Dominance of Replica Off-Diagonal Configurations and Phase Transitions in a *PT* Symmetric Sachdev-Ye-Kitaev Model

Antonio M. García-García,[1,*] Yiyang Jia (贾抑扬),[2,†] Dario Rosa,[3,4,‡] and Jacobus J. M. Verbaarschot[2,§]

[1]Shanghai Center for Complex Physics, School of Physics and Astronomy, Shanghai Jiao Tong University, Shanghai 200240, China

[2]Department of Physics and Astronomy, Stony Brook University, Stony Brook, New York 11794, USA

[3]Center for Theoretical Physics of Complex Systems, Institute for Basic Science(IBS), Daejeon 34126, Korea

[4]Department of Physics, Korea Advanced Institute of Science and Technology, 291 Daehak-ro, Yuseong-gu, Daejeon 34141, Republic of Korea

(Received 7 May 2021; revised 12 September 2021; accepted 28 January 2022; published 22 February 2022)

We show that, after ensemble averaging, the low temperature phase of a conjugate pair of uncoupled, quantum chaotic, non-Hermitian systems such as the Sachdev-Ye-Kitaev (SYK) model or the Ginibre ensemble of random matrices is dominated by saddle points that couple replicas and conjugate replicas. This results in a nearly flat free energy that terminates in a first-order phase transition. In the case of the SYK model, we show explicitly that the spectrum of the effective replica theory has a gap. These features are strikingly similar to those induced by wormholes in the gravity path integral which suggests a close relation between both configurations. For a nonchaotic SYK, the results are qualitatively different: the spectrum is gapless in the low temperature phase and there is an infinite number of second order phase transitions unrelated to the restoration of replica symmetry.

DOI: 10.1103/PhysRevLett.128.081601

The study of non-Hermitian effective Hamiltonians has a long history [1,2]. Perhaps the best known example is the effective Hamiltonian that describes resonances with a finite width, for example the one that enters in the calculation of the S matrix of open quantum systems such as quantum dots [3] or compound nuclei [4]. Another example is the Euclidean QCD Dirac operator at nonzero chemical potential, which is non-Hermitian with spectral support on a two-dimensional domain of the complex plane [5]. In Hermitian theories, a phase transition may arise due to the formation of a gap. This may also happen for non-Hermitian systems when the domain of eigenvalues splits into two or more pieces. However, another mechanism is possible. Because of the non-Hermiticity, the action is generally complex, and the saddle point with the largest real part of the free energy may get nullified after ensemble averaging. In QCD at nonzero baryon chemical potential, the pion condensation phase is nullified so that the phase transition to nonzero baryon density becomes visible [6]. The conclusion is that the phase diagram can be altered dramatically by the nullification of the leading saddle point [6,7].

A second point we wish to make is about the nature of quenched averages in non-Hermitian theories. Although alternatives are possible [4,8,9], quenched averages are often carried out by means of the replica trick [10]. However, because of Carlson's theorem [11], a naive application of the replica trick is not guaranteed to work [12]. The best known example of the failure of the replica trick is in the calculation of the quenched free energy of the Sherrington-Kirkpatrick model [13], a toy model for spin glasses, which in the low temperature limit yields a negative entropy [13]. This inconsistency was ultimately resolved by postulating a ground state that breaks the replica symmetry [14,15]. The problems with the replica trick are more dramatic for non-Hermitian theories as was first demonstrated for QCD at nonzero chemical potential μ [16]. In this case, the n replica (or n flavor) partition function is given by

$$Z_n = \langle \det^n D(\mu) \rangle, \tag{1}$$

where the averaging is over gauge field configurations weighted by the Euclidean Yang-Mills action. It was shown that the quenched approximation, where the determinant is put to unity, is not given by $\lim_{n\to 0} Z_n$ but rather by

