Eigenvalues of Random Matrices with Generalized Correlations: A Path Integral Approach

Joseph W. Baron,[1,*] Thomas Jun Jewell,[2] Christopher Ryder,[2] and Tobias Galla[1,2,†]
[1]Instituto de Física Interdisciplinar y Sistemas Complejos IFISC (CSIC-UIB), 07122 Palma de Mallorca, Spain
[2]Department of Physics and Astronomy, School of Natural Sciences, The University of Manchester, Manchester M13 9PL, United Kingdom

(Received 13 August 2021; revised 10 January 2022; accepted 23 February 2022; published 22 March 2022)

Random matrix theory allows one to deduce the eigenvalue spectrum of a large matrix given only statistical information about its elements. Such results provide insight into what factors contribute to the stability of complex dynamical systems. In this Letter, we study the eigenvalue spectrum of an ensemble of random matrices with correlations between any pair of elements. To this end, we introduce an analytical method that maps the resolvent of the random matrix onto the response functions of a linear dynamical system. The response functions are then evaluated using a path integral formalism, enabling us to make deductions about the eigenvalue spectrum. Our central result is a simple, closed-form expression for the leading eigenvalue of a large random matrix with generalized correlations. This formula demonstrates that correlations between matrix elements that are not diagonally opposite, which are often neglected, can have a significant impact on stability.

DOI: 10.1103/PhysRevLett.128.120601

Determining the factors that contribute to the stability of a dynamical system with many interacting components is a fundamental problem. The theory of large random matrices demonstrates that we can ascertain the stability of such a system given only statistical information about its microscopic interactions. As such, random matrix theory (RMT) has found myriad applications outside its original field of conception, that of nuclear and atomic physics [1,2], and has become a rich and active area in its own right [3–7]. Among the diverse range of fields where RMT enjoys a centrally important role are spin glasses [8–11], complex ecosystems [12–16], and neural networks [17–21].

As a result of the desire to apply RMT in a greater range of contexts, its remit has been expanded to encompass an evermore complete collection of random matrix ensembles. For example, it has been shown that Wigner's semicircle law [1,2] can be generalized for asymmetric matrices, which have eigenvalues that are uniformly distributed in an ellipse in the complex plane [22,23]. Allowing for a uniform nonzero mean for each of the matrix elements gives rise to an additional outlier eigenvalue [24–27]. Recently, the eigenvalue spectra of more elaborate block-structured matrices [28,29] and matrices with element-specific variability [17,19] have also been investigated.

Despite the aforementioned developments, typically only correlations between matrix elements that are diagonally opposite each other (i.e., a_{ij} and a_{ji}) are included in RMT calculations (with the notable exception of Ref. [30], where cyclic correlations are explored). This is a rather artificial restriction. In fact, correlations between matrix elements sharing only one index (e.g., a_{ij} and a_{ki}) have been shown to arise organically in dynamically evolved ecosystems [31], and they appear in contexts as wide-ranging as statistical inference [32], game theory [33,34], and data security [35].

The main result of this Letter is a surprisingly simple closed-form expression for the leading eigenvalue of an ensemble of large random matrices with generalized correlations. This includes correlations between elements that are not transpose pairs. The formula that we derive demonstrates directly that such correlations can have a significant impact on stability and therefore should not be dismissed.

To obtain this result, we have developed a new analytical approach. We exploit a correspondence between the resolvent of the random matrix and the response functions of a linear dynamical system [36]. With this duality in mind, one can write down the Martin-Siggia-Rose-Janssen-de Dominicis (MSRJD) [37–41] path integral for the dynamical system and use field-theoretic methods to find the response functions. We are thus able to find the resolvent matrix and, consequently, the leading eigenvalue.

Our method has several advantages. First, we are able to demonstrate explicitly that our results do not depend on the precise distribution from which the matrix elements are drawn (a property known as universality [42,43]). Further, the dynamical approach we use has no need for replicas [44]. Finally, we are able to include the effects of each of the different correlations in our ensemble one by one, greatly reducing the complexity of the calculation. Particularly because of this last property, we believe that

our analytical approach would lend itself to simplifying other similarly challenging problems in RMT.

