under simplifying assumptions, such as specific parameter regimes allowing for an approximate treatment [3] or to determine bounds [5] and certain shapes [6] and sizes [7,8] of the network reducing its complexity. In this Letter, we bridge these two approaches for the case of a sufficiently small quantum repeater chain up to about ten segments and present its exact, optimal solutions, generalizing and optimizing our previous, exact results on the statistics of repeater waiting times in various settings [9] (see also Refs. [10,11]).

It turns out that, depending on p and a, the "doubling" scheme does not always deliver the highest raw rate, and for some values of parameters other schemes perform better. The corresponding rate enhancement seems to increase for larger repeaters. Moreover, the assumption that n is a power of two is superfluous; the optimal scheme is defined for all combinations of n, p, and a. In this Letter, we show how this scheme can be found. We will also include the physically relevant case where the memory qubits have to wait for classical signals to obtain information regarding the distributions in other segments. We put no restriction on how long a state can be kept in memory [12].

Markov chains.—Consider a finite Markov chain with a single absorbing state. The set of states we denote as $\mathcal{S}$ and the transition probability matrix as $P = (p_{ss'})$, where $p_{ss'}$ is the transition probability from s to s'. With every state $s \in \mathcal{S}$ we associate a cost $r_s \geq 0$ of making a transition from this state. Assuming that the cost of the absorbing state is zero, we define the total cost of absorption T_s from any state $s \in \mathcal{S}$ as a sum of all costs $r_{s'}$ from s to the absorbing state. Clearly, T_s is a random variable whose distribution depends on the transition probabilities of the chain. What is the average value of this variable? Denoting $v_s = \mathbf{E}[T_s]$, it can be shown that these quantities satisfy the system of linear equations

$$v_s = \sum_{s' \in \mathcal{S}} p_{ss'} v_{s'} + r_s. \tag{1}$$

Following the convention that the absorbing state is the last one let Q be the matrix obtained from P by removing its last row and last column. The system [Eq. (1)] then reads as $\mathbf{v} = Q\mathbf{v} + \mathbf{r}$, and its solution is given by $\mathbf{v} = (I - Q)^{-1}\mathbf{r}$,

where I is the identity matrix of order $n-1$, $n=|\mathcal{S}|$, and $\mathbf{r}$ is the $(n-1)$-vector of transition costs (except the last component, which we assumed to be zero). It is known that $I-Q$ is invertible and thus the system [Eq. (1)] has a unique solution given by this expression; see Ref. [9]. For large n such an expression is impractical to deal with, so $\mathbf{v}$ must be computed numerically by solving the system of linear equations $(I-Q)\mathbf{v}=\mathbf{r}$.

Markov decision problems.—What if the transition probabilities and the associated transition cost in each state depend on a parameter, so-called action? These actions can be freely chosen at will, and any choice is referred to as a policy. A (finite) Markov decision process (MDP) consists of a finite set $\mathcal{S}$ of states and a finite set of actions $\mathcal{A}_s$ for each state $s\in\mathcal{S}$. For every policy we have an instance of the problem considered above, with the transition probabilities $p^{\alpha}_{ss'}$ and transition costs r^{α}_{s} now depending on the actions $\alpha\in\mathcal{A}_s$ chosen for all $s\in\mathcal{S}$. One can say that an MDP embeds many Markov chain absorption problems into one framework. How can one determine the policy with minimal average absorption cost?

A straightforward approach is to compute the average cost of all embedded problems [Eq. (1)] and take the best value. In total, there are $N=\prod_{s\in\mathcal{S}}|\mathcal{A}_s|$ embedded problems, and this number becomes ridiculously large even for problems of moderate size. We show that there is a more practical approach based on solving a proper linear optimization problem.

Theorem 1.—Any solution of the linear optimization problem which maximizes the sum $\sum_{s\in\mathcal{S}} v_s$ under the constraints

$$v_s \leqslant \sum_{s'\in\mathcal{S}} p^{\alpha}_{ss'} v_{s'} + r^{\alpha}_{s}, \qquad \alpha\in\mathcal{A}_s \tag{2}$$

is a solution of the following system of nonlinear equations:

$$v_s = \min_{\alpha\in\mathcal{A}_s}\left[\sum_{s'\in\mathcal{S}} p^{\alpha}_{ss'} v_{s'} + r^{\alpha}_{s}\right]. \tag{3}$$

As the objective function one can use any linear combination $\sum_{s\in\mathcal{S}} c_s v_s$ with positive coefficients c_s. The problem [Eq. (2)] has at least one solution.