$$\lim_{n\to 0} \langle \det^n [D(\mu) D^\dagger(\mu)] \rangle. \tag{2}$$

This partition function is dominated by the Goldstone modes of the spontaneous breaking of the (continuous) chiral symmetry between the flavors and conjugate flavors. A similar mechanism has been identified in the context of random matrix theory, for both Hermitian [12,17–20] and non-Hermitian [21] random matrix ensembles, in which saddle points that couple advanced and retarded replicas become dominant. We note that this is different from the replica symmetry breaking phenomenon in spin glasses where the replica symmetry is broken in the low temperature phase. More explicitly, the chiral condensate corresponding to (2) or the resolvent in random matrix theory remains flavor or replica symmetric. In the Sachdev-Ye-Kitaev (SYK) models we investigate, the analog of this type of symmetry breaking is the dominance of saddle points that couple the left (L) and right (R) replicas. However, there is no symmetry breaking among the L or R replicas.

The possibility of a partition function where the connected part dominates the disconnected part has recently received a great deal of attention in the analysis of wormhole solutions in Jackiw-Teitelboim (JT) [22,23] gravity and related theories [24–34] including different random matrix models representing different flavors of JT gravity [35–42]. The existence of these solutions in Lorentzian signature was first observed in [43] with the discovery of a low temperature traversable wormhole phase in a near AdS_2 background deformed by weakly coupling the two boundaries. As temperature increases, the system eventually undergoes a first-order wormhole-to-black-hole transition. By adding complex sources, it is possible to find [32] Euclidean wormhole solutions of JT gravity that undergo a similar transition at finite temperature.

Interestingly, a replica calculation [28] of the quenched free energy in JT gravity found that, in the low temperature limit, the contribution of replica wormholes is dominant. Likewise, the evaluation of the von Neumann entropy by the replica trick [29–31,44] revealed the existence of additional saddle points, wormholes connecting different copies of black holes in this context. These wormhole configurations are crucial to make the process of black hole evaporation consistent with unitarity [45].

A natural question to ask is whether these replica wormholes have a field theory analog. Solutions with coupled replicas have been explored in the usual Hermitian SYK model with real couplings [46,47]. However, there is no evidence that they dominate the partition function.

In this Letter, we answer the question posed in the previous paragraph affirmatively by identifying a pair of conjugate non-Hermitian SYK Hamiltonians whose sum is PT symmetric and where, after ensemble averaging, solutions that couple replicas and conjugate replicas are the leading saddle points of the action in the low temperature phase. Such solutions are permutation symmetric under pairs of conjugate partition functions. The non-Hermition SYK model has a phase transition from a low-temperature phase dominated by these coupled saddle points to a phase dominated by the disconnected part of the partition function where the replicas and conjugate replicas are not coupled.

The q-body SYK Hamiltonian [48–51] is defined by

$$H_{\mathrm{SYK}} = (i)^{q/2} \sum_{\alpha_1<\cdots<\alpha_q} J_{\alpha_1\cdots\alpha_q}\chi_{\alpha_1}\cdots\chi_{\alpha_q}, \tag{3}$$

where the χ_α represent N Majorana fermions, satisfying the anticommutation relations $\{\chi_\alpha,\chi_\beta\} = \delta_{\alpha\beta}$, and the $J_{\alpha_1\cdots\alpha_q}$ are real couplings, sampled from a Gaussian distribution having a vanishing mean value and a variance proportional to $1/N^{q-1}$. The coupled SYK model introduced by Maldacena and Qi (MQ) in [43] consists of a left (L) SYK model and a right (R) SYK model each with $N/2$ Majorana fermions, and a coupling term $i\mu\sum_{k=1}^{N/2}\chi_k^R\chi_k^L$. Although the left and right couplings, denoted by $J^{L(R)}_{\alpha_1\cdots\alpha_q}$ are chosen to be the same as in the MQ model, it is also possible, as was noted by the same authors, to take them different. One remarkable observation was made: the solution that couples the right and left SYK continues to exist in absence of an explicit coupling ($\mu = 0$) provided that

$$\langle J^L J^R\rangle > \langle J^L J^L\rangle = \langle J^R J^R\rangle, \tag{4}$$

where $\langle\ldots\rangle$ stands for ensemble average. Since the covariance matrix is no longer positive, this cannot be realized by real-valued J_L and J_R. However, this can be achieved for the complex couplings

$$J^L = J + ikK, \qquad J^R = J - ikK, \tag{5}$$

with J, K independent real Gaussian stochastic variables with the same variance and zero mean.

