Classical Pendulum Clocks Break the Thermodynamic Uncertainty Relation

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The thermodynamic uncertainty relation expresses a seemingly universal trade-off between the cost for driving an autonomous system and precision in any output observable. It has so far been proven for discrete systems and for overdamped Brownian motion. Its validity for the more general class of underdamped Brownian motion, where inertia is relevant, was conjectured based on numerical evidence. We now disprove this conjecture by constructing a counterexample. Its design is inspired by a classical pendulum clock, which uses an escapement to couple the motion of an oscillator to another degree of freedom (a "hand") driven by an external force. Considering a thermodynamically consistent, discrete model for an escapement mechanism, we first show that the oscillations of an underdamped harmonic oscillator in thermal equilibrium are sufficient to break the thermodynamic uncertainty relation. We then show that this is also the case in simulations of a fully continuous underdamped system with a potential landscape that mimics an escaped pendulum.

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Introduction.—Being able to tell the time precisely, regardless of astronomical observation, has ample importance for virtually all of human civilization. Ancient water clocks or hour glasses relied on the steady flow of matter. However, the precision of such devices was strongly limited, because of inhomogeneities or unpredictable external influences. A revolution in the history of timekeeping was the invention of the escapement. This mechanism uses coherent oscillations of a physical system to regulate the forward motion of a cog loaded with a weight and connected to a hand that displays time. Galileo realized that a swinging pendulum can provide such oscillations, since its period is (for small angles) independent of its amplitude. This inspired Huygens' invention of the pendulum clock in 1656, setting standards in precision for centuries to come [1]. This Letter shows that this well-established principle even allows for precision beyond the thermodynamic limits that have so far been believed to apply to classical systems.

The performance of a clock can be quantified by its precision and its turnover of energy. These quantities take center stage in the thermodynamic uncertainty relation (TUR) [2], formulated originally for biomolecular systems [3]. It describes a trade-off between the overall cost for driving a system and the precision observed in any output current.

More specifically, we consider a Markovian system in a steady state producing an integrated current $Y(t)$ (e.g., the accumulated angle of a clock hand). The energetic cost of driving is quantified by the entropy production rate σ. It corresponds (in the absence of chemical changes) to the heat dissipated into a surrounding heat bath, divided by its constant temperature T. This heat needs to be equal to the energy expended on the system's driving. The TUR states that

$$\frac{\mathrm{Var}\ Y(t)}{\langle Y(t)\rangle^2} t\sigma \geq 2, \tag{1}$$

where we set Boltzmann's constant $k_B = 1$ and define $\mathrm{Var}\ Y(t) \equiv \langle Y(t)^2\rangle - \langle Y(t)\rangle^2$, with averages $\langle \ldots \rangle$ taken in the steady state. This relation was first proven for the limit of large times t [4], and later generalized to finite times [5,6].

The TUR rests on the premise of local detailed balance, relating the log ratio of forward and backward transition rates between discrete states of a system to the entropy produced in a transition [7]. Brownian diffusion fits into this framework if it can be described as *overdamped*, meaning that momentum variables are assumed to relax instantly to a local equilibrium. Here, the TUR is recovered either through a fine discretization of the state space or directly from a Langevin description [8,9].

An underdamped description of Brownian dynamics explicitly retains the inertia that is present in every classical dynamical system. Yet, this more general dynamics lies beyond the original framework of the TUR. There, transitions in a finely discretized phase space appear irreversible without the simultaneous reversal of momenta, which