The proof of this theorem is given in the Supplemental Material [12]. If for any concrete choice of $\alpha\in\mathcal{A}_s$ for all $s\in\mathcal{S}$, i.e., for any policy π, we leave just one equation in Eq. (3), we get a system of linear equations of the form given by Eq. (1). The solution $\mathbf{v}^{\pi}$ of this system is the vector of the average costs of the absorption problem corresponding to the policy π. It is in this sense that an MDP embeds many absorption problems—every choice of actions produces a problem, and all these problems are contained in one framework described by Eq. (3). Note that any solution of Eq. (3) (which has at least one solution according to the previous theorem) corresponds to a policy—for any $s\in\mathcal{S}$ take an action $\alpha\in\mathcal{A}_s$ that minimizes the right-hand side of Eq. (3). For some s there can be more than one minimizing action, so the policy corresponding to a solution may not be unique. We now show that any solution of Eq. (3) is at least as optimal as the solution for any policy.

Theorem 2.—Let $\mathbf{v}^{*}$ be a solution of Eq. (3). Then for any policy π we have $\mathbf{v}^{*}\leqslant\mathbf{v}^{\pi}$, where this inequality is meant componentwise.

The proof is also given in the Supplemental Material [12]. From this theorem we derive the following property of optimal solutions: $v^{*}_{s}=\min_{\pi} v^{\pi}_{s}$, where the minimum is taken over all possible policies. We conclude that the system of nonlinear equations [Eq. (3)] has a unique solution, which can be obtained by solving the linear optimization problem given by Eq. (2). Having found the optimal solution $\mathbf{v}^{*}$ we can obtain an optimal policy corresponding to this solution by taking an action $\alpha\in\mathcal{A}_s$ that minimizes the right-hand side in Eq. (3) for all $s\in\mathcal{S}$. Such a scheme may not be unique.

Application to quantum repeaters.—We now apply the presented theory to the problem of finding the minimal waiting time in quantum repeaters. In a state where there are segments trying to distribute an entangled state and those that have already distributed, there is always a choice: either wait for nonready segments or try to swap a pair of neighboring ready segments. Clearly, different actions have different probabilistic evolutions, so the entanglement distribution process in a quantum repeater fits into an MDP model.

Here we focus on the case with equal segment lengths L_0 and connection efficiencies and thus identical characteristics, p and a, for all segments and connections. The more general and practical network scenario of unequal L_0 between the stations and unequal connection efficiencies can be treated as well with no real increase of complexity, and we give an explicit example for this in the Supplemental Material [12]. First, we need to list all possible states of a quantum repeater. We use a simple model where an attempt to distribute entanglement takes one unit of time and an attempt to swap segments takes no time at all. Under these assumptions the state of a repeater can be characterized by a string of nonnegative numbers, where 0 marks a segment trying to distribute entanglement, and $i>0$ marks a group of i successfully distributed and swapped segments. For the simplest case of a two-segment repeater the states are 00 (the initial state), 01, 10, 11, and 2 (the final, absorbing state). An optimal strategy must have identical actions on the states which are mirror images of each other, like the states 01 and 10 above, so we can apply the lumpability trick—we lump the mirror images into one new state and recompute the transition probabilities. This allows us to compress the size of the problem by reducing the number of states and actions, which will be very helpful for larger repeaters. In the case above from the two states 01 and 10 we form a new state $\{01,10\}$. So, in this simple case

we have four states $s_1 = 00$, $s_2 = \{01, 10\}$, $s_3 = 11$, and $s_4 = 2$. In each of these states only one action is possible, so our MDP reduces to the Markov chain problem of the form [Eq. (1)]:

$$\begin{aligned} v_1 &= q^2 v_1 + 2pqv_2 + p^2 v_3 + 1 \\ v_2 &= qv_2 + pv_3 + 1 \\ v_3 &= (1-a)v_1 + av_4, \end{aligned} \tag{4}$$

where $v_4 = 0$ (and we set $q = 1 - p$). Note that the transition probability $p_{12} = \mathbf{P}(s_1 \to s_2) = 2pq$ has a factor of 2, since $s_2 = \{01, 10\}$ and $s_1 = 00$ can go to s_2 in two ways—when either of the segments distributes entanglement. The probability of each path is pq, so the total transition probability is $2pq$. The constant terms on the right-hand side of the system [Eq. (4)] express our assumption that a distribution attempt costs one unit of time and a swapping attempt costs zero. Solving this system of linear equations, we obtain $v_1 = (3-2p)/[ap(2-p)]$, which is a well-known expression for the waiting time of a two-segment repeater.