Before continuing, let us analyze the quenched free energy of the single-site SYK we have just introduced:

$$\langle \log Z\rangle = \langle \log |Z|\rangle + i\langle \arg Z\rangle, \tag{6}$$

where Z is the partition function for a specific realization of the couplings. Since the phase of the partition function does not have a preferred direction, we expect that $\langle \arg Z\rangle = 0$. We conclude

$$\langle \log Z\rangle = \frac{1}{2}\langle \log ZZ^*\rangle \tag{7}$$

for a theory where Z and Z^* have equal probability. In particular, the quenched free energy is given by the replica limit

$$\frac{1}{2}\lim_{n\to 0}\left\langle \frac{(ZZ^*)^n - 1}{n}\right\rangle. \tag{8}$$

Therefore, we arrive naturally at a system of an SYK Hamiltonian and its conjugate only coupled through the probability distribution. The Hamiltonian corresponding to ZZ^* is given by

$$H_{\mathrm{SYK}} \otimes 1 + 1 \otimes H_{\mathrm{SYK}}^{\dagger}, \tag{9}$$

which is exactly the two-site Hamiltonian proposed in [43] with the explicit coupling turned off. This Hamiltonian is PT symmetric [1], where the P operator interchanges the L and R spaces and T is the tensor product of two copies of the time reversal operator for the standard SYK model [52–54].

Next, we calculate the partition function $\langle ZZ^* \rangle$ for large N. We will argue that this is sufficient to obtain the free energy. Moreover, it illustrates clearly the appearance of novel saddle points that couple the L replicas Z^n and the conjugate R replicas Z^{*n}.

We consider a pair of non-Hermitian SYK models with couplings (5) for $k = 1$. The spectral density is given by a disk with radius E_0 in the complex plane. Although the eigenvalue density is rotationally invariant, it is not constant as is the case for the large N limit of the Ginibre [55,56] ensemble or random matrices (see Fig. 1). However, we expect that eigenvalue correlations are in the universality class of the Ginibre model. If the averaged eigenvalue density is denoted by $\rho(z)$, the two-level correlation function is given by

$$\rho_2(z_1, z_2) = R_{2c}(z_1, z_2) + \delta^2(z_1 - z_2)\rho(z_1) + \rho(z_1)\rho(z_2),$$

where $R_{2c}(z_1, z_2)$ is the averaged connected two-point correlation function not including the self-correlations. The partition function is given by

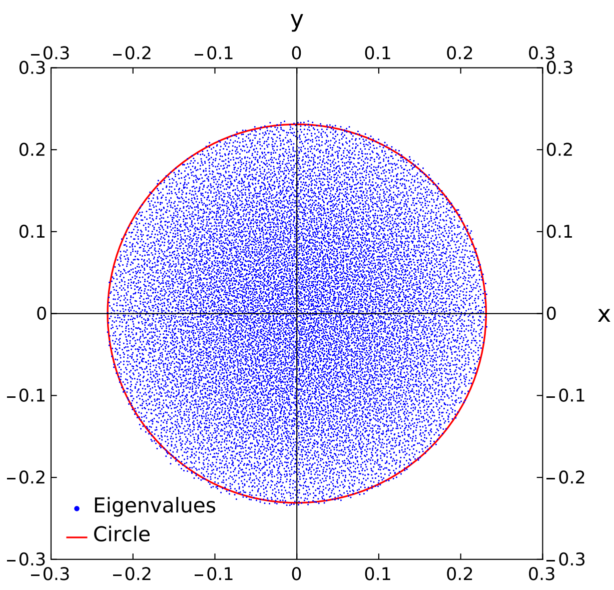

FIG. 1. The eigenvalue density, obtained from exact diagonalization, for one realization of the $q = 4$, $k = 1$ non-Hermitian SYK model with $N/2 = 30$, compared to a circle (red curve).

