Invariants in Polarimetric Interferometry: A Non-Abelian Gauge Theory

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The discovery of magnetic fields close to the M87 black hole using very long baseline interferometry by the Event Horizon Telescope collaboration utilized the novel concept of "closure traces," that are immune to element-based aberrations. We take a fundamentally new approach to this promising tool of polarimetric very long baseline interferometry, using ideas from the geometric phase and gauge theories. The multiplicative distortion of polarized signals at the individual elements are represented as gauge transformations by general 2×2 complex matrices, so the closure traces now appear as gauge-invariant quantities. We apply this formalism to polarimetric interferometry and generalize it to any number of interferometer elements. Our approach goes beyond existing studies in the following respects: (1) we use triangular combinations of correlations as basic building blocks of invariants, (2) we use well-known symmetry properties of the Lorentz group to transparently identify a complete and independent set of invariants, and (3) we do not need autocorrelations, which are susceptible to large systematic biases, and therefore unreliable. This set contains all the information, immune to corruption, available in the interferometer measurements, thus providing important robust constraints for interferometric studies.

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The measurement of coherence of fields is an important concept with applications in many disciplines of physics. Radio astronomers measure coherence by correlating radio signals received by an interferometer array and characterize the morphology of radio emission received from the sky [1]. The signals received at each telescope are usually corrupted due to propagation effects and local imperfections in the array receiver elements. Finding interferometric invariants, quantities that are immune to this corruption, and so accurately reflect the true structure of the sky emission, is of significant value. This is the principal focus of this Letter.

The subject of closure phases and closure amplitudes, two popular interferometric invariants in astronomy, has a long history [2,3]. Concepts analogous to closure invariants (interferometric invariants defined on closed loops) occur in other areas like speckle interferomentry [4], crystallography [5], and quantum mechanics [6,7]. Our approach, as in the recent work [8], is inspired by the geometric phase [7,9] of quantum physics and classical optics [10,11]. A common thread that ties all these diverse problems together is gauge theory. We adapt the general framework to the problem of closure invariants in astronomy.

Closure invariants in copolar correlations, in which signals of the same polarization are correlated between all pairs of telescopes, are well studied in both theory [1,12] and practice [13,14]. In a recent significant extension to polarimetric measurements, "closure traces" were introduced and explored in detail for a system of four array elements [15]. These closure traces were used in studying the magnetic properties near the event horizon of the supermassive black hole at the center of the galaxy M87 [16]. Generalizing and extending this work [15] on polarimetric interferometry and closure invariants, our Letter applies the gauge theory framework to the group of nonsingular 2×2 matrices [known to mathematicians as $\mathrm{GL}(2, \mathbb{C})$] and Lorentz groups. We go beyond earlier work by transparently determining the complete and independent set of invariants from an N-element interferometer array. We show that autocorrelation measurements, which are highly susceptible to systematic errors [17], are not necessary. If available, they are easily included in the scheme. This should be useful in future mapping of polarized sources, especially using long baseline interferometry, where calibration is challenging.

Notation.—We use indices $a, b = 0, \ldots, N-1$ to label the array elements, and indices p, $q = 1, 2$ to label the two polarizations. 2×2 and 4×4 matrices are shown in uppercase boldface. Two-element vectors and four-vectors are written using lowercase boldface in regular and italicized fonts, respectively. For four-vectors and 4×4 matrices written in component form, we use relativity conventions, like the Einstein summation convention for repeated Greek indices, which range over 0,1,2,3.

Polarimetric interferometry.—Consider an interferometer array with N elements. The pth component of polarization (in some orthonormal basis) of the electric field incident on element, a, is denoted by e_a^p, represented in amplitude and phase by a complex two-element column vector, $\mathbf{e}_a$. The true correlation matrix between the array elements is obtained via pairwise cross multiplication and averaging, $\mathbf{S}_{ab} \coloneqq \langle \mathbf{e}_a \mathbf{e}_b^\dagger \rangle$, where $\dagger$ denotes a conjugate-transpose operation and $\langle\cdot\rangle$ denotes the average.

Due to the propagation medium and the nonideal measurement process, the incoming amplitudes are corrupted by an element-based linear transformation—a general 2×2 complex matrix, $\mathbf{G}_a$. The off-diagonal entries of $\mathbf{G}_a$ represent leakage between the two polarized receiver channels at array element a. The measured amplitudes, $\mathbf{v}_a$, are related to the true amplitudes, $\mathbf{e}_a$, by $\mathbf{v}_a = \mathbf{G}_a \mathbf{e}_a$.