Let us now define more precisely the ensemble of random matrices on which we will focus. Consider a square matrix of size $N \times N$ whose elements $\{a_{ij}\}$ are drawn from a joint probability distribution. We decompose the matrix $\underline{\underline{a}}$ as follows:

$$a_{ij} = -\delta_{ij} + \frac{\mu}{N} + z_{ij}, \tag{1}$$

where μ/N is the mean of the off-diagonal elements and the $\{z_{ij}\}$ encode fluctuations about this mean such that $\langle z_{ij} \rangle = 0$ (angular brackets indicate an average over realizations of the random matrix). If $\underline{\underline{a}}$ were the Jacobian of a system linearized about a fixed point, the term $-\delta_{ij}$ in Eq. (1) would ensure the stability of the system in absence of interactions. Upon including the interactions encoded by the $\{z_{ij}\}$ and μ/N, one is able to deduce what statistical features of the interactions between components tend to make the system unstable.

We consider the most generic set of pairwise correlations for the elements z_{ij} that do not privilege any position in the matrix over any other. These are

$$\begin{aligned} \mathrm{var}(z_{ij}) &= \frac{\sigma^2}{N}, \\ \mathrm{corr}(z_{ij}, z_{ji}) &= \Gamma, \\ \mathrm{corr}(z_{ij}, z_{ki}) &= \frac{\gamma}{N}, \\ \mathrm{corr}(z_{ij}, z_{ik}) &= \frac{r}{N}, \\ \mathrm{corr}(z_{ji}, z_{ki}) &= \frac{c}{N}, \end{aligned} \tag{2}$$

where none of the indices i, j, or k take equal values, and where we use the shorthand $\mathrm{corr}(a,b) = [\langle ab \rangle - \langle a \rangle \langle b \rangle]/\sqrt{\mathrm{var}(a)\mathrm{var}(b)}$ for the correlation coefficients.

The scaling of the mean and the variance of each element with N in Eqs. (1) and (2) ensures a sensible thermodynamic limit [8,23,25].

We distinguish between two types of correlations in Eq. (2): type-1 correlations between diagonally opposite elements [these correlations are $\mathcal{O}(N^0)$ and quantified by Γ], and type-2 correlations between elements with one index in common (whose magnitudes are governed by γ, c, and r). The type-2 correlations must be of the order N^{-1}, again in order to ensure a sensible thermodynamic limit [31,33,34].

We do not consider explicitly correlations between elements that share no indices; the presence of such correlations can be shown to be produced by a value of μ that fluctuates between realizations of the random matrix (see Section I.B of the Supplemental Material (SM) [45]).

We now turn our attention to the eigenvalue spectrum. Consider the disorder-averaged eigenvalue density, which is defined as

$$\rho(\omega) = \left\langle \frac{1}{N} \sum_{i=1}^{N} \delta(\omega - \lambda_i) \right\rangle, \tag{3}$$

where the $\{\lambda_i\}$ are the eigenvalues of any one realization of the random matrix. The eigenvalue density is normalized such that $\int d^2\omega \rho(\omega) = 1$, where the integral covers the entire complex plane.

The nonzero mean of the elements of the matrix $\underline{\underline{a}}$ constitutes a rank-1 perturbation to the matrix $-\underline{\underline{\mathbb{1}}} + \underline{\underline{z}}$ [24–27]. Previous work thus allows us to anticipate the general form that the eigenvalue density $\rho(\omega)$ will take [14,25,26]. There are two contributions to the eigenvalue spectrum: a bulk region, to which the vast majority of the eigenvalues are confined, and a single outlier that results from the nonzero mean. An example is shown in Fig. 1. More precisely, we write

$$\rho(\omega) = \rho_{\mathrm{bulk}}(\omega) + \frac{1}{N}\delta(\omega - \lambda_{\mathrm{outlier}}). \tag{4}$$

The primary tool that we use for calculating the outlier eigenvalue and the bulk eigenvalue density is the resolvent matrix, defined in our case by

$$\underline{\underline{G}} = \langle [(1+\omega)\underline{\underline{\mathbb{1}}} - \underline{\underline{z}}]^{-1} \rangle. \tag{5}$$

