Kramers-Wannier-like Duality Defects in $(3+1)D$ Gauge Theories

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We introduce a class of noninvertible topological defects in $(3+1)D$ gauge theories whose fusion rules are the higher-dimensional analogs of those of the Kramers-Wannier defect in the $(1+1)D$ critical Ising model. As in the lower-dimensional case, the presence of such noninvertible defects implies self-duality under a particular gauging of their discrete (higher-form) symmetries. Examples of theories with such a defect include SO(3) Yang-Mills (YM) at $\theta = \pi$, $\mathcal{N} = 1$ SO(3) super YM, and $\mathcal{N} = 4$ SU(2) super YM at $\tau = i$. We also introduce an analogous construction in $(2+1)D$, and give a number of examples in Chern-Simons–matter theories.

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Symmetries have been a driving force behind modern advances in theoretical physics. Recent developments have led to several extensions of the notion of global symmetry. One such example is higher-form symmetry [1], which has had numerous applications such as constraining the IR phases of pure Yang-Mills (YM) theory [2].

Another type of generalized symmetry is noninvertible symmetry. The prototypical example of such a symmetry is the one arising from the Kramers-Wannier self-duality of the $(1+1)D$ Ising model at the critical point. This duality can be implemented by a topological defect line $\mathcal{N}$ [3–5]. If one performs the duality twice, one projects out the $\mathbb{Z}_2$-odd operators, meaning that the composition rule of the topological defect satisfies

$$\mathcal{N} \times \mathcal{N} = 1 + \eta_{\mathbb{Z}_2}, \tag{1}$$

with $\eta_{\mathbb{Z}_2}$ the symmetry defect implementing the $\mathbb{Z}_2$ twist of the spin system. The only topological defects in the Ising CFT are $\mathcal{N}$ and $\eta_{\mathbb{Z}_2}$, which means that there is no inverse $\mathcal{N}^{-1}$ such that $\mathcal{N}^{-1} \times \mathcal{N} = 1$, and therefore the defect $\mathcal{N}$ cannot be thought of as implementing a group action.

The basic idea of noninvertible symmetry is to consider *any* topological defect as a form of generalized symmetry. This means that one must extend the notion of symmetry beyond groups, leading in $(1+1)D$ to a mathematical construction known as a fusion category [6–8]. The Ising category is one of the simplest such fusion categories.

Though noninvertible symmetries are relatively well studied in $(1+1)D$ (see, e.g., Refs. [9–18] for recent developments in continuum QFTs), examples in dimensions greater than two remain limited, except in topological QFTs. We mention just one example here [19], in which noninvertible lines were used to study the string tension in $(2+1)D$ $U(1)\rtimes S_N$ semi-Abelian gauge theory.

In this Letter, we provide a general procedure for obtaining noninvertible defects in $3+1$D, starting with any theory with 't Hooft anomaly for discrete higher-form symmetries of a particular form. By gauging a subset of the symmetries appearing in the anomaly, the defect associated with the remaining symmetry becomes noninvertible (such a construction was first suggested in Ref. [20]). For the particular cases we study, the resulting noninvertible defect will be shown to be a generalization of the Kramers-Wannier defect in $(1+1)D$. Our defects will have similar implications for self-duality of the gauge theories. Our result can be seen as a continuum analog of the Kramers-Wannier duality of lattice $\mathbb{Z}_2$ gauge theories in $(3+1)$D [5], whose corresponding defect has been studied in Ref. [21].

We illustrate the existence of these defects and their potential dynamical applications through the example of

SO(3) gauge theory with zero and one supercharges, as well as $\mathcal{N}=4$ SU(2) super YM. This Letter is accompanied by Supplemental Material, in which we give details as well as various generalizations of our construction, including to $(2+1)$-dimensional theories and $(3+1)$-dimensional theories with symmetries besides $\mathbb{Z}_2$ [22].