The corrupted correlation matrix for a pair of elements, (a, b), is constructed from the measurements as

$$\mathbf{C}_{ab} = \langle \mathbf{v}_a \mathbf{v}_b^\dagger \rangle = \mathbf{G}_a \langle \mathbf{e}_a \mathbf{e}_b^\dagger \rangle \mathbf{G}_b^\dagger = \mathbf{G}_a \mathbf{S}_{ab} \mathbf{G}_b^\dagger. \qquad (1)$$

$\mathbf{G}_a$, $\mathbf{C}_{ab}$, and $\mathbf{S}_{ab}$ are 2×2 matrices. Generically, their eigenvalues are nonzero and we will assume this to be true. These matrices are then invertible. The autocorrelations, $\mathbf{A}_{aa} \coloneqq \mathbf{C}_{aa}$, are Hermitian and positive definite (strictly positive eigenvalues). We use a special symbol to distinguish them from general cross-correlations, $\mathbf{C}_{ab}$.

Our objective is to construct quantities that are immune to the corruptions and actually reflect the true coherence properties of the source of radio emission, rather than local conditions (represented by $\mathbf{G}_a$) at the interferometer elements. This problem is mathematically related to the "gauge" theories of fundamental interactions, such as electromagnetism or the nuclear force. We regard multiplication of the true signal by the local gains, $\mathbf{G}_a$, as gauge transformations. We wish to eliminate these spurious effects introduced by $\mathbf{G}_a$ and identify a maximal, independent set of gauge-invariant quantities (independent of $\mathbf{G}_a$), which we will call "closure invariants." The gains are not unitary, resulting in features not seen in the usual gauge theories of particle physics. The case of copolar observations is described in detail in [1,12], where the gains are just nonzero complex numbers. These form a commutative group. In a companion paper [18], we have introduced these ideas in a simpler context, which serves as a stepping stone to the polarized theory. The current Letter deals with full polarimetric measurements. An important difference between the two cases is that matrices do not, in general, commute. This introduces a level of complexity not seen in the copolar case [18].

Counting invariants.—How many independent closure invariants can we expect to find? The number of true cross-correlations, $\mathbf{S}_{ab}$, in the measured cross-correlations, $\mathbf{C}_{ab}$, is equal to the number of array element pairs, $N(N-1)/2$, which are assumed to be nonredundant in this Letter. Each $\mathbf{C}_{ab}$ gives us one complex 2×2 matrix, with eight real parameters. If autocorrelations are also measured, then we would have to add $4n_A$ to this count, since $\mathbf{A}_{aa}$ is Hermitian and has four real parameters, and n_A is the number of true autocorrelations in the measurements. When the object occupies a small fraction of the field of view, all elements would measure essentially the same autocorrelation. Thus, $\mathbf{S}_{aa} = \mathbf{S}_{bb}$ for all a, b, and we can set n_A to zero or one without loss of generality. Thus, if $\mathcal{M}$ is the space of all measured correlations, its dimension is the total number of real parameters in the measured correlations, $\dim(\mathcal{M}) = 8N(N-1)/2 + 4n_\mathrm{A}$.

Each of the N gain matrices, $\mathbf{G}_a$, is described by four complex numbers (or eight real numbers), resulting in $8N$ real parameters in total. The gains $\mathbf{G}_a$ from the N array elements corrupt the signals according to Eq. (1) and the $\mathbf{C}_{ab}$ could lie anywhere on a surface of points containing $\mathbf{S}_{ab}$ (see Fig. 1). Any function of the measurements that is constant on each surface is immune to corruption by the gains. These are the interferometric invariants we seek. The number of independent interferometric invariants is the dimension of $\mathcal{M}$ minus the dimension of the surface.

