Anderson Localization in the Fractional Quantum Hall Effect

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The interplay between interaction and disorder-induced localization is of fundamental interest. This article addresses localization physics in the fractional quantum Hall state, where both interaction and disorder have nonperturbative consequences. We provide compelling theoretical evidence that the localization of a single quasiparticle of the fractional quantum Hall state at filling factor $\nu = n/(2n+1)$ has a striking quantitative correspondence to the localization of a single electron in the $(n+1)$th Landau level. By analogy to the dramatic experimental manifestations of Anderson localization in integer quantum Hall effect, this leads to predictions in the fractional quantum Hall regime regarding the existence of extended states at a critical energy, and the nature of the divergence of the localization length as this energy is approached. Within a mean field approximation, these results can be extended to situations where a finite density of quasiparticles is present.

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Introduction.—The scaling theory of localization [1] made the remarkable prediction that arbitrary weak random disorder localizes all eigenstates of a single electron in two dimensions (2D), implying an absence of a metallic phase for noninteracting electrons in 2D. The physics of localization also played a central role in Laughlin's explanation [2] of the origin of plateaus in the integer quantum Hall effect (IQHE) [3], leading to extensive experimental and theoretical investigation of Anderson localization of a single electron in the presence of a magnetic field. The observation of IQHE implies, unlike at zero magnetic field, existence of extended single particle states in the presence of a magnetic field. It has been argued that, in the asymptotic limit, extended states occur at a single critical energy E_c in each Landau level (LL). Experimental measurements of the temperature dependence of the width of the transition region from one plateau to the next suggest a power law divergence of the localization length as the critical energy is approached [4,5], although slightly dissimilar exponents have been observed in different experiments and for different transitions. Many theoretical studies have attempted to determine the value of the critical exponent characterizing the divergence of the localization length [6–13]. We note that recent work has called the notion of scaling into question and suggested that the exponent observed in numerical calculations and experiments is only an "effective" exponent which is not universal but model dependent [14,15].

Interaction between particles significantly complicates the problem. For example, interaction is thought to be responsible for a metal-insulator transition in 2D electron systems at zero magnetic field [16]. The objective of this Letter is to address the localization physics in the fractional quantum Hall effect (FQHE) regime, where both the interaction and disorder cause highly nontrivial, nonperturbative phenomenology. From a microscopic perspective, beginning with interacting electrons in a disordered potential is not practical or fruitful, and exact diagonalization studies are inadequate because they are limited to very small systems, whereas, as we see below, it is necessary to go to rather large systems to capture the thermodynamic behavior. Fortunately, the composite fermion (CF) theory [17–19] opens a new avenue for tackling this problem. According to the CF theory, the nonperturbative role of interaction is to create composite fermions, and the problem of strongly interacting electrons in the FQHE regime maps into that of weakly interacting composite fermions in the IQHE regime. This suggests the possibility that the localization physics of the FQHE is analogous to that of the IQHE [20,21], but a microscopic confirmation of such a correspondence has been lacking. That question has motivated the present Letter. The primary goal of our Letter is to demonstrate that in the presence of a disorder potential, a composite fermion in the nth CF-LL [called Λ level (ΛL)] behaves, to a surprising degree, as an electron in the nth LL, implying that the localization physics in the FQHE corresponds to that in the IQHE in its universal as well as nonuniversal aspects.

Model.—We begin by considering a single quasiparticle (qp) or quasihole (qh) in the presence of a disorder potential $H = \sum_{\alpha,k} \epsilon_k \delta(r_\alpha - w_k)$, which represents a random distribution of short-range impurities at random positions $\{w_k\}$ with random on site energies ϵ_k. Here, the lengths are measured in units of the magnetic length $\ell = \sqrt{\hbar c/eB}$, and the disorder strength ϵ_k in units of e^2/ℓ. We will assume that disorder strength is weak compared to the FQHE gap, in the sense that it neither creates qp's or qh's out of the FQHE vacuum nor significantly distorts the wave function of the already present qp's or qh's [see Supplemental Material (SM) [22] for a quantitative discussion of how the state hybridizes with the neutral roton excitation in the presence of the disorder [23–26]]. The objective then reduces to diagonalizing the above problem in the basis $\{\Psi_{w_j}\}$, where Ψ_{w_j} is the wave function of the many particle state with a quasiparticle or quasihole localized at w_j. Remembering that the basis is not orthogonal, the eigenfunctions $\Phi = \sum_i c_i \Psi_{w_i}$ and the eigenenergies E satisfying $H\Phi = E\Phi$ are given by the solutions of the matrix equation $O^{-1}Hc = Ec$, with $O_{ij} = \langle \Psi_{w_i}|\Psi_{w_j}\rangle$, $H_{ij} = \langle \Psi_{w_i}|H|\Psi_{w_j}\rangle$, and $c = (c_1, \ldots, c_N)^{\mathrm{T}}$. The quantity H_{ij} is the tunneling amplitude for $i \neq j$. Note that ϵ_j can be positive or negative; we keep only the Ψ_{w_i} which are localized at the "attractive" impurities in our basis, although all impurities contribute to H_{ij}. We will assume that the number of impurities is smaller than the total number of available orbitals, ensuring that the Ψ_{w_j} in our basis are linearly independent. H_{ij} and O_{ij} are, in general, complex due to the breaking of time reversal invariance by the magnetic field, which is fundamentally responsible for the strikingly different behavior in a magnetic field.