Example.—Now consider a three-segment repeater. In this case there are nine states: $s_1 = 000$, $s_2 = 001$, $s_3 = 010$, $s_4 = 011$, $s_5 = 101$, $s_6 = 111$, $s_7 = 02$, $s_8 = 12$, and $s_9 = 3$, where any nonsymmetric sequence like 001 denotes the corresponding class $\{001, 100\}$ in order to not overload the notation. In the state $s_4 = 011$ (which denotes $\{011, 110\}$) two actions are possible: waiting while the last segment distributes entanglement, which costs one time unit per attempt, or trying to swap the other two segments, which costs nothing (in the state 111 the two possible swappings represent one action in the compressed system). The MDP equations in this case read as

$$\begin{aligned} v_1 &= q^2(qv_1 + 2pv_2 + pv_3) + p^2(2qv_4 + qv_5 + pv_6) + 1 \\ v_2 &= q^2 v_2 + pqv_4 + pqv_5 + p^2 v_6 + 1 \\ v_3 &= q^2 v_3 + 2pqv_4 + p^2 v_6 + 1 \\ v_4 &\leqslant qv_4 + pv_6 + 1 \\ v_4 &\leqslant (1-a)v_1 + av_7 \\ v_5 &= qv_5 + pv_6 + 1 \\ v_6 &= (1-a)v_2 + av_8 \\ v_7 &= qv_7 + pv_8 + 1 \\ v_8 &= (1-a)v_1, \end{aligned}$$

and $v_9 = 0$. If we remove the first inequality for v_4, we get the scheme where we always swap in the states 011 and 110, removing the second inequality we get the scheme where we always try to distribute entanglement in these states. Maximizing the sum $\sum_{i=1}^{8} v_i$ under the constraints given above, for each p and a we obtain the best waiting time v_1^* and the optimal scheme (which may depend on p and a). It turns out that for all p and a the former scheme (always swapping when ready) is better, and the analytical expression for the waiting time is the same as we have already given in Ref. [9], where it is denoted as $\bar{K}_3^{(\mathrm{dyn})}$.

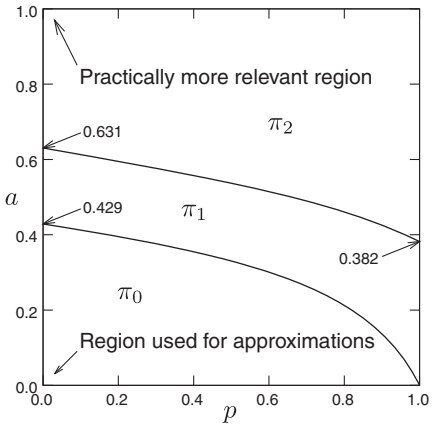

FIG. 1. Regions of different optimal schemes, $n = 4$.

Nondoubling optimal schemes.—For a four-segment repeater the corresponding MDP has 20 variables (excluding the variable for the absorbing state, whose value is zero) and 29 constraints. For every state there is at most one "wait for distribution" action and zero or more "swapping" actions. Somewhat surprisingly, the "doubling" scheme is not always the best one. We have solved the MDP for all $0.01 \leqslant p$, $a \leqslant 1.0$, and for each pair of probabilities p and a we determined the best scheme for these parameters. We found that there are three schemes that are optimal in different regions of the probability square; see Fig. 1. The equations of the curves between the regions and exact values of their intersections with the square boundary are given in the Supplemental Material [12].

In the lower-left corner of the square, which corresponds to small p and a, the optimal scheme is "doubling," denoted as π_0. A practically more relevant range of parameters may be at small p and large a, which corresponds to the upper-left corner of the square, and the optimal scheme there differs from "doubling," denoted as π_2. In between these two regions there is a third optimal scheme, π_1. These schemes are described in the Supplemental Material [12]. For some relevant fixed p and a values, Fig. 2 illustrates that the optimized raw waiting time is a linear function of the repeater size n [19].

Classical communication.—We can extend our model to include classical communication (CC), assuming that it takes one unit of time to restart a segment (and the swapping process itself takes no time). For example, for $n = 4$ in the state 0110 we can try to swap the inner pair of segments. If the swapping fails, then in the previous model the system transitions to the initial state 0000, but in this model it goes to a new state 0(1)(1)0, where the number in brackets denotes the number of time units after which this segment returns to the initial state 0. If we make a swapping

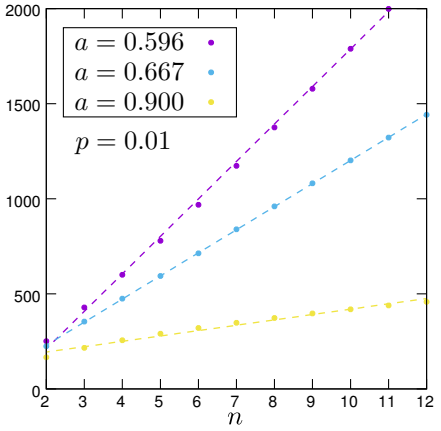

FIG. 2. Optimal raw waiting time as a function of n. For some values of p and a it is approximately linear.