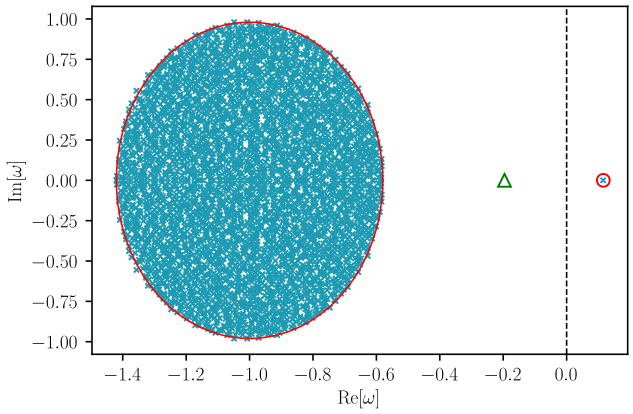

FIG. 1. An example eigenvalue spectrum. Crosses represent the results of numerical diagonalization of computer-generated random matrices from the ensemble with generalized correlations. The solid line is the ellipse $(1+x)^2/(1+\Gamma)^2 + y^2/(1-\Gamma)^2 = \sigma^2$ where $\omega = x + iy$. The red circle is the prediction for the outlier from Eq. (14). The green triangle is the prediction one would obtain ignoring type-2 correlations [$\lambda_\Delta = -1 + \mu + \Gamma\sigma^2/\mu$]. Parameters are $N = 4000$, $\sigma = 0.7$, $\Gamma = -0.4$, $\mu = 1$, $\gamma = 0.7$, $r = 1.4$, and $c = 1.6$.

The bulk eigenvalue density can be calculated from only the trace of the resolvent matrix. We define $G(\omega,\omega^\star)=(1/N)\sum_i G_{ii}(\omega,\omega^\star)$, noting that the resolvent may not necessarily be an analytic function of ω. The bulk spectrum can then be obtained as [23]

$$\rho_{\text{bulk}}(\omega)=\frac{1}{2\pi}\text{Re}\left[\frac{\partial G}{\partial\omega^\star}\right]. \tag{6}$$

Using established techniques [23,46–48], we find (see Section III of the SM) that the bulk eigenvalue density is independent of r, c, and γ (i.e., the type-2 correlations), and is given by the familiar elliptical law [22,23]

$$\rho_{\text{bulk}}(\omega)=\begin{cases}[\pi\sigma^2(1-\Gamma^2)]^{-1} & \text{if } \frac{(1+x)^2}{(1+\Gamma)^2}+\frac{y^2}{(1-\Gamma)^2}<\sigma^2,\\ 0 & \text{otherwise,}\end{cases} \tag{7}$$

where $\omega=x+iy$. This is verified in Fig. 1.

We now turn our focus to finding the outlier eigenvalue λ_{outlier}, for which the effect of the type-2 correlations is significant. The outlier eigenvalue satisfies the known relation [26,49] (see also Section II.A of the SM)

$$1-\frac{\mu}{N}\sum_{ij}G_{ij}(\lambda_{\text{outlier}})=0. \tag{8}$$

Notably, the off-diagonal elements of the resolvent matrix are required to find the outlier. Often, the resolvent matrix is presumed to be diagonal [9,19,29,41], but in the more general case considered here, the off-diagonal elements turn out to be crucial.

Two observations aid us in evaluating the resolvent matrix, including its off-diagonal elements: (i) The resolvent is an analytic function of ω outside the bulk region of the eigenvalue spectrum [see Eq. (6) and Refs. [19,23,50]]; (ii) When the resolvent is analytic, we can show that the elements of the resolvent matrix correspond to the response functions of the following linear dynamical system,

$$\dot{x}_i=-x_i+\sum_j z_{ij}x_j+h_i, \tag{9}$$

where the $\{h_i(t)\}$ are external fields. That is, the Laplace transforms of the response functions $R_{ij}(t-t')=\langle\delta x_i(t)/\delta h_j(t')\rangle$ (where we exploit time-translation invariance) are equal to the elements of the resolvent matrix defined in Eq. (5),

$$\hat{R}_{ij}(u)=G_{ij}(u). \tag{10}$$

This observation is the basis for our calculation. It means that if we can calculate the response functions of the dynamics in Eq. (9), we can use Eq. (8) to find λ_{outlier}.

To find the response functions we begin with the MSRJD generating functional [37–41] of the system in Eq. (9), and then carry out the disorder average along the lines of Refs. [44,51,52]. One thus obtains the following path integral expression for the response functions (see Section II.C of the SM):

$$R_{ij}(t-t')=-i\langle x_i(t)\hat{x}_j(t')\rangle_S,\qquad \langle\cdots\rangle_S=\int D[\mathbf{x},\hat{\mathbf{x}}][\cdots]e^{S_0+S_{\text{int}}}, \tag{11}$$

where $\int D[\mathbf{x},\hat{\mathbf{x}}]$ indicates a functional integral over all possible trajectories of the coordinates $\{x_i(t)\}$ and their conjugates $\{\hat{x}_i(t)\}$ [40,41].