$$\langle ZZ^* \rangle = \langle Z \rangle \langle Z^* \rangle + \langle ZZ^* \rangle_c, \tag{10}$$

where $\langle ZZ^* \rangle_c = \int d^2 z_1 d^2 z_2 \rho_{2c}(z_1, z_2) e^{-\beta(z_1 + z_2^*)}$, $\rho_{2c}(z_1, z_2) = R_{2c}(z_1, z_2) + \delta^2(z_1 - z_2)\rho(z_1)$, and $\langle Z \rangle = \int d^2 z \rho(z) e^{-\beta z}$. Because the eigenvalue density has rotational invariance, we can use the mean value theorem to show that the partition function is independent of β and given by the normalization of $\rho(z)$, which we denote by D, $\langle Z \rangle = D$.

To evaluate the second term of (10), we use the sum rule

$$\int d^2 z_2 [R_{2c}(z_1, z_2) + \delta^2(z_1 - z_2)\rho(z_1)] = 0, \tag{11}$$

and the fact that the correlations are short range [55,56] with the connected correlator taking the universal form

$$R_{2c}(z_1, z_2) = R_{2c}^{\mathrm{unv}}[\sqrt{\rho(\bar{z})}(z_1 - z_2)]\rho(\bar{z})^2, \tag{12}$$

where $\bar{z} = (z_1 + z_2)/2$. We thus have that $|z_1 - z_2| < 1/\sqrt{D}$ region gives the dominant contribution and we can Taylor expand the exponent in $\langle ZZ_c^* \rangle$ in powers of $\beta \mathrm{Im}(z_1 - z_2)$. The zero order term vanishes because of the sum rule (11), the linear term vanishes because the probability distribution is even under complex conjugation. After performing the integral over $z_1 - z_2$, we obtain the connected partition function

$$\begin{aligned} \langle ZZ^* \rangle_c &= \frac{1}{2}\beta^2 \langle \zeta^2 \rangle \int_{|\bar{z}| < E_0} d^2 \bar{z} e^{-\beta(\bar{z} + \bar{z}^*)} \\ &= \frac{\pi}{2}\beta E_0 \langle \zeta^2 \rangle I_1(2\beta E_0), \end{aligned} \tag{13}$$

where $\langle \zeta^2 \rangle = \langle [\mathrm{Im}(z_1 - z_2)]^2 \sqrt{\rho(\bar{z})} \rangle \sim D^0$, the ground state energy $-E_0 \sim N$ and $D = 2^{N/4}$.

We are now ready for the calculation of the free energy since expect that in the large N limit

$$\langle (ZZ^*)^n \rangle = \langle ZZ^* \rangle^n \tag{14}$$

so that the replica limit (8) only requires the result for $\langle ZZ^* \rangle$ calculated above.

Including the disconnected part of the partition function and using the asymptotic expression of I_1, we obtain a free energy

$$F = -T \log[e^{2\beta E_0} + 2^{N/2}], \tag{15}$$

where we have neglected prefactors that are subleading in N. In the strict large N limit, it simplifies to

$$\frac{F(T)}{N} = \frac{-2E_0}{N}\theta(T_c - T) - \frac{T \log 2}{2}\theta(T - T_c), \tag{16}$$

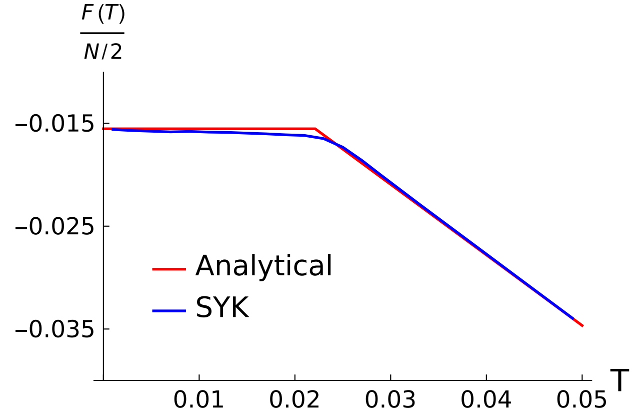

FIG. 2. The temperature dependence of the free energy of the non-Hermitian SYK model for $N/2 = 30$, $q = 4$, and $k = 1$ (blue curve) compared to the analytical result (16). The value of E_0 is the radius of the circle in Fig. 1.

with $T_c = 4E_0/(N \log 2)$. In Fig. 2, we show the numerical quenched free energy of the SYK Hamiltonian (9) for $N/2 = 30$, $q = 4$, and $k = 1$ (black curve) and compare it to the analytical result (16). The deviation in the constant part seems to scale as $1/N$. The free energy can also be worked out for $k < 1$, where we also find a first-order phase transition with $T_c \sim k^4$ for small k.