The dimension of the surface can at most be the dimension of the gains $\mathbf{G}_a$, which is $8N$. It could be smaller if there are gain variations that do not corrupt the measured correlations. We refer to these as noncorrupting gains (NCGs). As a result, we would expect

$$N_\mathcal{I} = 4N^2 - 12N + 4n_A + s \qquad (2)$$

independent invariants, where s is the number of parameters in NCGs. One well-known example of an NCG is a constant phase shift applied to all the signals on all the interferometer elements, which would not affect $\mathbf{C}_{ab}$. This is the only NCG ($s = 1$) in all cases except $N = 3$, $n_A = 0$ (see below). For $N = 4$ this yields 17 and 21 invariants, for $n_A = 0$ and $n_A = 1$, respectively. We note that our count differs from that of [15], where NCGs were not considered and the dimension of the surface was assumed to be $8N$ (from $4N$ complex gain parameters).

Gauge theory on a graph.—Given the measured correlations, we wish to find a complete and independent set of invariants. In a standard gauge theory, this is done by fixing a base point and considering parallel transport along closed loops that start and end at the base point. Such parallel transport measures curvature and is called a Wilson loop. Wilson loops generate all the gauge invariant quantities in a gauge theory and are the key to constructing closure invariants. We fix a base vertex at one of the array elements

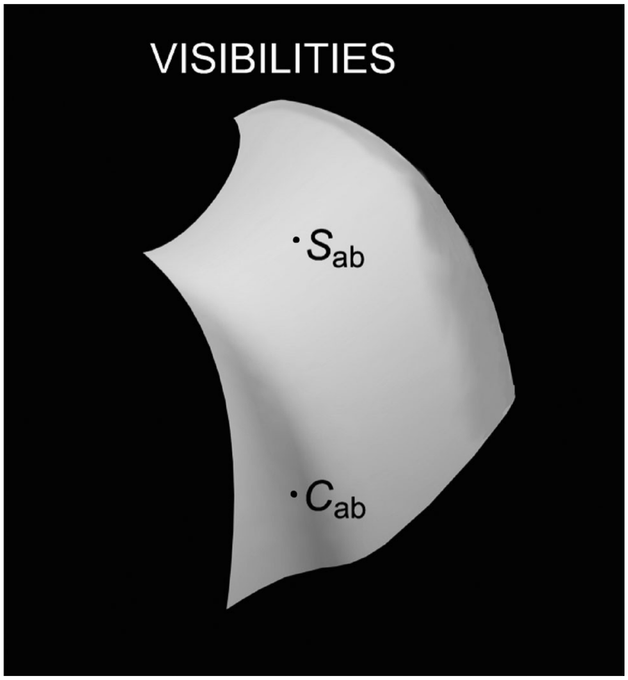

FIG. 1. A schematic representation of the space of correlations using a dark three-dimensional region. If the true correlation due to the source structure is at the point marked S_{ab}, then the measured correlation, C_{ab}, could lie anywhere on the gray two-dimensional surface because of gain distortions. For $N > 3$, $n_A = 0$, the correlation space and the surface have real dimensions $4N(N-1)$ and $8N-1$, respectively. The figure shows these spaces as three and two dimensional, respectively, for illustrative purposes.

labeled 0. We regard each array element, a, as a vertex in a graph with N vertices. Each element pair, (a, b), is a directed edge or a "link" in the graph, carrying the variable, $\mathbf{C}_{ab}$, a 2×2 complex matrix. This link variable is called a "connection" and defines parallel transport from a to b. The gains, $\mathbf{G}_a$, are local (i.e., element-based) gauge transformations.

In standard gauge theories, such as SU(2) for example, the gauge group describing $\mathbf{G}_a$ is unitary and Hermitian conjugation in Eq. (1) is the same as matrix inversion. Since our gauge group is not unitary, we have to proceed differently. So, we define a hat operator on invertible matrices, $\widehat{\mathbf{P}} = (\mathbf{P}^\dagger)^{-1}$. This convenient notation lets us adapt methods employed by [15]. We first define covariants, $\boldsymbol{\mathcal{C}}_{\Gamma_0}$, as products of an even number of $\mathbf{C}_{ab}$ matrices around a closed loop, Γ_0, pinned at vertex 0, with the even terms "hatted." For example, for $\Gamma_0 = 0abc0$, $\boldsymbol{\mathcal{C}}_{\Gamma_0} = \mathbf{C}_{0a}\widehat{\mathbf{C}}_{ab}\mathbf{C}_{bc}\widehat{\mathbf{C}}_{c0}$. Under gauge transformations, covariants transform as

$$\boldsymbol{\mathcal{C}} \overset{\mathcal{G}}{\mapsto} \mathbf{G}_0 \boldsymbol{\mathcal{C}} \mathbf{G}_0^{-1}. \tag{3}$$

Note that by defining covariants pinned at vertex 0, we have considerably reduced the number of gains contributing to corruptions. All that remains is a single gain $\mathbf{G}_0$ at the base vertex 0. Closure invariants can be found by taking traces, $\mathcal{I} = \mathrm{tr}(\boldsymbol{\mathcal{C}})$, of covariants as in [15] or indeed in gauge theories. These invariants will be unchanged under any gauge transformation with $\mathbf{G}_a$. However, the need for an even number of links to construct covariants would seem to force us towards rectangles. The problem now is that it is far from clear which set will give us a complete and independent set of invariants.