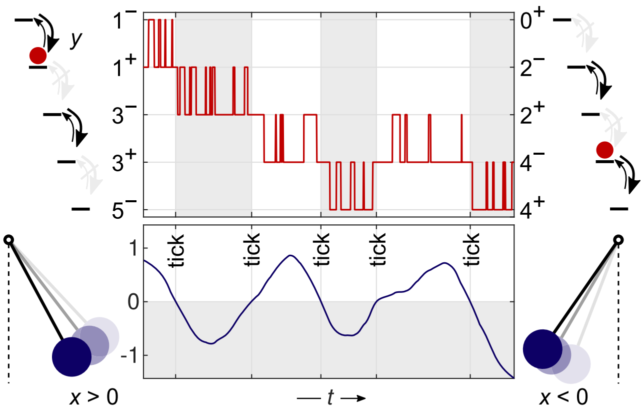

FIG. 1. Minimal model for a pendulum clock subject to thermal noise. An oscillating degree of freedom x (bottom), is coupled to a discrete counter degree of freedom (top), where y labels pairs of adjacent states y^- and y^+. A typical trajectory is shown in the middle, with snapshots of the configuration at its beginning (left) and end (right). Transitions of the counter are generally biased downward. When $x > 0$, transitions between even pairs of states $y^\pm$ are strongly suppressed and transitions between odd ones are very fast, and vice versa for $x < 0$. This way, transitions from one pair of states y to a neighboring one are possible only upon a "tick," when the pendulum crosses through $x = 0$.

leads to a formally divergent entropy production, despite the actual entropy production being finite [10]. Bounds on the precision of irreversible currents in underdamped systems have been derived [10–14], however, they are weaker than the TUR or require additional information about the system (beyond the entropy production rate). Violations of the TUR in its original form have been observed where an external magnetic field breaks time-reversal symmetry [15,16]. It is also known that ballistic motion on short timescales spoils the finite-time variant of the TUR. Yet, numerical evidence suggested that the TUR would hold for large times [17], in line with the intuition that sufficient driving is needed to overcome the time-reversal symmetry of thermal equilibrium.

In this Letter, we provide a disproof of the TUR for underdamped dynamics. As it turns out, the principle of an escapement, originally conceived to ward off environmental perturbations, is effective even if these are of purely thermal origin. We develop a minimal thermodynamically consistent model for a pendulum clock, which yields at given energetic cost high precision beyond the limits of the TUR.

Once wound, pendulum clocks operate autonomously. Hence, they are different from periodically driven systems, for which refined TUR's have been derived [18–22]. Taken for themselves, periodic systems do not obey the original TUR. It nonetheless holds on a global scale taking into account the infinite cost required to generate a deterministic protocol in a Markovian framework [23]. We take this idea further, describing an escapement as a way to couple a discrete system to an oscillating system that is not perfectly precise, but comes at small (or even zero) energetic cost.

General setting.—We consider a system consisting of two subsystems, as shown in Fig. 1. The first, which we call an "oscillator," could be an arbitrary physical system described by a state $x(t)$ (in our main example and Fig. 1 we choose this to be a pendulum subject to thermal noise). The other subsystem, which we call a "counter," is a one-dimensional degree of freedom on an infinite discrete lattice. For notational convenience we label pairs of adjacent states by integers y, denoting the upper state as y^- and the lower state as y^+, and identifying y^+ and $(y+1)^-$ as the same states of the counter. In the picture of a pendulum clock, the discrete state space of the counter corresponds to orientations of the seconds hand, mapped to the infinite number line by keeping account of full revolutions around the clock face.

Jumps between any y^- and the corresponding y^+ occur continuously in time and are biased toward the latter by a nonconservative force (e.g., provided by a weight on a cord). The work delivered by this force in a forward jump divided by the temperature T defines the affinity A, fixing the log ratio of the rates for forward and backward transitions through the local detailed balance condition

$$\ln\frac{k(y^- \to y^+)}{k(y^+ \to y^-)} = A. \tag{2}$$

Since the states y^- and y^+ have the same internal energy, the first law requires the work to be dissipated as heat, increasing the entropy in the environment by A. Conversely, a backward step decreases this entropy by A.