We have thus observed that the leading exponent of the disconnected part of the partition function is nullified by the phase of the Boltzmann factor so that the contribution due to the connected part of the two-point correlation function becomes dominant. The free energy behaves as if the system has a gap. This is a novel mechanism to induce the formation of a wormhole solution.

In order to make this connection more explicit, we show that these results can also be obtained by solving the Schwinger-Dyson equations, in the ΣG formulation of the two-site SYK model [43,57] which is equivalent to performing the replica trick and then solving the model in the saddle point approximation. For $T > T_c$ and $k = 1$, the solution with the free G and Σ is dominant so that only the kinetic term of the Lagrangian remains. As a consequence, the free energy is $-T \log 2/2$ in agreement with the spectral calculation above. For $T < T_c$, a nontrivial solution that breaks the symmetry between L and R becomes dominant which results in a constant free energy up to very small corrections. Similar results can be derived for $k < 1$, where, in agreement with the previous analytical calculation, we have also found $T_c \sim k^4$. Indeed, this feature is shared by both Euclidean [32] and traversable [43,58] wormholes. Details of this and the previous analytical calculation will be given elsewhere [59].

For traversable wormholes [43], the spectrum is gapped. Physically, it is related to the interaction-driven tunneling between the left and right sites. The existence of the energy gap can be demonstrated [43] directly from the effective boundary gravity action or from the exponential decay of the left-right Green's function $G_{LR}(\tau)$ of the two-site SYK model for low temperatures. We study whether a similar gap exists in the two-site non-Hermitian SYK. We stress the gap in this case is not a property of the microscopic Hamiltonian (9) but only of the resulting replica field theory after ensemble averaging. Since $G_{LR}(\beta/2 - \tau) = -G_{LR}(\tau - \beta/2)$ we employ the ansatz [43]

$$G_{LR}(\tau) \sim \sinh E_g(\beta/2 - \tau), \tag{17}$$

where the gap E_g is a fitting parameter. The fit is excellent except for very small times [60], see Fig. 3. This reinforces the picture that solutions that couple the conjugate replicas mediate tunneling between the two sites even though there is no direct coupling term in the Hamiltonian. We note that G_{LL} and G_{RR} show a similar decay. In Fig. 3, we also show