The quantities H_{ij} and O_{ij} can be evaluated (numerically) for the FQHE provided we know Ψ_{w_j}. One can imagine resorting to exact diagonalization studies, but those are restricted to systems that are too small to be meaningful for the issue at hand, given that even a single localized quasiparticle or quasihole has a rather large size. Instead, we use the CF theory, which has clarified that, microscopically, the quasiparticles are composite fermions in an almost empty ΛL, and quasiholes are missing composite fermions in an almost full ΛL [17,19,27–29]. To construct their wave function, we recall that the wave function of a single electron localized at $w = w_x + iw_y$ in the nth LL at effective magnetic field B^* is given by

$$\phi_w^{(n)}(z, B^*) = \sum_m [\eta_m^{(n)}(w, B^*)]^* \eta_m^{(n)}(z, B^*), \tag{1}$$

where the particle coordinate is defined as $z = x + iy$, and m is the angular momentum index. The single particle angular momentum orbitals in the nth LL in the symmetric gauge are given by

$$\eta_m^{(n)}(z, B^*) = \frac{1}{\ell^*}\sqrt{\frac{n!}{2\pi 2^m (m+n)!}}\frac{z^m}{\ell^{*m}} L_n^m\left(\frac{|z|^2}{2\ell^{*2}}\right) e^{-\frac{|z|^2}{4\ell^{*2}}}, \tag{2}$$

with the effective magnetic length $\ell^* = \sqrt{\hbar c/eB^*}$ and $L_n^m(t)$ is the associated Laguerre polynomial in Rodrigues definition. The wave function $\Psi_w^{\mathrm{qp}-n}$ for the CF quasiparticle of the $\nu = n/(2pn+1)$ state, located at w, is obtained from the state with n filled LLs and one additional electron localized at w in the $(n+1)$th LL by composite fermionization. To give an explicit example, the wave function of a quasiparticle of the $\nu = 2/(4p+1)$ state is given by

$$\Psi_w^{\mathrm{qp}-2} = \mathcal{P}_{\mathrm{LLL}} \begin{vmatrix} \phi_w^{(2)}(z_1, B^*) & \phi_w^{(2)}(z_2, B^*) & \ldots \\ \eta_{-1}^{(1)}(z_1, B^*) & \eta_{-1}^{(1)}(z_2, B^*) & \ldots \\ \eta_0^{(1)}(z_1, B^*) & \eta_0^{(1)}(z_2, B^*) & \ldots \\ \vdots & \vdots & \ldots \\ \eta_{N_1-2}^{(1)}(z_1, B^*) & \eta_{N_1-2}^{(1)}(z_2, B^*) & \ldots \\ \eta_0^{(0)}(z_1, B^*) & \eta_0^{(0)}(z_2, B^*) & \ldots \\ \eta_1^{(0)}(z_1, B^*) & \eta_1^{(0)}(z_2, B^*) & \ldots \\ \vdots & \vdots & \ldots \\ \eta_{N_0-1}^{(0)}(z_1, B^*) & \eta_{N_0-1}^{(0)}(z_2, B^*) & \ldots \end{vmatrix}$$
$$\times \left[\prod_{i<k=1}^{N} (z_i - z_k) e^{-\sum_j |z_j|^2/4\ell_1^2}\right]^{2p}. \tag{3}$$