The escapement mechanism is realized by exploiting the freedom of choice of a common prefactor to both rates, which can be made dependent on the state x of the oscillator. We model that the escapement can be in either of two states $i(x) \in \{0, 1\}$. If x is the angular displacement of a pendulum, an obvious choice is $i(x) = 1$ for $x \geq 0$ and $i(x) = 0$ for $x < 0$. For $i(x) = 1$, we set the transition rates $k(y^\mp \to y^\pm) = k^\pm$ for y odd and $k(y^\mp \to y^\pm) = \varepsilon k^\pm$ for y even. Vice versa, for $i(x) = 0$, we set $k(y^\mp \to y^\pm) = \varepsilon k^\pm$ for y odd and $k(y^\mp \to y^\pm) = k^\pm$ for y even. The rates k^+ and $k^- = k^+ \exp(-A)$ are both chosen to be much larger than the inverse of the fastest relevant timescale of the oscillator. In contrast, the factor $\varepsilon > 0$ is chosen sufficiently small such that the rates $\varepsilon k^\pm$ are much smaller than the inverse of the slowest timescale of the oscillator. This choice of rates ensures that, after a change of the state $i(x)$, the counter is effectively constrained to a single pair of states $y^\pm$, between which it equilibrates quickly. It can then be found in either of the states at the conditional probability

$$p_\pm = k^\pm/(k^+ + k^-) = \exp(\pm A/2)/[2\cosh(A/2)]. \tag{3}$$

Next time the state $i(x)$ changes, the counter will have ended up in either of the states $y^\pm$, whose link then gets effectively broken. Depending on this outcome, the counter then proceeds to fluctuate either between states $(y-1)^{\pm}$ or $(y+1)^{\pm}$.

We see that upon every change of $i(x)$, called a "tick" in the following, the variable y, labeling the pair of states between which the counter currently fluctuates, performs a single step of an asymmetric random walk. In this random walk, the number of ticks $N(t)$ up to time t plays the role of a discrete time. The steps taken by y upon subsequent ticks are independent and identically distributed, such that (given $y=0$ at time $t=0$) the central limit theorem yields the conditional probability $p(y|N)$ as a Gaussian with mean and variance

$$NJ_{y|N} \equiv N(p^{+}-p^{-}) = N\tanh(A/2), \tag{4}$$

$$2ND_{y|N} \equiv N[p^{+}+p^{-}-(p^{+}-p^{-})^{2}] = N/\cosh^{2}(A/2). \tag{5}$$

While the dynamics of the counter depends strongly on the dynamics of the oscillator, there is no feedback in the other direction. This is possible in our model because a change of $i(x)$ leaves the energy levels of the counter intact, such that no work is transferred between the two subsystems. In practice, where the discrete dynamics of the counter is derived from the diffusion in a corrugated potential, the reduction of rates by the factor ε requires the insertion of some potential barrier. This can be achieved at the expense of a vanishingly small amount of work, by finely tuning the width and height of the barrier [18].

With this insight, we see that the process $N(t)$ counting the number of ticks generated by the oscillator up to time t is *a priori* independent of the counter y. Given the distribution $p(N,t)$, the distribution of the state of the counter at time t follows as

$$p(y,t) = \sum_{N} p(y|N)p(N,t). \tag{6}$$

We now assume that the dynamics of the oscillator is such that $N(t)$ satisfies a central limit theorem with mean $\langle\dot{N}\rangle t$ and variance $2D_N t$, as will be the case for the harmonic oscillator considered below. Then, the distribution $p(y,t)$ is Gaussian as well with mean and variance

$$\langle y(t)\rangle = \langle\dot{N}\rangle J_{y|N} t, \tag{7}$$

$$\mathrm{Var}\ y(t) = 2(D_{y|N}\langle\dot{N}\rangle + D_N J_{y|N}^{2})t. \tag{8}$$

The entropy production rate of the counter is given by

$$\sigma_{\mathrm{ctr}} = A\langle\dot{N}\rangle J_{y|N}, \tag{9}$$

which is the affinity of a single step multiplied by the net rate of forward steps. The entropy production rate for the total system $\sigma = \sigma_{\mathrm{ctr}} + \sigma_{\mathrm{osc}}$ follows by adding the entropy production of the oscillator σ_{osc}.