Our solution to this problem is to define and use quantities called "advariants," which we introduced in [18]. They are constructed by multiplying an odd number of correlations, $\mathbf{C}_{ab}$, around a closed loop, Γ_0, pinned at vertex 0 with even terms hatted. For example, for $\Gamma_0 = 0abcd0$, $\boldsymbol{\mathcal{A}}_{\Gamma_0} = \mathbf{C}_{0a}\widehat{\mathbf{C}}_{ab}\mathbf{C}_{bc}\widehat{\mathbf{C}}_{cd}\mathbf{C}_{d0}$. Under gauge transformations, advariants transform as

$$\boldsymbol{\mathcal{A}} \overset{\mathcal{G}}{\mapsto} \mathbf{G}_0 \boldsymbol{\mathcal{A}} \mathbf{G}_0^{\dagger}. \tag{4}$$

One special case of an advariant is the autocorrelation, $\boldsymbol{\mathcal{A}}_0 \coloneqq \mathbf{A}_{00}$, which obeys Eq. (4). The next nontrivial example is a three-vertex advariant defined on an elementary triangle $\Delta_{(0ab)} \equiv (0, a, b)$, $\boldsymbol{\mathcal{A}}_{ab} \coloneqq \mathbf{C}_{0a}\widehat{\mathbf{C}}_{ab}\mathbf{C}_{b0}$. Note that $\boldsymbol{\mathcal{A}}_{ab}^{\dagger}$ is also an advariant. Advariants are the building blocks for covariants. In fact, any covariant can be obtained by multiplying an even number of such elementary advariants with even terms hatted as described earlier. For example, a four-vertex covariant on a closed loop, $\Gamma_0 = 0abc0$, can be written as $\boldsymbol{\mathcal{C}}_{\Gamma_0} = \boldsymbol{\mathcal{A}}_{ab}\widehat{\boldsymbol{\mathcal{A}}}_{bc}$. Note that all these covariants transform in accordance with Eq. (3) for a closed loop pinned at vertex 0. The traces of these covariants are the closure invariants [15]. The key advance in this Letter is that we use elementary triangular advariants to arrive at a complete and independent set of closure invariants, using general arguments based on Lorentz and scaling symmetries. An advariant, $\boldsymbol{\mathcal{A}}$, can be expanded in terms of the identity matrix, $\boldsymbol{\sigma}_0 \coloneqq \mathbf{I}$, and Pauli matrices, $\boldsymbol{\sigma}_m, m = 1, 2, 3$ as

$$\boldsymbol{\mathcal{A}} = z^{\mu}\boldsymbol{\sigma}_{\mu}, \tag{5}$$

where z^μ are complex coefficients. The z^μ can be thought of as similar to complex Stokes parameters $\{I, Q, U, V\}$. In the case of the autocorrelation advariant, which is Hermitian, these are real and are the ordinary Stokes parameters of polarization optics [19].

Under gauge transformations of $\boldsymbol{\mathcal{A}}$, the action of the gauge group, $\mathbf{G}_0 \in \mathrm{GL}(2, \mathbb{C})$, on advariants can be split into two parts. First, the subgroup, $\mathbf{G}_0 = \lambda\mathbf{I}$, acts on z^μ by scaling

$$z^{\mu} \mapsto |\lambda|^2 z^{\mu}. \tag{6}$$

Second, the subgroup $\mathrm{SL}(2, \mathbb{C})$ of matrices with unit determinant preserves $\det(\boldsymbol{\mathcal{A}}) = I^2 - Q^2 - U^2 - V^2 = z^{\mu}\eta_{\mu\nu}z^{\nu}$, which shows that it acts on z^μ by a Lorentz transformation,

$$z^\mu \mapsto \Lambda^\mu_\nu z^\nu, \tag{7}$$

where $\Lambda \equiv \Lambda^\mu_\nu$ is a real 4×4 matrix representing a Lorentz transformation. Note that the real and imaginary parts of $\mathbf{z}$ separately transform as four-vectors. Closure invariants must remain unchanged under transformations by both these subgroups.