Here, ℓ_1 is the magnetic length at $\nu = 1$, and N_0 and N_1 are the number of composite fermions in the lowest two ΛLs (we set $n = 0$ for the lowest Λ level), and the total number of particles is $N = N_0 + N_1 + 1$. The symbol $\mathcal{P}_{\mathrm{LLL}}$ represents projection into the lowest LL (LLL), which we evaluate using the standard Jain-Kamilla method [30,31]. The relation $\ell^{*-2} + 2p\ell_1^{-2} = \ell^{-2}$ ensures that the product wave function has the Gaussian factor corresponding to the actual external magnetic field. The wave functions for quasiparticles and quasiholes at other fractions can be constructed analogously; see SM [22]. These wave functions have been tested against exact wave functions and found to be extremely accurate representations of the exact Coulomb wave functions [19]. As a result, the conclusions derived below from these wave functions can be considered to be reliable.

The key question is whether the above FQHE problem is equivalent, modulo a rescaling of parameters, to an IQHE problem, for which H_{ij} and O_{ij} can be obtained analytically. Such a correspondence, should it exist, would be powerful, because it would allow us to carry our knowledge about the localization physics of the IQHE over to the FQHE. We demonstrate below a close correspondence between

$$H(w_j, \epsilon_j, B) \Leftrightarrow \mathcal{H}(w_j, \epsilon_j^* = \epsilon_j B^*/B, B^*). \tag{4}$$

$$O(w_j, \epsilon_j, B) \Leftrightarrow \mathcal{O}(w_j, \epsilon_j^* = \epsilon_j B^*/B, B^*). \tag{5}$$

We use H and O for the matrix elements in FQHE, and $\mathcal{H}$ and $\mathcal{O}$ for the matrix elements of IQHE. We note that the sample size and the impurity positions remain identical in

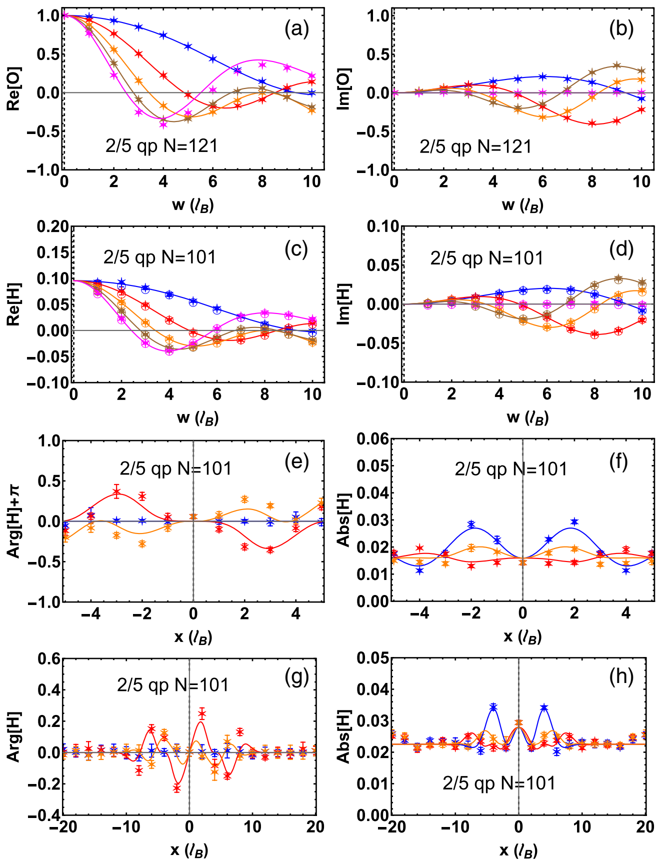

FIG. 1. This figure shows various matrix elements for the quasiparticle of the $\nu = 2/5$ state (stars) and compares them with the corresponding analytical results for an electron in the third Landau level at an effective magnetic field (solid lines). (We stress that the solid lines have no fitting parameters in this and subsequent figures.) Panels (a) and (b) show the real and imaginary parts of the overlap matrix elements $\langle \Psi^{\mathrm{qp}-2}_{(w/2)e^{i\theta}} | \Psi^{\mathrm{qp}-2}_{(w/2)} \rangle$, for $\theta = \pi/6$ (blue); $\theta = \pi/3$ (red); $\theta = \pi/2$ (orange); $\theta = 2\pi/3$ (brown); and $\theta = \pi$ (magenta). Panels (c) and (d) show the real and imaginary parts of the tunneling matrix elements $\langle \Psi^{\mathrm{qp}-2}_{(w/2)e^{i\theta}} | \sum_i \delta[r_i - (w/2)e^{i\theta}] | \Psi^{\mathrm{qp}-2}_{(w/2)} \rangle$ with the same color code. (The solid lines are not fits.) Panels (e)–(h) show the phase and the modulus of the impurity assisted tunneling matrix element $\langle \Psi^{\mathrm{qp}-2}_{-(w/2)} | \sum_i \delta(r_i - xe^{i\theta'}) | \Psi^{\mathrm{qp}-2}_{(w/2)} \rangle$, for $\theta' = 0$ (blue); $\theta' = \pi/4$ (orange); and $\theta' = \pi/2$ (red). Panels (e) and (f) correspond to $w = 5$, and panels (g) and (h) correspond to $w = 8$. All lengths are quoted in units of the magnetic length at $\nu = 2/5$. The number of particles N in the Monte Carlo calculation is shown on each panel; the results represent the thermodynamic limit. We take $N_0 = N_1$.