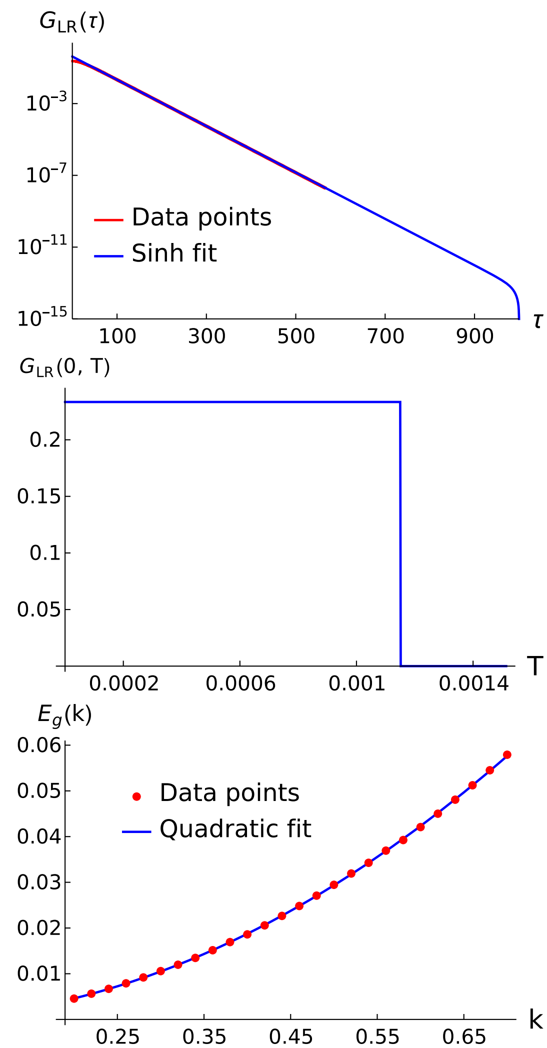

FIG. 3. Top: $G_{LR}(\tau)$ from the solution of the Schwinger-Dyson equations for the SYK model (9) with $q = 4$, $T = 0.0005$, and $k = 0.5$ fitted by (17). Middle: the order parameter $G_{LR}(0)$, versus temperature for $k = 0.5$. Bottom: the energy gap E_g for $T = 0.0005 \ll T_c$ as a function of k from the fit (17).

that the gap E_g depends quadratically on k. It would be interesting to understand this exponent from the gravity side. We propose $G_{LR}(0)$ as the order parameter of the transition since a nonvanishing value of G_{LR} is the distinctive feature of spontaneous LR symmetry breaking. Results depicted in in Fig. 3, confirm that $G_{LR}(0)$ remains almost constant in the wormhole phase and vanishes for $T > T_c$.

The studied $q = 4$ SYK model is quantum chaotic [51]. We expect very similar results in other quantum chaotic systems such as $q > 2$ SYK models and the Ginibre ensemble of random matrices [55] because the spontaneous breaking of LR symmetry depends on the connected two-level correlation function which is universal in this case [61]. However, it is unclear whether quantum chaos is a necessary condition for replica nondiagonal solutions to become dominant. In order to further elucidate this issue, we study the, nonquantum chaotic, $q = 2$ non-Hermitian SYK model which admits an explicit analytical solution of the Schwinger-Dyson equations. It can also be solved, see [59] for details, by mapping it onto a model of free fermions. The free energy can be expressed, see also [59], as a sum over Matsubara frequencies. Each time a new Matsubara frequency enters the sum, by lowering the temperature, a second order phase transition occurs. Kinks in $-dF/dT$, depicted in Fig. 4, indicate the positions of the critical temperatures. The propagator $G_{LR}(\tau)$ can be also expressed as a finite sum over Matsubara frequencies so it does not depend exponentially on τ. Because of the absence of a gap, there is no direct relation between LR coupling of replicas and wormholes which further suggests that the physics is qualitatively different from the quantum chaotic case. Since it is worrisome that $-dF/dT$ becomes negative, we have checked that the free energy obtained from the SD equations is identical to the one from an independent free fermion calculation and agrees with the result from diagonalization of the SYK Hamiltonian. Further research is needed to delimit the importance of quantum chaos for the existence of symmetry breaking between the L and R replicas as well as to understand the effect of non-Hermiticity on the sign of dF/dT. In summary, we have provided evidence that solutions that couple conjugate replicas dominate the low temperature phase of the partition function of pairs of random non-Hermitian, quantum chaotic systems whose sum is PT symmetric. These solutions mimic the contributions of wormholes in the gravitational path integral. In both cases, a first-order transition occurs when replica symmetric saddle points take control of the partition function.

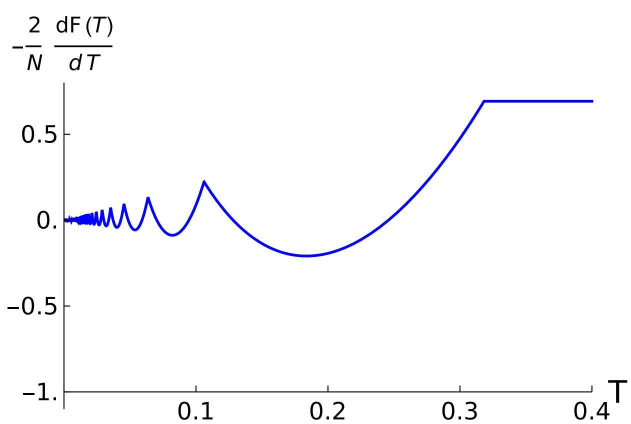

FIG. 4. Derivative of the free energy for the non-Hermitian $q = 2$, $k = 1$ SYK model. On the way to $T = 0$, the system undergoes infinitely many second order phase transitions. For $T > 1/\pi$, it becomes a constant which is a typical feature of free fermions.