We identify two contributions to the action in Eq. (11): a "bare" action, which would still be present if we were to set $r=c=\gamma=0$, and an "interaction" term, which comes about due to the type-2 correlations

$$\begin{aligned}S_0=&\,i\sum_i\int dt[\hat{x}_i(t)(\dot{x}_i(t)+x_i(t)-h_i(t))]\\&-\frac{\sigma^2}{2N}\sum_{(i,j)}\int dtdt'[\hat{x}_i(t)\hat{x}_i(t')x_j(t)x_j(t')\\&+\Gamma\hat{x}_i(t)x_i(t')\hat{x}_j(t')x_j(t)],\\S_{\text{int}}=&-\frac{\sigma^2}{2N^2}\sum_{(i,j,k)}\int dtdt'[2\gamma\hat{x}_i(t)x_j(t)\hat{x}_k(t')x_i(t')\\&+r\hat{x}_i(t)x_j(t)\hat{x}_i(t')x_k(t')+c\hat{x}_i(t)x_j(t)\hat{x}_k(t')x_j(t')].\end{aligned} \tag{12}$$

The notation (i,j,k) indicates that only combinations where none of the indices take equal values contribute to the sum. We note that one benefit of the dynamic formalism presented here is that this action, and therefore the outlier eigenvalue, can be seen to be universal with relative ease [42,43] (see SM Section II C).

We now expand the exponential in Eq. (11) in powers of S_{int} and write the series expansion for the "dressed" response functions $R_{ij}(t-t')$ in terms of the bare response functions $R^{(0)}_{ij}(t-t')$ (the response functions of the system with $S_{\text{int}}=0$). This is accomplished using dynamic mean-field theory, which simultaneously yields an expression for $R^{(0)}_{ij}(t-t')$. Conveniently, our dynamic approach allows us to consider each contribution to the interaction action S_{int} one by one.

The terms of the series for $R_{ij}(t-t')$ can be represented efficiently using diagrams [41] (see Fig. 2). These "rainbow" diagrams have a structure similar to those representing quark-gluon interactions [53,54].

The dynamic formulation of the random matrix problem drastically reduces the complexity of the diagrammatic series that one has to evaluate. This is the main advantage of

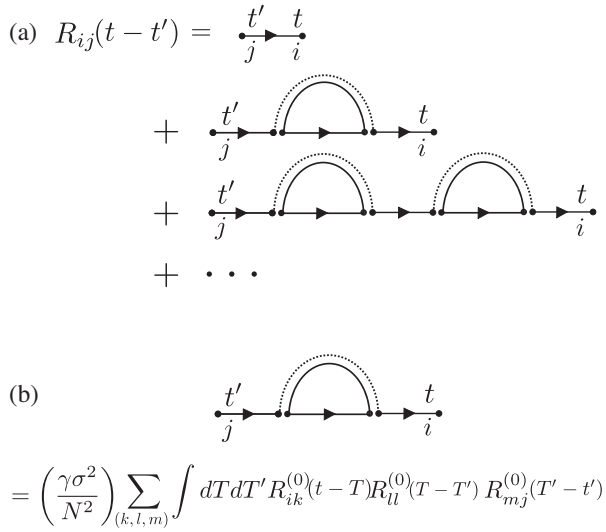

FIG. 2. Panel (a) depicts the diagrammatic representation of the series used to evaluate the response function $R_{ij}(t-t')$. Panel (b) indicates the value of the second diagram in panel (a) to leading order in N^{-1}. Each node is associated with a site index (i, j, k, etc.). Nodes that are placed closely together have the same time coordinate and each such pair carries a factor of $(\gamma\sigma^2/N^2)^{\frac{1}{2}}$. Directed edges each carry a factor of the bare response function. Solid undirected arcs connect indices which share the same value and are summed over [l in the example in panel (b)]. Dashed arcs indicate indices that are summed, but cannot take the same value [e.g., k and m in panel (b)].

our approach over established diagrammatic techniques in RMT [19,54]. A more detailed discussion can be found in Section II of the SM.