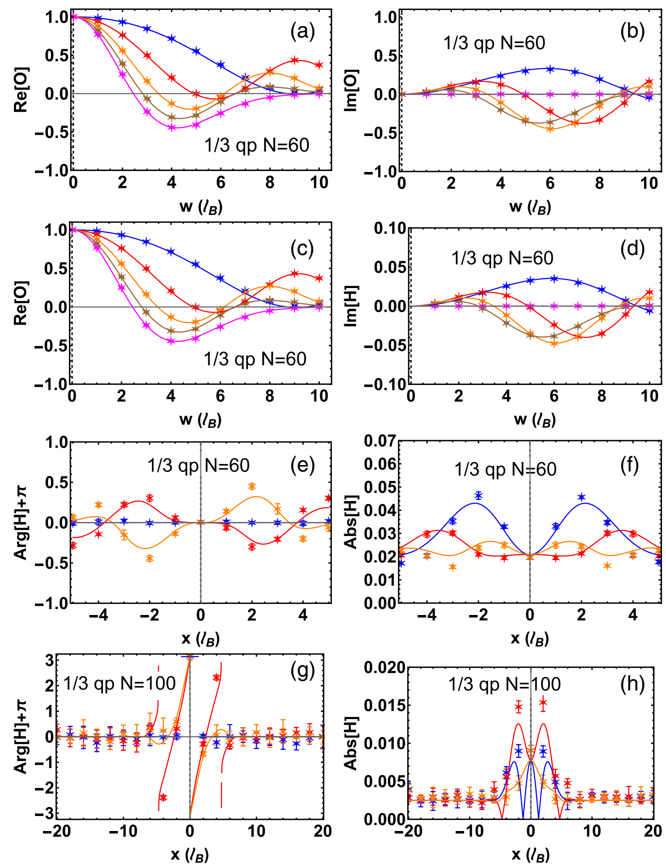

FIG. 2. Same as in Fig. 1 but for the quasiparticle of $\nu = 1/3$.

the FQHE and the IQHE systems in laboratory units; if the impurity positions are measured in units of the magnetic length, then we must scale them by a factor (ℓ^*/ℓ).

Results.—The natural basis set in our problem, consisting of states with the quasiparticle localized at different impurity positions, is not orthogonal. Apart from relating the matrix elements of the Hamiltonian, establishing a correspondence between the single quasiparticle localization problems in IQHE and FQHE also entails relating the overlap matrix of the basis states.

With the wave functions in hand, first, we proceed to evaluate the overlap matrix $O^{\mathrm{qp}-n}_{ww'} = \langle \Psi^{\mathrm{qp}-n}_{w} | \Psi^{\mathrm{qp}-n}_{w'} \rangle$, assuming $2p = 2$ below. The top two panels in Figs. 1 and 2 display the real and imaginary parts of the matrix element $O^{\mathrm{qp}-2}_{ww'}$ for the quasiparticle of $\nu = 2/5$ and $\nu = 1/3$ (stars). For comparison, we also show the IQHE counterpart, $\mathcal{O}^{\mathrm{qp}-\mathrm{n}}_{ww'}$, namely, the overlap an electron localized at w and that with that at w' (solid lines). This can be analytically evaluated to be (see SM [22])

$$\mathcal{O}^{\mathrm{qp}-\mathrm{n}}_{ww'} = \sqrt{2}\pi\eta^{(n)}_0(w - w', B^*)e^{\iota\frac{\mathrm{Im}(w\bar{w}')}{2\ell^{*2}}}. \tag{6}$$

The correspondence between the overlaps for CF quasiparticle in the nth ΛL and for electron in the nth LL is strikingly close (the discussion in the SM [22] provides further insight into this result).