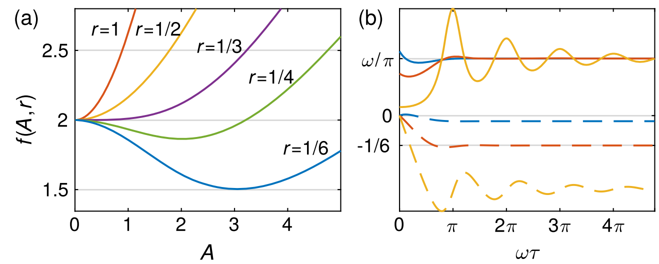

FIG. 2. (a) Plot of the function $f(r,A)$ of Eq. (11), for selected values of the parameter r. The value $r=1/3$ is a critical one, below which a minimum below one occurs at nonzero A. (b) Correlation function $\langle\dot{N}(0)\dot{N}(t)\rangle/\langle\dot{N}\rangle$ (solid) and its integral $\int_{0^+}^{t} d\tau[\langle\dot{N}(0)\dot{N}(\tau)\rangle/\langle\dot{N}\rangle - \langle\dot{N}\rangle]$ (dashed). The parameter $\gamma/(m\omega)$ has values 1.5 (blue/dark gray), 0.981 (red/medium gray), and 0.2 (yellow/light gray).

Given the three relevant quantities for the oscillator, σ_{osc}, $\langle\dot{N}\rangle$, and D_N, we can now check whether a TUR of the form (1) holds for the observable y for all values of the affinity A. If it does not, then the TUR cannot be valid for the type of dynamics underlying the oscillator.

In particular, if the oscillator system is in thermal equilibrium, the only entropy production is that of the counter, and the product of relative uncertainty and entropy production becomes

$$\lim_{t\to\infty} \frac{\mathrm{Var}\ y(t)}{\langle y(t)\rangle^{2}} t\sigma = f(A, D_N/\langle\dot{N}\rangle) \tag{10}$$

with the function

$$f(A,r) = 2A[1/\sinh(A) + r\tanh(A/2)], \tag{11}$$

shown in Fig. 2(a). For $r \geq 1/3$, it has the global minimum 2, attained for $A \to 0$. Hence, for $D_N/\langle\dot{N}\rangle \geq 1/3$, an inequality in the form of the TUR holds. However, for $D_N/\langle\dot{N}\rangle < 1/3$ the relation is broken for sufficiently small values of A.

Harmonic oscillator in equilibrium.—We now specify the oscillator system to be a pendulum, modeled as an underdamped harmonic oscillator in a heat bath at the same temperature T as the heat bath of the counter. The position x and velocity v obey the Langevin equation

$$\dot{x} = v, \qquad m\dot{v} = -m\omega^{2}x - \gamma v + \xi(t), \tag{12}$$

where the dot denotes a time derivative, m is the mass, ω the undamped angular frequency, and γ the damping coefficient. The Gaussian white noise $\xi(t)$ has zero mean and correlations $\langle\xi(t)\xi(t')\rangle = 2\gamma T\delta(t-t')$. The equilibrium state is the Gaussian $p^{\mathrm{eq}} = \exp[-E(x,v)/T]/Z$ with the energy $E(x,v) = m(\omega^{2}x^{2}+v^{2})/2$ and normalization Z.

Partitioning the state space into $i(x)=1$ for $x \geq 0$ and $i(x)=0$ for $x<0$, ticks occur whenever x crosses through zero. The total number of ticks up to time t is

$$N(t) = \int_0^t d\tau |v(\tau)| \delta[x(\tau)], \tag{13}$$

where the factor $|v|$ ensures that every crossing of $x = 0$ at speed v increments $N(t)$ by one. The average rate of ticks follows readily from the equilibrium distribution as

$$\langle \dot{N} \rangle = \int dx \int dv p^{\mathrm{eq}}(x, v) |v| \delta(x) = \omega/\pi. \tag{14}$$