Complete and independent set.—The triangular advariants (pinned at vertex 0) can be used to build covariants for every closed loop with even links in the graph. Each triangular advariant, $\boldsymbol{\mathcal{A}}_{ab}$, can be expanded as $\boldsymbol{\mathcal{A}}_{ab} = z^\mu_{ab}\boldsymbol{\sigma}_\mu$ as in Eq. (5), where, z^μ_{ab} are complex four-vectors.

We first consider only the Lorentz transformations in Eq. (7). Each triangle, $\Delta_{(0ab)}$, gives us a complex four-vector, z^μ_{ab}, which can be decomposed into two real four-vectors. The number of triangles with one vertex at 0 is $N_\Delta = \binom{N-1}{2}$. In addition, for $n_A = 1$, there will be one more Hermitian advariant, $\boldsymbol{\mathcal{A}}_0$, which gives us one more real four-vector. Thus, we have a set of $M = 2N_\Delta + n_A$ real, independent, four-vectors, $\boldsymbol{y}_m$, $m = 1, 2, \ldots M$. Invariants of the Lorentz subgroup (Lorentz invariants), can only be functions of Minkowskian inner products among these vectors.

We construct an independent set of closure invariants (for $N > 3$) as follows: choose any four real four-vectors to be the basis set, say $\boldsymbol{h}_k = \mathbf{y}_k$, $k = 1, 2, 3, 4$. Between these four independent basis vectors, there are 10 Minkowskian inner products, $\widetilde{\mathcal{I}}_{k\ell} = \boldsymbol{h}_k \cdot \boldsymbol{h}_\ell$, with $k, \ell = 1, 2, 3, 4$ and $\ell \geq k$. Further, the remaining $(M-4)$ four-vectors give us $4(M-4)$ Minkowski inner products with the four basis vectors, $\widetilde{\mathcal{I}}_{kn} = \boldsymbol{h}_k \cdot \mathbf{y}_n$ for $n = 5, \ldots M$. Together, we have $10 + 4(M-4) = 4N^2 - 12N + 2 + 4n_A$ real Minkowski inner products. There is no need to take any more inner products since a four-vector is completely specified by its components in a basis. The M triangular advariants are independent and all covariants can be constructed from these triangular advariants. This ensures that these Minkowskian inner products are both complete and independent.

The inner products (Lorentz invariants) are still not the invariants we seek since they acquire a scale factor from the element-based gains [Eq. (6)]. We can obtain true invariants by forming ratios of the Lorentz invariants $\{\widetilde{\mathcal{I}}_{k\ell}, \widetilde{\mathcal{I}}_{kn}\}$, for example, dividing the set by any one of them, say $\widetilde{\mathcal{I}}_{11}$. Thus, the number of closure invariants will be $4N^2 - 12N + 1 + 4n_A$, that is, one less than the number of independent inner products. Comparing with Eq. (2), we find that $s = 1$ (one NCG corresponding to a uniform phase across all elements) for $N \geq 4$.

The algorithmic summary for finding the closure invariants from the measured visibilities is as follows: (1) choose a base element 0 and construct the $N_\Delta = \binom{N-1}{2}$ complex triangular advariants. (2) Use these to construct $2N_\Delta$ real four-vectors. (3) If $n_A = 1$, then include one more real four-vector to get $M = 2N_\Delta + n_A$ four-vectors. (4) Pick four of these four-vectors as a basis and compute the 10 Minkowski dot products between them. (5) Compute the Minkowski dot products between the remaining $(M-4)$ four-vectors and the basis vectors. (6) Divide the $10 + 4(M-4)$ Minkowski dot products by any one of them to find all the closure invariants. Finding a complete and independent set of closure invariants in polarized interferometry is the main new result of this Letter.