General construction.—

Kramers-Wannier-like duality defect. Our starting point is a $(3+1)D$ theory $\mathcal{T}$ with zero-form symmetry $\mathbb{Z}_2^{(0)}$ (which can be either linear or antilinear) and one-form symmetry $\mathbb{Z}_2^{(1)}$. Associated with these symmetries are codimension-1 and -2 topological defects. We will denote the background fields of $\mathbb{Z}_2^{(0)}$ and $\mathbb{Z}_2^{(1)}$ as $A^{(1)}$ and $B^{(2)}$, respectively. The $\mathbb{Z}_2^{(0)}$ symmetry defect inserted on M_3 in the presence of $B^{(2)}$ will be denoted by $D(M_3, B^{(2)})$. Throughout we will assume that M_3 is oriented.

For simplicity, assume that the spacetime manifold X_4 is spin. The two symmetries can have a mixed 't Hooft anomaly, captured by a 5d integral built from the background gauge fields $A^{(1)}$ and $B^{(2)}$. We will be interested in the particular case of a 't Hooft anomaly of the form

$$\pi \int_{X_5} A^{(1)} \cup \frac{\mathcal{P}(B^{(2)})}{2}, \qquad (2)$$

with $\mathcal{P}(B^{(2)})$ the Pontrjagin square of $B^{(2)}$ and $\partial X_5 = X_4$. The mixed anomaly Eq. (2) implies that the $\mathbb{Z}_2^{(0)}$ defect $D(M_3, B^{(2)})$ is anomalous under $\mathbb{Z}_2^{(1)}$ transformations, and hence only the combination

$$D(M_3, B^{(2)}) e^{i\pi \int_{M_4} \mathcal{P}(B^{(2)})/2} \qquad (3)$$

with $\partial M_4 = M_3$ is invariant under gauge transformations of the background field $B^{(2)}$. Note that in Eq. (3) the dependence on M_4 is only through a term involving the classical background $B^{(2)}$, so Eq. (3) should still be regarded as a genuine 3d invertible defect.

We will be interested in understanding the gauging of $\mathbb{Z}_2^{(1)}$. Upon gauging, the background field $B^{(2)}$ is promoted to a dynamical field $b^{(2)}$. From Eq. (3), we see that $D(M_3, b^{(2)})$ is no longer well defined since it is not invariant under the dynamical gauge transformations of $b^{(2)}$. To make it well defined, we must either couple to a dynamical bulk, or couple to a 3d TQFT $\mathfrak{T}(M_3, b^{(2)})$, which cancels the anomaly, thereby absorbing the bulk dependence. Since we will be interested in intrinsically 3d defects, we will pursue the latter strategy. The TQFT canceling the anomaly is not unique. However, it was shown in Ref. [33] that any such TQFT can be factorized into the decoupled tensor product of two theories $\mathfrak{T}(M_3, b^{(2)}) = \hat{\mathfrak{T}}(M_3) \otimes \mathcal{A}^{2,1}(M_3, b^{(2)})$, where $\hat{\mathfrak{T}}(M_3)$ does not couple to the dynamical field $b^{(2)}$. The theory $\mathcal{A}^{2,1}(M_3, b^{(2)})$ is simply $\mathrm{U}(1)_2$ Chern Simons theory, i.e., the minimal TQFT that lives on the boundary of $e^{i\pi \int_{M_4} \mathcal{P}(b^{(2)})/2}$. Since tensoring a decoupled TQFT only changes the overall normalization of the defect, we choose the TQFT to be the minimal $\mathcal{A}^{2,1}$ for simplicity. Hence we find a well-defined genuinely 3d defect

$$\mathcal{N}(M_3) \coloneqq D(M_3, b^{(2)}) \mathcal{A}^{2,1}(M_3, b^{(2)}). \qquad (4)$$

Note that $\mathcal{N}(M_3)$ explicitly depends on the dynamical field $b^{(2)}$. The defect $\mathcal{N}$, when regarded as an operator, is linear (antilinear) if and only if $\mathbb{Z}_2^{(0)}$ before gauging is linear (antilinear).