in the state 012 and fail, this state goes to 0(1)(1)(2). With probability q the next state is 000(1), and with probability p it is 100(1). In the former case the next state will be $***0$, where $***$ is any combination of three zeros and ones, and in the latter case it will be $1**0$. So, the general rule is $(i) \to (i-1)$ if $i > 1$ and $(1) \to 0$. Let us illustrate possible transitions from the initial state 13: $13 \to (1)(1)(2)(3) \to 00(1)(2) \to 110(1) \to (1)(1)0(1) \to 0000$. We first try to swap and fail, restarting all the segments. Then, two segments are in the initial state and the other two are in progress still waiting for classical signals. Next, the ready segments both distribute entanglement, simultaneously succeeding here, another segment goes to the initial state, and the last segment is still in progress. Then, we try to perform swapping and fail, restarting the first two segments (the other two are in the same state since swapping itself takes no time). Finally, all segments are in the initial state, since the third segment failed to distribute entanglement. Note that this is only one of the possible transition sequences between repeater states in our model. These transitions illustrate that in multisegment repeaters several "waves" of restarting are possible—an earlier restart signal still in progress when a newer one starts to propagate. The influence of the classical communication on the raw waiting time is illustrated in the Supplemental Material [12] for various repeater sizes. It is interesting to compare the optimal waiting time with the "doubling" waiting time. The ratio of the two quantities is shown in Fig. 3 for the model including CC. This figure shows that there is a small but noticeable advantage of the optimal scheme. This advantage becomes more visible for larger repeaters, as Fig. 3 (right) demonstrates for an eight-segment repeater, in the practically highly relevant regime of small p and large a ($\sim$1.5% for $n = 4$ and $\sim$5% for $n = 8$).

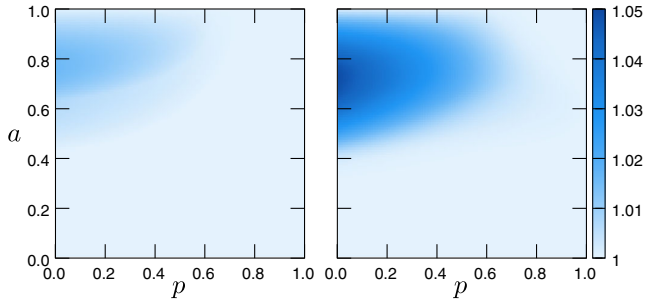

FIG. 3. Ratio of the "doubling" to the optimal waiting time for $n = 4$ (left) and $n = 8$ (right).

Conclusion.—In conclusion, we presented a method to determine the most efficient entanglement swapping scheme in a quantum repeater and demonstrated that the "doubling" scheme is not always the best. Moreover, our approach shows that when additional elements such as entanglement distillation on higher levels are excluded the power-of-two number of segments is not a distinguished case, since the best scheme can be constructed for any number of segments. We showed that for small repeaters the best scheme has a tiny, but noticeable, advantage over the "doubling" scheme, but this advantage seems to increase with the repeater's size. Our most general model leading to this conclusion includes all necessary classical communication times, while we were able to treat repeater sizes up to the order of ten segments. It is currently intractable to treat 16 repeater segments or more for a direct comparison.

Our algorithm has exponential complexity and thus is applicable to fairly "small" repeaters only, but even these repeaters are still beyond current technological capabilities, and so our approach here is fully applicable to current experiments to have meaningful physical benchmarks. Moreover, a ten-segment repeater can cover a distance of around 1000 km, which is already of practical interest. On the other hand, great progress has been made in algorithms for solving linear optimization problems. A study of different versions of an optimization software, CPLEX, performed in Ref. [23], shows a speedup of a factor of 29 000 only due to algorithmic advantages. Combined with hardware advances that happened during this time (two decades), we get an even more impressive performance boost factor. Of course, no technological advance can turn an exponential algorithm to a subexponential one, but what seems intractable now could become feasible in the near future.

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[1] In a typical quantum repeater system, the parameter p is primarily given by the probability that a photonic qubit is successfully transmitted via a fiber channel of length L_0 connecting two stations, $e^{-L_0/22\ \mathrm{km}}$. It also includes local

state preparation/detection, fiber coupling, frequency conversion, and memory write-in efficiencies. The parameter a can be related to the memory read-out or an optical Bell measurement efficiency.

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[19] The value $p = 0.01$ corresponds to a segment length of $L_0 = 100$ km, only considering fiber channel loss. Thus, through segment numbers $n = 2, \ldots, 12$ we can cover total distances from 200 km to 1200 km. A swapping probability value of, for example, $a = 2/3$ is meaningful, because it corresponds to the efficiency of a linear-optics Bell measurement enhanced by means of four auxiliary photons [20]. The smallest value considered in Fig. 2 for the swapping probability, $a = 0.596$, corresponds to the Bell measurement efficiency that is obtainable with the help of active linear mode transformations, i.e. when the auxiliary photons are replaced by squeezing operations [21,22].

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