The series expansion of the response function depicted in Fig. 2 can be evaluated exactly in the limit $N \to \infty$, without the need for further approximation. We find

$$\frac{1}{N}\sum_{ij}\hat{R}_{ij} = \hat{R}^{(0)}[1-\gamma\sigma^2(\hat{R}^{(0)})^2]^{-1}, \tag{13}$$

where $\hat{R}^{(0)}(u)$ is the average diagonal element of the bare resolvent matrix, given by $\hat{R}^{(0)}(u) = [(1+u)/2\Gamma\sigma^2] \times [1+\sqrt{1-4\Gamma\sigma^2/(1+u)^2}]$ for $|1+u| > \sigma(1+\Gamma)$ and $u \in \mathbb{R}$.

Using the correspondence in Eq. (10) we substitute the result of Eq. (13) into Eq. (8) and solve for λ_{outlier}. We finally obtain our central result: a formula for the outlier eigenvalue,

$$\lambda_{\text{outlier}} = -1 + \mu + \frac{\mu}{2}\left(1+\frac{\Gamma}{\gamma}\right)\left(\sqrt{1+\frac{4\gamma\sigma^2}{\mu^2}}-1\right). \tag{14}$$

This expression is valid when $|\mu| > \sigma(1-\gamma)$; there is no outlier eigenvalue when $|\mu| \le \sigma(1-\gamma)$. We observe that $\lambda_{\text{outlier}} \to -1 \pm (1+\Gamma)\sigma$ when $\mu \to \pm\sigma(1-\gamma)$. That is, the points at which the expression in Eq. (14) becomes invalid correspond to the outlier being absorbed into the bulk region. In the limit $\gamma \to 0$, we recover from Eq. (14) the known result $\lambda_{\text{outlier}} = -1 + \mu + \Gamma\sigma^2/\mu$ [25,26,29], which is valid when $|\mu| > \sigma$.

We note that the in-row and in-column correlations, quantified by r and c, respectively, do not appear in Eq. (14). As such, these correlations do not alter the eigenvalue spectrum in the thermodynamic limit. This is in contrast to the correlations quantified by γ, which affect the outlier eigenvalue significantly. The reason for the absence of r and c in Eq. (14) is discussed in SM Section II G.

We test the prediction of Eq. (14) in Fig. 3 using computer-generated random matrices. The method used

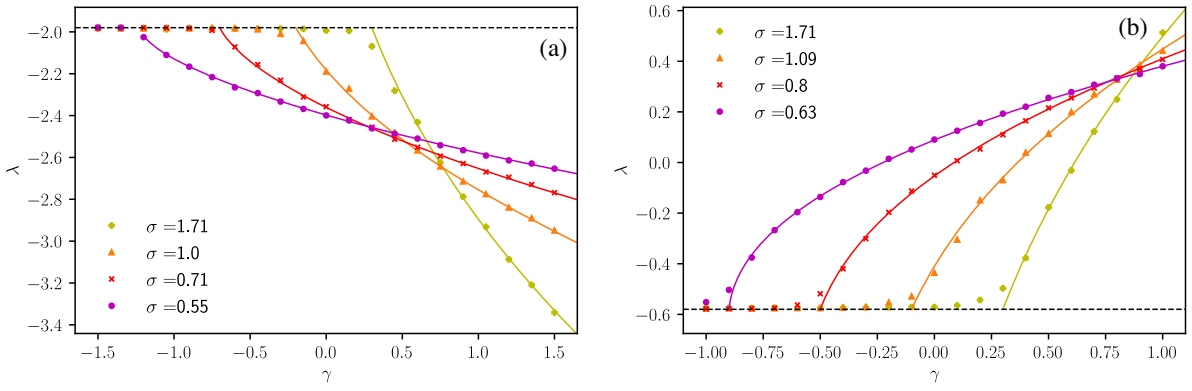

FIG. 3. The leftmost eigenvalue [panel (a)] and the rightmost eigenvalue [panel (b)] as a function of the strength of the type-2 correlations γ for different combinations of the parameters Γ and σ. Symbols are from numerical diagonalization of computer-generated random matrices, solid lines are the prediction in Eq. (14). The horizontal dashed lines show the edges of the bulk spectrum at $\lambda^{\pm}_{\text{edge}} = -1 \pm (1+\Gamma)\sigma$. The values of σ and Γ are such that $(1+\Gamma)\sigma = 0.98$ in panel (a), and $(1+\Gamma)\sigma = 0.42$ in panel (b). In panel (a), matrix entries are from a Bernoulli distribution with $\mu = -1.2$, $r = 1.4$, and $c = 1.6$. In panel (b), matrix entries are constructed from uniformly distributed random numbers such that $\mu = 1.2$, $r = 1$, and $c = 1$. Numerical results are for $N = 10\,000$, averaged over 10 trials.

to produce random matrices with the correlations in Eqs. (2) is described in Section IV of the SM.