We now show that $\mathcal{N}$ satisfies Kramers-Wannier-like fusion rules; in particular, it is noninvertible. To begin, consider the case in which $\mathcal{N}$ is linear. In this case the dual $\bar{\mathcal{N}}$ is just equal to $\mathcal{N}$ itself. This follows since $D = \bar{D}$ from $D^2 = 1$ and $\bar{\mathcal{A}}^{2,1} = \mathcal{A}^{2,-1} \cong \mathcal{A}^{2,1}$, with the last equality following from the fact that $\mathrm{U}(1)_2$ is time-reversal symmetric as a spin-TQFT [34,35]. We also note that the tensor product theory $\mathcal{A}^{2,1} \otimes \mathcal{A}^{2,-1}$, often called the "double-semion" theory [36,37], is equivalent to $\mathbb{Z}_2$ Dijkgraaf-Witten (DW) theory [38] with nontrivial DW twist. The DW twist can be written as $(-1)^{\int_{M_3} a^3}$ where $a \in H^1(M_3, \mathbb{Z}_2)$ is the $\mathbb{Z}_2$ gauge field. Indeed, the K matrix for the double-semion theory, $K = \mathrm{diag}(2, -2)$, can be rotated to that of the BF representation of the DW theory, $K = \begin{pmatrix} 0 & 2 \\ 2 & 1 \end{pmatrix}$, by an $\mathrm{SL}(2, \mathbb{Z})$ transformation.

These considerations motivate the following result: upon fusion of two $\mathcal{N}(M_3)$ defects, one obtains a nontrivial $\mathbb{Z}_2$ DW theory living on M_3. Poincaré duality allows one to exchange the sum over $\mathbb{Z}_2$ 1 co-cycles a with a sum over two-cycles Σ, upon which the DW twist $e^{i\pi \oint_{M_3} a^3}$ becomes the triple intersection number $Q(\Sigma)$ of Σ in M_3 [39]. This gives the following fusion rules:

$$\mathcal{N}(M_3) \times \mathcal{N}(M_3) = \frac{1}{|H^0(M_3, \mathbb{Z}_2)|} \sum_{\Sigma \in H_2(M_3, \mathbb{Z}_2)} (-1)^{Q(\Sigma)} L(\Sigma), \qquad (5)$$

where $L(\Sigma) \coloneqq e^{i\pi \oint_\Sigma b^{(2)}}$. The normalization factor is related to the volume of the gauge group of the DW theory. A more explicit derivation of both the normalization and the fusion rules will be given in the Supplemental Material [22]. Since this is a sum of more than one operator, we see that $\mathcal{N}$ is a noninvertible defect.

On the other hand, in the case when $\mathcal{N}$ is antilinear, we have $\mathcal{N} \times \mathcal{N} = D\mathcal{A}^{2,1} \times D\mathcal{A}^{2,1} = D^2 \mathcal{A}^{2,-1} \times \mathcal{A}^{2,1}$, since D flips the orientation of $\mathcal{A}^{2,1}$ as the former passes though the latter. Combining this with $D^2 = 1$, we see that $\mathcal{N} \times \mathcal{N}$

again hosts the double-semion theory, giving the same fusion rules as in Eq. (5). In summary, $\mathcal{N}$ satisfies the fusion rule (5) regardless of whether it is linear or antilinear.

We may now consider the fusion of $\mathcal{N}(M_3)$ and $L(\Sigma)$ for some Σ embedded in M_3. Note that in Eq. (4) the one-form symmetry of $\mathcal{A}^{2,1}$ is coupled to the bulk dynamical field $b^{(2)}$. This means that the Wilson line of $\mathcal{A}^{2,1}$, which has a nontrivial one-form charge, has to be bounded by the bulk global one-form symmetry generator L. In other words, L can end on $\mathcal{N}$ without costing energy, and furthermore it can be absorbed. In the process of absorption, the boundary of L, identified with the Wilson line, sweeps out a surface Σ in M_3, producing a sign $(-1)^{Q(\Sigma)}$ from the framing anomaly of the Wilson line. This effect is derived in the first section of the Supplemental Material [22]. Hence we obtain the fusion rules

$$\mathcal{N}(M_3)\times L(\Sigma) = (-1)^{Q(\Sigma)}\mathcal{N}(M_3), \tag{6}$$

where Σ is embedded in M_3.