Next, we come to the comparison of matrix elements of the Hamiltonian $H = \sum_{\alpha,k} \epsilon_k \delta(r_\alpha - w_k)$. The tunneling matrix element $H^{\mathrm{qp}-n}_{ij} = \langle \Psi^{\mathrm{qp}-n}_{w_i} | H | \Psi^{\mathrm{qp}-n}_{w_j} \rangle$ is given by $H^{\mathrm{qp}-n}_{ij} = \epsilon_i H^{\mathrm{qp}-n}_{w_i w_j, w_i} + \epsilon_j H^{\mathrm{qp}-n}_{w_i w_j, w_j} + \sum_{k \neq i,j} \epsilon_k H^{\mathrm{qp}-n}_{w_i w_j, w_k}$, where $H^{\mathrm{qp}-n}_{w_i w_j, w_k} \equiv \langle \Psi_{w_i} | \sum_\alpha \delta(r_\alpha - w_k) | \Psi_{w_j} \rangle$. We consider, separately, the two types of terms: $H^{\mathrm{qp}-n}_{w_i w_j, w_i}$, and $H^{\mathrm{qp}-n}_{w_i w_j, w_k}$ with

$k \neq i, j$. In panels (c) and (d) of Figs. 1 and 2, we show $H_{w_1 w_2, w_1}$ for the quasiparticle at $\nu = 2/5$ and $\nu = 1/3$. Panels (e)–(h) show $H_{w_1 w_2, w}$ with $w \neq w_1, w_2$. In the numerical calculations, we have approximated the δ functions as normalized Gaussian functions of a small width 0.1ℓ. The corresponding tunneling matrix elements for the quasiparticle of the IQHE state can be calculated analytically as shown in the SM [22]. The explicit formula for $\mathcal{H}^{\mathrm{qp}-n}_{w_1 w_2, w}$ is

$$\mathcal{H}^{\mathrm{qp}-n}_{w_i w_j, w} = \frac{\nu}{\sqrt{2\pi}} \eta_0^{(n)}(w_i - w_j) e^{\frac{i}{2}\mathrm{Im}[\overline{w_j} w_i]} + \eta_0^{(n)}(w - w_j) e^{\frac{i}{2}\mathrm{Im}[\overline{w_j} w]} \eta_0^{(n)}(w_i - w) e^{\frac{i}{2}\mathrm{Im}[\overline{w} w_i]}. \quad (7)$$

These are also shown in Figs. 1 and 2 (solid lines). Again a close correspondence can be seen between the FQHE and the IQHE results. Analogous results for the quasiholes at $\nu = 1/3$ and $2/5$ are given in the SM [22].

These studies provide a quantitative confirmation of the "law of corresponding states" [20,21]. We should remember that our quasiparticle and quasihole wave functions [Eq. (3) and SM [22]] are extremely accurate approximations of the actual quasiparticles and quasiholes in the FQHE, and thus, the calculated O_{ij} and H_{ij} are faithful representations of the actual overlap and tunneling matrix elements for the exact quasiparticles and quasiholes. In other words, if an exact diagonalization study were possible for large system sizes, it would have yielded very nearly the same overlap and tunneling matrix elements. While the CF theory provides a natural framework for understanding this correspondence, there was not *a priori* reason why the overlap and tunneling matrix element between the two collective, many-particle states in the FQHE should have such close correspondence with the overlap and tunneling matrix element of a single electron. The correspondence is particularly striking given that these quantities depend sensitively on the filling factor as well as on the distance across which a qp or qh tunnels. We expect that a slight modification of the qp or qh wave function due to disorder would produce a greater deviation between the calculated and analytical matrix elements, but not change the primary conclusions of our Letter.

Implications for experiments.—The above correspondence between the overlap matrix elements implies that the problem of Anderson localization of a quasiparticle of the $\nu = n/(2pn + 1)$ FQHE state is essentially identical to that of the Anderson localization of an electron in the $\nu = n$ IQHE state, provided that the on site binding energies are appropriately rescaled. Therefore, these two systems are predicted to have identical critical properties. Thus, we can export *mutatis mutandis* the results from the study of noninteracting electrons in a magnetic field to make predictions for the nature of localization in the FQHE. We list these, along with the underlying assumptions.