Figure 3 not only serves to verify the theoretical predictions in Eqs. (14) and (7) and their universality, but it also demonstrates the extent to which the type-2 correlations between a_{ij} and a_{ki} (quantified by γ) can affect the position of the outlier. In particular, the presence of these correlations can give rise to an outlier eigenvalue when there would not ordinarily be one. As is shown in Fig. 3(b) in particular, this outlier eigenvalue can become positive, resulting in instability, purely as a result of varying γ (see also Fig. 1).

To conclude, correlations between pairs of nontranspose elements (e.g., a_{ij} and a_{ki}) are routinely neglected in many problems in random matrix theory. In this work, we developed a dynamic path-integral approach to take these more subtle correlations into account. The diagrammatic series we derived using this approach is far simpler than what one would obtain with established methods. We thus arrived at an explicit formula for the leading eigenvalue of an ensemble of random matrices with correlations between any pair of matrix elements. Hence, we demonstrated directly that such correlations can impact stability and therefore should not be ignored.

It is known that correlations between transpose pairs of interaction coefficients can significantly affect stability in complex ecosystems [14,52,55,56]. However, the effect of the correlations that we study has not been explored in this context (to our knowledge). Interestingly, correlations between nontranspose interaction coefficients can arise quite organically in models of complex ecosystems [31]. This indicates one immediate opportunity for the application of our results. Additional avenues for future inquiry present themselves in the context of neural networks, where increasingly more general interactions between neurons are being studied [17,19].

We anticipate that the dynamic field-theoretic approach that we developed would also lend itself to solving other challenging problems in random matrix theory. For instance, our method complements replica [57–59] or cavity techniques [60,61] used in the calculation of the spectra of sparse random matrices, and would perhaps allow for the generalization of present results. The calculation of other more complicated ensembles of dense random matrix (such as those with block structure [15,28,29,62]) could also be simplified and extended with our approach.

We acknowledge partial financial support from the Agencia Estatal de Investigación (AEI, MCI, Spain) and Fondo Europeo de Desarrollo Regional (FEDER, UE), under Project PACSS (RTI2018-093732-B-C21) and the Maria de Maeztu Program for Units of Excellence, Grant No. MDM-2017-0711 funded by MCIN/AEI/10.13039/501100011033.