The case $N = 3$ is special. For $n_A = 1$, we get five invariants as expected (from $4N^2 - 12N + 1 + 4n_A$). For $n_A = 0$, we may likewise expect to get one invariant. Instead, we get two invariants: the three inner products of the two four-vectors associated with $\boldsymbol{\mathcal{A}}_{12}$ give us two invariant ratios. The "extra" invariant appears because there is an extra one parameter family of NCGs. These are Lorentz transformations in the plane orthogonal to the two four-vectors. We identified a uniform phase shift on all elements as an example of an NCG. Are there others? The work of this Letter and comparison with Eq. (2) clearly rules this out in all cases but one ($N = 3$, $n_A = 0$). A simple proof of this general statement can be given: any Lorentz transformation that preserves all the visibilities also preserves the advariants and the associated four-vectors. The only Lorentz transformation that preserves three (or more) independent vectors is the identity. The only case where there are fewer than three independent four-vectors occurs when $N = 3$, $n_A = 0$, as was noted earlier as an example of an NCG. For $N > 3$, there are no NCGs in the Lorentz subgroup. There is just one in the scaling subgroup, a constant phase factor [18].

Comparison with earlier work.—We now return to the formulation of invariants in terms of closure traces introduced in [15], and relate it to the framework of Lorentz invariants constructed from four-vectors developed in this Letter. An elementary calculation shows that

$$\widehat{\boldsymbol{\mathcal{A}}} = \frac{z^{0*}\boldsymbol{\sigma}_0 - z^{1*}\boldsymbol{\sigma}_1 - z^{2*}\boldsymbol{\sigma}_2 - z^{3*}\boldsymbol{\sigma}_3}{(z \cdot z)^*}. \tag{8}$$

Given two advariants, $\boldsymbol{\mathcal{A}}$ and $\boldsymbol{\mathcal{A}}'$, we form a covariant, $\boldsymbol{\mathcal{A}}\widehat{\boldsymbol{\mathcal{A}}'}$. Taking the trace and using Eq. (8), we find an invariant,

$$\frac{1}{2}\mathrm{tr}(\boldsymbol{\mathcal{A}}\widehat{\boldsymbol{\mathcal{A}}'}) = \frac{z \cdot z'^*}{(z' \cdot z')^*}, \tag{9}$$

which is a ratio of Minkowskian inner products. This equation provides the link between the closure trace formulation [15] and the Minkowskian one presented here.

We have performed numerical tests to check the analytical theory described in this Letter. The expected number of independent invariants was confirmed by computing the rank of the Jacobian of the numerical partial derivatives of the invariants with respect to the components of the correlation matrices. We used multiple realizations of the correlations, and examined the rank of the Jacobian using its singular values. The rank agreed with our analytic results in all cases, including for $N > 4$.

In the case of $N = 4$, $n_A = 1$, a set of covariants is exhibited in Sec. 3.2 and Appendix D of [15]. Because they did not account for the NCGs, they expected to find only 20 real independent invariants from 10 complex closure traces. We have numerically verified that only 18 of their 20 real invariants are independent and further that the complete and independent set consists of 21 real invariants when the NCGs are properly accounted for. This illustrates the difficulty of ensuring independence and completeness within the closure trace formalism. We conclude that the new ideas and discussions in [15] are valid and valuable, but the details regarding the independent loops need reexamination.

Discussion.—The four-vector formalism gives us an elegant criterion for deciding if the object has any polarization structure at all. In the absence of polarization structure, the four-vectors corresponding to the true correlations have only a 0th component (which depends only on the Stokes total intensity components of the correlations), which renders them all collinear. As a result, y_m^μ are all collinear, since this property is preserved by Lorentz transformations and scaling. The dimension of the space spanned by the four-vectors thus gives a strong statistical test for evidence of polarized structure.

This Letter opens up immediate applications, and areas for further investigation. In cutting-edge very long baseline interferometry work on polarized emission, the availability of a full set of calibration-independent constraints will provide a valuable confirmation of derived images. Interferometry has been used, even without imaging, to study the statistics of random fields. These studies will also gain by a knowledge of closure invariants.

In this context, the four-vectors introduced in this Letter have the advantage that they belong to a linear space, so that the same information can be encoded in linear combinations, ranked by signal to noise, via singular value decomposition. Given that direct geometric interpretation of invariants is still being explored even in the case of copolar measurements [8], there is clearly much work to be done on the four-vector approach. Finally, we anticipate that the general concept of gauge invariance we have introduced may have wider applicability, in other fields where general linear transformations of multichannel data by unknown or ill-constrained factors have corrupted the correlations.

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