Finally, the $L\times L$ fusion rule is obvious,

$$L(\Sigma)\times L(\Sigma) = 1. \tag{7}$$

The fusion rules (5), (6), and (7) are reminiscent of the fusion rules of the Ising fusion category in two dimensions. For this reason, we refer to the noninvertible defect $\mathcal{N}$ in the 4d theory $\mathcal{T}/\mathbb{Z}_2^{(1)}$ as a "Kramers-Wannier-like defect." Though we do not work it out here, we expect these fusion rules to yield a fusion three-category [42,43]. As we now explain, $\mathcal{N}$ implements a self-duality transformation on $\mathcal{T}/\mathbb{Z}_2^{(1)}$.

Self-duality. We now explain why the gauged theory $\mathcal{T}/\mathbb{Z}_2^{(1)}$ has a notion of self-duality. We begin by considering the partition function of $\mathcal{T}/\mathbb{Z}_2^{(1)}$ [44],

$$Z_{\mathcal{T}/\mathbb{Z}_2^{(1)}}[C^{(2)}] = \int \mathcal{D}b^{(2)} Z_{\mathcal{T}}[b^{(2)}] e^{i\pi\int_{X_4} C^{(2)}b^{(2)}}, \tag{8}$$

where $C^{(2)}$ is the background field for the quantum $\hat{\mathbb{Z}}_2^{(1)}$ symmetry, whose corresponding defect is L. If we further gauge $C^{(2)}$, the last factor becomes a delta functional for $b^{(2)}$ and we reobtain the original theory $\mathcal{T}$.

To find self-duality in $\mathcal{T}/\mathbb{Z}_2^{(1)}$, we first include a Dijkgraaf-Witten term $(\pi/2)\mathcal{P}(C^{(2)})$ and then gauge. This gives

$$\begin{aligned}&\int \mathcal{D}c^{(2)} Z_{\mathcal{T}/\mathbb{Z}_2^{(1)}}[c^{(2)}] e^{i\int_{X_4}\frac{\pi}{2}\mathcal{P}(c^{(2)})+\pi c^{(2)}A^{(2)}}\\ &\quad = \int \mathcal{D}c^{(2)}\mathcal{D}b^{(2)} Z_{\mathcal{T}}[b^{(2)}] e^{i\int_{X_4}\pi c^{(2)}b^{(2)}+\frac{\pi}{2}\mathcal{P}(c^{(2)})+\pi c^{(2)}A^{(2)}}\\ &\quad = \int \mathcal{D}b^{(2)} Z_{\mathcal{T}}[b^{(2)}] e^{-i\frac{\pi}{2}\int_{X_4}\mathcal{P}(b^{(2)}+A^{(2)})},\end{aligned} \tag{9}$$

where we have made a change of variables $c^{(2)}\to c^{(2)}-b^{(2)}-A^{(2)}$ and dropped a contribution from the TQFT $\int\mathcal{D}c^{(2)}e^{i(\pi/2)\int\mathcal{P}(c^{(2)})}$, which can be continuously deformed to the trivial theory. We next use the following anomalous transformation law of $\mathcal{T}$ under *global* $\mathbb{Z}_2^{(0)}$ transformations:

$$Z_{\mathcal{T}}[b^{(2)}]e^{-i\frac{\pi}{2}\int_{X_4}\mathcal{P}(b^{(2)})} = \begin{cases} Z_{\mathcal{T}}[b^{(2)}], & \mathbb{Z}_2^{(0)} \text{ is linear} \\ Z^*_{\mathcal{T}}[b^{(2)}], & \mathbb{Z}_2^{(0)} \text{ is antilinear.}\end{cases} \tag{10}$$