For a single electron in a Landau level, it is believed that the localization length diverges at some critical energy E_c. Furthermore, the divergence of the localization length as E_c is approached is described, at least approximately, by $\xi \sim |E - E_c|^{-\alpha}$ where α is the effective localization length exponent. These features have found support from direct numerical studies that solve the Schrödinger equation on a disordered strip of finite width, and then use finite size scaling to deduce the localization length in the thermodynamic limit. The correspondence established above implies that the localization length of a single CF quasiparticle also diverges at a critical energy, with the same exponent as that for electrons. Numerical calculations have shown that the divergence of the localization length within the lowest Landau level (relevant to $1 \to 2$ transition) is characterized by an exponent $\alpha \approx 2.3$–2.5 [32,33]. In the second LL, where the numerical calculations are less reliable, a higher value $\alpha \approx 5.5$ is obtained for short range disorder [34–36], and the exponent has been found to depend on the range of the disorder. It has been argued [37] that, in higher LLs, there is a large irrelevant length, and only for system sizes and localization lengths larger than this irrelevant length, not readily accessible to numerics, can a single parameter scaling behavior be observed. It is generally believed that, in the asymptotic, critical regime, the localization length exponent is $\alpha \approx 2.3$–2.5 in all LLs. This is consistent with calculations using a quantum percolation model that allows tunneling across saddle points [8,38]. This value is also consistent with that measured in the most reliable experiments [5]. Translating these results to composite fermions, we also predict $\alpha \approx 2.3$–2.5 for the FQHE. In addition, this Letter may lead to generalizations of studies of the disorder-induced quasiparticle lifetime or broadening in the IQHE regime [39,40] to the FQHE.

So far, we have considered a single quasiparticle. Increasing the electron filling factor amounts to creating a finite density of CF quasiparticles in the topmost ΛL. Now, the problem is more complex, because one must deal with the interaction between composite fermions as well as their fractional braiding statistics. It appears reasonable to assume that composite fermions are weakly interacting. This may be justified given that a large portion of the Coulomb interaction has been spent into creating composite fermions, which are themselves much more weakly interacting than electrons; the interaction pseudopotentials for composite fermions are reduced by an order of magnitude or more relative to the Coulomb pseudopotentials for electrons [41,42]. Alternatively, one can note that the charge associated with the CF quasiparticle or CF quasihole has magnitude $e/(2pn \pm 1)$, leading to an inter-CF interaction that is reduced by a factor of $(2pn \pm 1)^{-2}$ relative to the Coulomb interaction between electrons. The fractional braiding statistics can be incorporated by mapping into an IQHE problem with the effective magnetic field given by $B^* = B - 2\pi\rho(r)$ where the spatial dependence of the density $\rho(r)$ encodes information on fractional braiding statistics. One must find eigenstates of "noninteracting" composite fermions in the presence of disorder, which must be done in a self-consistent manner as the Hamiltonian depends on the density [43] (composite fermions are not really noninteracting, as the solution for any eigenstate depends on other eigenstates). The problem is simplified if we make the mean field approximation of replacing $\rho(r)$ by its average value and, thus, treat composite fermions in a uniform B^*. It appears to us, although we cannot prove it, that this approximation is likely to become valid for single-CF wave functions whose extent is much larger than the separation between quasiparticles; such wave functions with large localization lengths are of primary interest in the vicinity of the phase transition. (As shown in Refs. [44,45], a charged particle in the presence of a lattice of fractional flux tubes behaves similarly to a particle in a uniform magnetic field.) Then, the problem maps into that of weakly interacting electrons in a uniform effective magnetic field, and one can translate the above results into a behavior as a function of the filling factor, because increasing the filling factor simply fills successively higher energy single particle CF orbitals. This implies the power law behavior $\xi \sim |\nu - \nu_c|^{-\alpha}$. This provides insight into the result by Engel *et al.* [46], who find the same exponent for the $1/3 \to 2/5$ transition as for the IQHE transitions.

Within the mean field approximation, for symmetric disorder, the extended state is predicted to occur at the critical filling factor $\nu_c^* = n + 1/2$, which determines the position of the peak separating the incompressible states at $\nu^* = n$ and $\nu^* = n + 1$ to be $\nu_c = (n + 1/2)/[2p(n + 1/2) \pm 1]$. There is experimental support for this prediction [47]. We note that field theoretical treatments have also suggested universality of the localization physics in the FQHE and the IQHE [48–51].

In summary, we have shown that, in spite of the strongly correlated nature of the FQHE, it is possible to make detailed quantitative predictions regarding the nature of localization, exploiting the observation that the analogy between the FQHE and the IQHE also extends to the localization physics.

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