The dispersion of ticks is calculated as [24]

$$D_N = \frac{1}{2} \langle \dot{N} \rangle + \int_{0^+}^{\infty} d\tau [\langle \dot{N}(0) \dot{N}(\tau) \rangle - \langle \dot{N} \rangle^2]. \tag{15}$$

The lower limit 0^+ indicates that the trivial self-correlation of every tick at $\tau = 0$ is excluded, it produces the first term. The correlation function $\langle \dot{N}(0) \dot{N}(\tau) \rangle / \langle \dot{N} \rangle$ [shown in Fig. 2(b)] can be interpreted as the probability density of a tick occurring at time τ given a tick at time 0. We calculate it analytically from the Gaussian propagator [25] and evaluate the integral in Eq. (15) numerically. The mean and variance of the counter variable $y(t)$ can then be calculated using Eqs. (7) and (8), provided the timescale separation $k^{\pm} \gg \omega/\pi$ and $k^{\pm} \gg \gamma/m$ [25]. For small damping γ, the correlation function exhibits oscillations with maxima at subsequent ticks, which are initially sharply peaked and then become broader. The time in between these peaks can have a sufficient negative contribution, so that the overall integral becomes less than $-\langle \dot{N} \rangle/6$, yielding $D_N/\langle \dot{N} \rangle < 1/3$. This is the case for the damping below a certain critical value, determined numerically as $\gamma/(m\omega) \simeq 0.981$. Remarkably, this critical value presents still a fairly strong damping, with just one coherent oscillation discernible in Fig. 2(b). For any damping weaker than that, the TUR is violated for matching affinity A.

In the limit of vanishing damping, $\gamma \to 0$, the sequence of ticks becomes deterministic (regardless of the energy, which is sampled initially from p^{eq}). The counter system then behaves as a discrete-time Markov process, for which the possibility of violating the TUR (1) is well known [19,26]. Yet, we show here that such a discrete-time process can be realized as a limiting case of a continuous one, without additional entropic cost. In accordance with the discrete-time TURs of Refs. [19] and [27], our model allows for a vanishing uncertainty product (10) for $D_N/\langle \dot{N} \rangle \to 0$ and $A \to \infty$. The latter entails either divergent entropy production or vanishing speed $\langle \dot{y} \rangle$. For clocks that require the hand to move forward at a nonvanishing speed, a recent study shows that precision does indeed come at a minimal energetic cost [28].

Continuous model.—So far, we have shown that the TUR does not hold for systems consisting of a discrete and an underdamped continuous degree of freedom. We now show numerically that the TUR can also be broken with two continuous degrees of freedom. Moreover, we consider an escapement that, like in actual pendulum clocks, provides a feedback on the oscillator to sustain amplitudes beyond those of equilibrium oscillations.

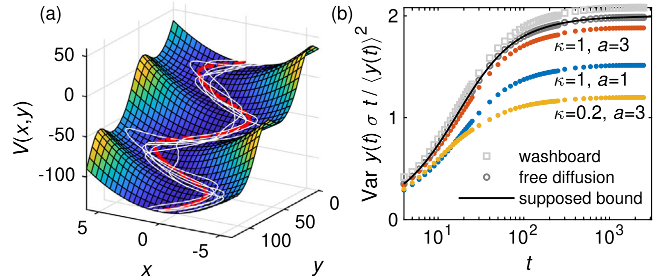

FIG. 3. (a) Potential landscape $V(x, y)$ serving as a continuous model for an escapement. The external force $f^{\mathrm{ex}} = 1$ is indicated as a tilting in the y direction. The ideal curve $\hat{x}(y)$ is shown as red dashed, with sample trajectories in white. Parameters: $\kappa = 1$, $a = 3$, $\omega = 1$, $\gamma = 0.1$, $T = 1$, and $m = 1$. (b) Uncertainty product as a function of the length t of the time window. The bound conjectured in Ref. [17] (solid line) corresponds to free diffusion in y ($\kappa = 0$, dark gray). This bound holds for the washboard potential $V_c(\boldsymbol{r}) = (f^{\mathrm{ex}}\gamma/\omega) \sin(\omega y \gamma / f^{\mathrm{ex}})$ (light gray). For the escapement potential the bound is broken. In particular, the long-time limit is below the value 2 relevant for the TUR (1) for all three combinations of the parameters κ, a shown (red, blue, yellow). All other parameters as in (a).