Equation (9) then reduces to

$$Z_{\mathcal{T}/\mathbb{Z}_2^{(1)}}[A^{(2)}]e^{-i\frac{\pi}{2}\int_{X_4}\mathcal{P}(A^{(2)})} \tag{11}$$

if $\mathbb{Z}_2^{(0)}$ is linear, and

$$Z^*_{\mathcal{T}/\mathbb{Z}_2^{(1)}}[A^{(2)}]e^{i\frac{\pi}{2}\int_{X_4}\mathcal{P}(A^{(2)})} \tag{12}$$

if $\mathbb{Z}_2^{(0)}$ is antilinear. To complete the self-duality, we need only add a compensating counterterm (and in the antilinear case, do a complex conjugation K).

We now recall the operations S and T defined in Ref. [1] (see also Refs. [45,46]); in the current context S corresponds to gauging of a $\mathbb{Z}_2^{(1)}$ form symmetry and T corresponds to coupling to an invertible phase $(\pi/2)\mathcal{P}(B^{(2)})$. From the above, we conclude that $\mathcal{T}/\mathbb{Z}_2^{(1)}$ is self-dual under TST if $\mathbb{Z}_2^{(0)}$ is linear, and under $TKST$ if $\mathbb{Z}_2^{(0)}$ is anti-linear.

We have seen that $\mathcal{T}/\mathbb{Z}_2^{(1)}$ has a self-duality, as well as a non invertible defect with Kramers-Wannier-like fusion rules. We now argue that the two facts are related, following Ref. [23]. First, it is simple to argue that the existence of a self-duality should imply a noninvertible defect of Kramers-Wannier type. Indeed, note that gauging, and hence the full operation TST (or $TKST$), can be implemented by a codimension-1 topological defect. We will denote the total defect by $\mathcal{N}(X_3)$. By stacking two copies of $\mathcal{N}(X_3)$, we are left with a condensate as in Fig. 1. Thus we obtain fusion rules

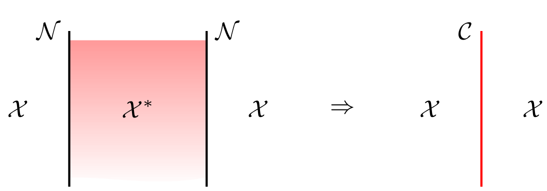

FIG. 1. Any theory $\mathcal{X}$ with a self-duality admits a noninvertible defect with Kramers-Wannier–type fusion rules. $\mathcal{X}^*$ is the TST (or $TKST$) transform of $\mathcal{X}$.

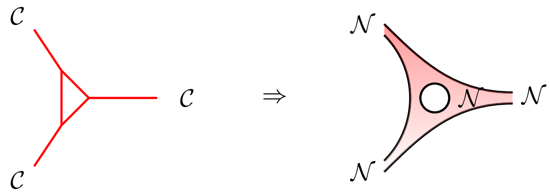

FIG. 2. A theory with a Kramers-Wannier defect has a self-duality since the mesh of $\mathcal{C}$ can be replaced by a set of topologically trivial loops of $\mathcal{N}$.

$$\mathcal{N}(M_3) \times \mathcal{N}(M_3) = \mathcal{C}(M_3). \quad (13)$$

The condensate $\mathcal{C}(M_3)$ can be understood by taking the one-form gauge theory to live in a small tubular neighborhood of M_3 with Dirichlet boundary conditions. As shown in the Supplemental Material [22], this can be reduced to a zero-form gauge theory living on M_3 itself, with the condensate taking the form

$$\mathcal{C}(M_3) = \frac{1}{|H^0(M_3, \mathbb{Z}_2)|} \sum_{\Sigma \in H_2(M_3, \mathbb{Z}_2)} (-1)^{Q(\Sigma)} L(\Sigma). \quad (14)$$

This reproduces the fusion rules of Eq. (5). Note that the factor of $(-1)^{Q(\Sigma)}$ descends from the T operation before S in the self-duality, as derived explicitly in the Supplemental Material [22]. It is easily verified that $\mathcal{C}(M_3)$ squares to itself up to normalization.