*josephbaron@ifisc.uib-csic.es
†tobias.galla@ifisc.uib-csic.es

[1] E. P. Wigner, On the distribution of the roots of certain symmetric matrices, Ann. Math. **67**, 325 (1958).
[2] E. P. Wigner, Random matrices in physics, SIAM Rev. **9**, 1 (1967).
[3] T. Tao, *Topics in Random Matrix Theory*, Graduate Studies in Mathematics Vol. 132 (American Mathematical Society, Providence, 2012).
[4] M. L. Mehta, *Random Matrices* (Elsevier, London, 2004).
[5] G. W. Anderson, A. Guionnet, and O. Zeitouni, *An introduction to Random Matrices*, Cambridge Studies in Advanced Mathematics, Series Number 118 (Cambridge University Press, Cambridge UK, 2010).
[6] G. Livan, M. Novaes, and P. Vivo, *Introduction to Random Matrices Theory and Practice* (Springer International Publishing, Cham, Switzerland, 2018).
[7] M. Potters and J.-P. Bouchaud, *A First Course in Random Matrix Theory for Physicists, Engineers and Data Scientists* (Cambridge University Press, Cambridge, UK, 2020).
[8] M. Mézard, G. Parisi, and M. Virasoro, *Spin Glass Theory and Beyond: An Introduction to the Replica Method and Its Applications* (World Scientific Publishing Company, London, 1987), Vol. 9.
[9] A. J. Bray and M. A. Moore, Evidence for massless modes in the 'solvable model' of a spin glass, J. Phys. C **12**, L441 (1979).
[10] J. M. Kosterlitz, D. J. Thouless, and R. C. Jones, Spherical model of a spin-glass, Phys. Rev. Lett. **36**, 1217 (1976).
[11] C. De Dominicis and I. Kondor, Eigenvalues of the stability matrix for parisi solution of the long-range spin-glass, Phys. Rev. B **27**, 606 (1983).
[12] R. M. May, Will a large complex system be stable?, Nature (London) **238**, 413 (1972).
[13] S. Allesina and S. Tang, The stability–complexity relationship at age 40: A random matrix perspective, Popul. Ecol. **57**, 63 (2015).
[14] S. Allesina and S. Tang, Stability criteria for complex ecosystems, Nature (London) **483**, 205 (2012).
[15] D. Gravel, F. Massol, and M. A. Leibold, Stability and complexity in model meta-ecosystems, Nat. Commun. **7**, 12457 (2016).
[16] G. Barabás, M. J. Michalska-Smith, and S. Allesina, Self-regulation and the stability of large ecological networks, Nat. Ecol. Evol. **1**, 1870 (2017).
[17] J. Aljadeff, M. Stern, and T. Sharpee, Transition to Chaos in Random Networks with Cell-Type-Specific Connectivity, Phys. Rev. Lett. **114**, 088101 (2015).
[18] A. C. Coolen, P. Sollich, and R. Kühn, *Theory of Neural Information Processing Systems* (Oxford University Press, Oxford, UK, 2005).
[19] A. Kuczala and T. O. Sharpee, Eigenvalue spectra of large correlated random matrices, Phys. Rev. E **94**, 050101(R) (2016).
[20] K. Rajan and L. F. Abbott, Eigenvalue Spectra of Random Matrices for Neural Networks, Phys. Rev. Lett. **97**, 188104 (2006).
[21] C. Louart, Z. Liao, and R. Couillet, A random matrix approach to neural networks, Ann. Appl. Probab. **28**, 1190 (2018).
[22] V. Girko, Elliptic law, Theory Probab. Appl. **30**, 677 (1986).

[23] H.-J. Sommers, A. Crisanti, H. Sompolinsky, and Y. Stein, Spectrum of large random asymmetric matrices, Phys. Rev. Lett. **60**, 1895 (1988).

[24] T. Tao, Outliers in the spectrum of iid matrices with bounded rank perturbations, Probab. Theory Relat. Fields **155**, 231 (2013).

[25] S. F. Edwards and R. C. Jones, The eigenvalue spectrum of a large symmetric random matrix, J. Phys. A **9**, 1595 (1976).

[26] S. O'Rourke, D. Renfrew *et al.*, Low rank perturbations of large elliptic random matrices, Electron. J. Pro **19**, 1 (2014).

[27] D. Féral and S. Péché, The largest eigenvalue of rank one deformation of large wigner matrices, Commun. Math. Phys. **272**, 185 (2007).

[28] J. Grilli, M. Adorisio, S. Suweis, G. Barabás, J. R. Banavar, S. Allesina, and A. Maritan, Feasibility and coexistence of large ecological communities, Nat. Commun. **8**, 14389 (2017).

[29] J. W. Baron and T. Galla, Dispersal-induced instability in complex ecosystems, Nat. Commun. **11**, 6032 (2020).

[30] P. V. Aceituno, T. Rogers, and H. Schomerus, Universal hypotrochoidic law for random matrices with cyclic correlations, Phys. Rev. E **100**, 010302(R) (2019).

[31] G. Bunin, Interaction patterns and diversity in assembled ecological communities, arXiv:1607.04734.

[32] G. I. Allen and R. Tibshirani, Inference with transposable data: Modelling the effects of row and column correlations, J. R. Stat. Soc. **74**, 721 (2012).

[33] J. Berg and A. Engel, Matrix Games, Mixed Strategies, and Statistical Mechanics, Phys. Rev. Lett. **81**, 4999 (1998).

[34] T. Galla, Relative population size, cooperation pressure and strategy correlation in two-population evolutionary dynamics, Philos. Mag. **92**, 324 (2012).

[35] S. P. Kasiviswanathan, M. Rudelson, A. Smith, and J. Ullman, The price of privately releasing contingency tables and the spectra of random matrices with correlated rows, in *Proceedings of the Forty-Second ACM Symposium on Theory of Computing* (Association for Computing Machinery, New York, 2010), pp. 775–784.

[36] W. Cui, J. W. Rocks, and P. Mehta, The perturbative resolvent method: Spectral densities of random matrix ensembles via perturbation theory, arXiv:2012.00663.