We use an underdamped Langevin equation of the form

$$m\ddot{\boldsymbol{r}} = -\nabla V(\boldsymbol{r}) - \gamma \dot{\boldsymbol{r}} + f^{\mathrm{ex}} \boldsymbol{e}_y + \boldsymbol{\xi}(t) \tag{16}$$

for the configuration $\boldsymbol{r} = (x, y)^T$, with the mass m, damping γ, a driving force f^{ex} acting in the y direction, and a noise term $\boldsymbol{\xi}(t)$ with two independent components with the same properties as for the harmonic oscillator above. The potential is harmonic in x, with an additional coupling term, $V(\boldsymbol{r}) = m\omega^2 x^2/2 + V_c(\boldsymbol{r})$.

We choose the coupling term such that it reinforces the harmonic motion of an undamped oscillator of frequency ω and a certain amplitude a in the x direction while moving steadily at terminal velocity f^{ex}/γ in the y direction. This ideal motion traces out the curve $\hat{x}(y) = a \sin(\omega y \gamma / f^{\mathrm{ex}})$, and we choose the coupling potential such that this curve is favored, setting $V_c(\boldsymbol{r}) = \kappa [x - \hat{x}(y)]^2/2$, with some stiffness κ. Figure 3(a) shows this potential and the ideal curve. For suitably chosen κ and a, sample trajectories follow the ideal curve closely. The potential landscape acts as an escapement, providing potential barriers that impede the motion in the y direction when it is too fast, and accelerates it when it is too slow.

We simulate the steady state of the system by numerically integrating the Langevin equation (16), using the method of Ref. [29]. The variance of $y(t)$ is calculated for samples of time windows of length t taken from a long trajectory. The current $J_y = \langle y(t) \rangle / t$ (for any t) is evaluated as the average speed over the whole trajectory, and the entropy production rate follows as $\sigma = f^{\mathrm{ex}} J_y / T$.

As a result, the escapement mechanism suppresses fluctuations in the y direction, compared to the fluctuations of an underdamped particle diffusing freely, see Fig. 3(b). In Ref. [17], it had been conjectured that free diffusion sets a lower bound on the uncertainty product [the left hand side of the TUR (1)] for underdamped dynamics on finite timescales. While this supposed bound holds true for most generic potential landscapes (and in particular for uncoupled dynamics, e.g., a washboard potential in the y direction independent of x), it is broken for our design of an escapement. Violations occur for any $t > 0$, and in particular in the long-time limit, over a robust range of the parameters κ and a.

Outlook.—We have used a simple design of an escapement coupled to a pendulum to construct a counterexample to the TUR for underdamped dynamics. The considerations that have led to Eqs. (7)–(9) are specific for the model of the escapement, but completely general about the oscillator producing the ticks. The application to other physical systems may also be fruitful. For instance, for quantum systems exhibiting coherent oscillations, a general TUR of the form (1) could also be ruled out, in line with previous observations [15,30,31]. An atomic clock could hence yield precision beyond the limitations of the TUR as well.

Likewise, a thermodynamically consistent analysis of resistor-inductor-capacitor (RLC) circuits [32] could reveal coherent oscillations similar to those of the underdamped oscillator, showing that the TUR is broken not only for constant, external magnetic fields but also for fluctuating magnetic fields generated by the system itself. Steady state thermoelectric devices exploiting this fact could evade the trade-off between power, efficiency, and constancy that follows from the TUR [33], similar to cyclically driven heat engines [34,35].

Future research, systematically comparing different designs of escapement mechanisms and oscillator systems, may reveal ultimate thermodynamic bounds on the precision of autonomous clocks and complement thermodynamic uncertainty relations for underdamped dynamics.

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Correction: The fifth sentence of the abstract contained an error and has been fixed.