Conversely, assuming that we have a defect with fusion rules given in Eq. (5), there must be a corresponding self-duality. Indeed, we may begin by inserting a fine mesh of the condensate $\mathcal{C}(M_3)$ (which is itself a fine mesh of surfaces), and then replacing it with pairs of $\mathcal{N}(M_3)$ as shown schematically in Fig. 2. But assuming that we started with a fine enough mesh, each loop of $\mathcal{N}(M_3)$ is now contractible, and may be evaluated to a number. Thus with appropriate normalization we reobtain the original theory.

Examples.—We now give some examples of theories with Kramers-Wannier–type noninvertible defects, and hence with self-dualities.

SO(3) Yang-Mills theory at $\theta = \pi$. As a first example, take the theory $\mathcal{T}$ to be a pure SU(2) Yang-Mills theory at $\theta = \pi$. This theory has a $\mathbb{Z}_2^{(1)}$ 1-form symmetry, as well as a time-reversal symmetry T. These two symmetries are known to have an anomaly of the form (2), with $A^{(1)}$ replaced by the first Stiefel-Whitney class $w_1^{TX_5}$ of the tangent bundle of X_5 [49]. Our general construction tells us that upon gauging $\mathbb{Z}_2^{(1)}$, the codimension-1 defect implementing T becomes non-invertible. Indeed, the resulting theory $\mathcal{T}/\mathbb{Z}_2^{(1)}$ is SO(3) Yang-Mills at $\theta = \pi$, which lacks the usual time-reversal symmetry since θ is 4π periodic. Instead, it contains the noninvertible defect $\mathcal{N}$ implementing a self-duality transformation under $TKST$.

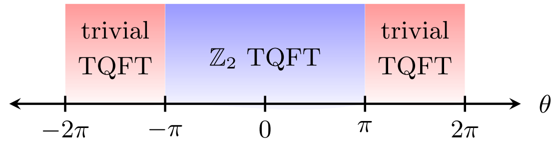

FIG. 3. Phase diagram of SO(3) YM as a function of θ.

The noninvertible defect $\mathcal{N}$ also suggests the structure of phases of SO(3) Yang-Mills as a function of the theta angle. From the UV perspective, we find that

$$TKST\colon [\mathrm{SO}(3), \theta] \to [\mathrm{SO}(3), 2\pi - \theta]. \quad (15)$$

Hence $\theta = \pm\pi$ mod 4π is invariant under $TKST$. At these points the defect implementing the transformation $TKST$ between different theories at generic theta becomes one implementing self-duality of a single theory. This suggests that there should be a phase transition at these fixed values of theta. Indeed, just such a transition is expected on the basis of, e.g., soft supersymmetry breaking. See Fig. 3 for a schematic phase diagram.

Let discuss this phase diagram further. In Ref. [24] it was argued via soft supersymmetry breaking that for $|\theta| < \pi$ the theory flows to a $\mathbb{Z}_2$ TQFT, while for $\pi < |\theta| < 2\pi$, the theory flows to a trivially gapped phase. The phase transitions at $\theta = \pm\pi$ mod 4π are transitions between these two low-energy phases. Our result is in agreement with [21], where a noninvertible defect was found in a lattice model which exhibits the same phase transition.

$\mathcal{N} = 1$ SO(3) super Yang-Mills theory. Next we take $\mathcal{T}$ to be $\mathcal{N} = 1$ SU(2) super Yang-Mills. The symmetry of this theory is $\mathbb{Z}_4^{(0)} \times \mathbb{Z}_2^{(1)}$, where $\mathbb{Z}_4^{(0)}$ contains the fermion parity $\mathbb{Z}_2^F$ as a normal subgroup. There is a mixed anomaly between $\mathbb{Z}_4^{(0)}$ and $\mathbb{Z}_2^{(1)}$ with the anomaly inflow action as in Eq. (2), where $A^{(1)}$ is now the background field of $\mathbb{Z}_4^{(0)}$. Note that $\mathbb{Z}_2^F \subset \mathbb{Z}_4^{(0)}$ is anomaly free. The low-energy dynamics are well known: the $\mathbb{Z}_4^{(0)}$ is spontaneously broken to $\mathbb{Z}_2^F$, and there are two gapped vacua related by $\mathbb{Z}_2^{(0)} \coloneqq \mathbb{Z}_4^{(0)}/\mathbb{Z}_2^F$. Each vacuum is trivially gapped.