[37] P. C. Martin, E. D. Siggia, and H. A. Rose, Statistical dynamics of classical systems, Phys. Rev. A **8**, 423 (1973).

[38] H.-K. Janssen, On a lagrangean for classical field dynamics and renormalization group calculations of dynamical critical properties, Z. Phys. B **23**, 377 (1976).

[39] C. De Dominicis, Techniques de renormalisation de la théorie des champs et dynamique des phénomènes critiques, J. Phys. (Paris), Colloq. **37**, C1 (1976).

[40] A. Altland and B. D. Simons, *Condensed Matter Field Theory* (Cambridge University Press, Cambridge, England, 2010).

[41] J. A. Hertz, Y. Roudi, and P. Sollich, Path integral methods for the dynamics of stochastic and disordered systems, J. Phys. A **50**, 033001 (2017).

[42] T. Tao and V. Vu, Random matrices: Universality of local eigenvalue statistics up to the edge, Commun. Math. Phys. **298**, 549 (2010).

[43] T. Tao, V. Vu, and M. Krishnapur, Random matrices: Universality of ESDs and the circular law, Ann. Probab. **38**, 2023 (2010).

[44] C. De Dominicis, Dynamics as a substitute for replicas in systems with quenched random impurities, Phys. Rev. B **18**, 4913 (1978).

[45] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.128.120601 for further details of the calculation of both the outlier eigenvalue and bulk eigenvalue density, as well as a method for the numerical production of random matrices with generalized correlations.

[46] F. Haake, F. Izrailev, N. Lehmann, D. Saher, and H.-J. Sommers, Statistics of complex levels of random matrices for decaying systems, Z. Phys. B Condens. Matter **88**, 359 (1992).

[47] K. Anand and T. Galla, Stability and dynamical properties of material flow systems on random networks, Eur. Phys. J. B **68**, 587 (2009).

[48] J. Hubbard, Calculation of Partition Functions, Phys. Rev. Lett. **3**, 77 (1959).

[49] F. Benaych-Georges and R. R. Nadakuditi, The eigenvalues and eigenvectors of finite, low rank perturbations of large random matrices, Adv. Math. **227**, 494 (2011).

[50] J. Feinberg and A. Zee, Non-hermitian random matrix theory: Method of hermitian reduction, Nucl. Phys. **B504**, 579 (1997).

[51] T. R. Kirkpatrick and D. Thirumalai, Dynamics of the Structural Glass Transition and the p-Spin—Interaction Spin-Glass Model, Phys. Rev. Lett. **58**, 2091 (1987).

[52] M. Opper and S. Diederich, Phase Transition and 1/f Noise in a Game Dynamical Model, Phys. Rev. Lett. **69**, 1616 (1992).

[53] E. Brézin and A. Zee, Correlation functions in disordered systems, Phys. Rev. E **49**, 2588 (1994).

[54] R. A. Janik, M. A. Nowak, G. Papp, and I. Zahed, Non-hermitian random matrix models, Nucl. Phys. **B501**, 603 (1997).

[55] T. Galla, Dynamically evolved community size and stability of random lotka-volterra ecosystems (a), Europhys. Lett. **123**, 48004 (2018).

[56] M. Barbier, J.-F. Arnoldi, G. Bunin, and M. Loreau, Generic assembly patterns in complex ecological communities, Proc. Natl. Acad. Sci. U.S.A. **115**, 2156 (2018).

[57] G. Semerjian and L. F. Cugliandolo, Sparse random matrices: The eigenvalue spectrum revisited, J. Phys. A **35**, 4837 (2002).

[58] G. J. Rodgers and A. J. Bray, Density of states of a sparse random matrix, Phys. Rev. B **37**, 3557 (1988).

[59] R. Kühn, Spectra of sparse random matrices, J. Phys. A **41**, 295002 (2008).

[60] T. Rogers, I. P. Castillo, R. Kühn, and K. Takeda, Cavity approach to the spectral density of sparse symmetric random matrices, Phys. Rev. E **78**, 031116 (2008).

[61] V. A. Susca, P. Vivo, and R. Kühn, Cavity and replica methods for the spectral density of sparse symmetric random matrices, SciPost Phys. Lect. Notes 33 (2021) 10.21468/SciPostPhysLectNotes.33.

[62] S. Allesina, J. Grilli, G. Barabás, S. Tang, J. Aljadeff, and A. Maritan, Predicting the stability of large structured food webs, Nat. Commun. **6**, 7842 (2015).