After gauging $\mathbb{Z}_2^{(1)}$, we obtain $\mathcal{N} = 1$ SO(3) super YM. Since $\mathbb{Z}_2^{(0)}$ is extended by a nonanomalous $\mathbb{Z}_2^F$, the fusion rules for the Kramers-Wannier duality defect discussed in the previous section are modified. Denote the 3d defect implementing $\mathbb{Z}_2^F$ by $F(M_3)$, and the $\mathbb{Z}_2^{(0)}$ defect before gauging as $D(M_3)$. The fact that they together generate $\mathbb{Z}_4^{(0)}$ implies $D \times D = F$ and $F \times F = 1$. Correspondingly, the fusion rule (5) becomes

$$\mathcal{N}(M_3) \times \mathcal{N}(M_3) = \frac{F(M_3)}{|H^0(M_3, \mathbb{Z}_2)|} \sum_{\Sigma \in H_2(M_3, \mathbb{Z}_2)} (-1)^{Q(\Sigma)} L(\Sigma). \quad (16)$$

The other fusion rules (6) and (7) are unmodified. Equation (16) can be understood as an extension of the noninvertible symmetry by an invertible symmetry $\mathbb{Z}_2^F$.

There are still two vacua in the SO(3) theory [24]. One vacuum remains trivially gapped, while the other vacuum supports a nontrivial $\mathbb{Z}_2$ TQFT. Thus the two vacua are not exchanged by a conventional 0-form symmetry. Instead, they are exchanged by acting with a noninvertible line $\mathcal{N}$ implementing the self-duality TST. The existence of the two vacua related by self-duality can be viewed as spontaneous breaking of the noninvertible symmetry.

$\mathcal{N}=4$ SU(2) super Yang-Mills theory at $\tau=i$.—Finally, we note that $\mathcal{N}=4$ $\mathrm{SO}(3)_-$ super Yang-Mills theory at $\tau=i$ is invariant under S-duality, which effectively maps $\tau\to-1/\tau$ [24]. This S duality is an invertible $\mathbb{Z}_2^{(0)}$ symmetry. There is also a $\mathbb{Z}_2^{(1)}$ one-form symmetry, which has a mixed anomaly with S duality

$$Z_{\mathrm{SO}(3)_-}[-1/\tau,B^{(2)}]=e^{\frac{i\pi}{2}\int_{X_4}\mathcal{P}(B^{(2)})}Z_{\mathrm{SO}(3)_-}[\tau,B^{(2)}], \quad (17)$$

following from $Z_{\mathrm{SU}(2)}[\tau,B^{(2)}]=Z_{\mathrm{SO}(3)_+}[-1/\tau,B^{(2)}]$. Hence, the general results of the previous section apply here as well.

Concretely, we take $\mathcal{T}$ to be $\mathcal{N}=4$ $\mathrm{SO}(3)_-$ super YM at $\tau=i$. Then $TS\mathcal{T}$ is SU(2) super YM at $\tau=i$. Our general results imply that SU(2) super YM at $\tau=i$ contains a noninvertible defect $\mathcal{N}$ implementing the Kramers-Wannier self-duality under S. More generally, on the conformal manifold parametrized by $\tau=2\pi i/g^2$, SU(2) super YM theories come in pairs related by

$$S:[\mathrm{SU}(2),g]\to[\mathrm{SU}(2),2\pi/g]. \quad (18)$$

The theory at the fixed point $g=\sqrt{2\pi}$ is the 4D analog of the topological transition studied in Ref. [53].

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Note added.—Recently, we were informed that work on a similar topic will appear in Ref. [48].

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