This work was supported by NSFC Grant No. 11874259 (A. M. G.), the National Key R&D Program of China (Project ID 2019YFA0308603) (A. M. G.), a Shanghai talent program (A. M. G.), NRF Grant No. NRF-2020R1C1C1007591 (D. R.), the Institute for Basic Science in Korea (IBS-R024-D1) (D. R.), and U.S. DOE Grant No. DE-FAG-88FR40388 (Y. J. and J. J. M. V.). We thank Juan Diego Urbina, Jeff Murugan, Tomoki Nosaka, Klaus Richter and Zhenbin Yang for illuminating correspondence.

*amgg@sjtu.edu.cn
†yiyang.jia@weizmann.ac.il
Present address: Department of Particle Physics and Astrophysics, Weizmann Institute of Science,Rehovot 7610001, Israel.
‡dario_rosa@ibs.re.kr
§jacobus.verbaarschot@stonybrook.edu

[1] C. M. Bender and S. Boettcher, Phys. Rev. Lett. **80**, 5243 (1998).
[2] N. Hatano and D. R. Nelson, Phys. Rev. Lett. **77**, 570 (1996).
[3] Y. Alhassid, Rev. Mod. Phys. **72**, 895 (2000).
[4] J. J. M. Verbaarschot, H. A. Weidenmuller, and M. R. Zirnbauer, Phys. Rep. **129**, 367 (1985).
[5] I. Barbour, N.-E. Behilil, E. Dagotto, F. Karsch, A. Moreo, M. Stone, and H. W. Wyld, Nucl. Phys. **B275**, 296 (1986).
[6] S. Hands, Prog. Theor. Phys. Suppl. **168**, 253 (2007).
[7] J. C. Osborn, K. Splittorff, and J. J. M. Verbaarschot, Phys. Rev. Lett. **94**, 202001 (2005).
[8] K. Efetov, Adv. Phys. **32**, 53 (1983).
[9] E. Kanzieper, Phys. Rev. Lett. **89**, 250201 (2002).
[10] S. F. Edwards and P. W. Anderson, J. Phys. F **5**, 965 (1975).
[11] F. Carlson, Sur une classe de séries de Taylor, Ph.D. dissertation, Uppsala University, Uppsala, Sweden, 1914.
[12] J. J. M. Verbaarschot and M. R. Zirnbauer, J. Phys. A **18**, 1093 (1985).
[13] D. Sherrington and S. Kirkpatrick, Phys. Rev. Lett. **35**, 1792 (1975).
[14] G. Parisi, Phys. Lett. **73A**, 203 (1979).
[15] M. Mézard, G. Parisi, N. Sourlas, G. Toulouse, and M. Virasoro, Phys. Rev. Lett. **52**, 1156 (1984).
[16] M. A. Stephanov, Phys. Rev. Lett. **76**, 4472 (1996).

[17] L. Schäfer and F. Wegner, Z. Phys. B Condens. Matter **38**, 113 (1980).
[18] A. Kamenev and M. Mézard, J. Phys. A **32**, 4373 (1999).
[19] K. Okuyama, Phys. Lett. B **803**, 135280 (2020).
[20] A. Altland and J. Sonner, SciPost Phys. **11**, 034 (2021).
[21] S. M. Nishigaki and A. Kamenev, J. Phys. A **35**, 4571 (2002).
[22] R. Jackiw, Nucl. Phys. **B252**, 343 (1985).
[23] C. Teitelboim, Phys. Lett. **126B**, 41 (1983).
[24] P. Gao, D. L. Jafferis, and A. Wall, J. High Energy Phys. 12 (2017) 151.
[25] P. Saad, S. H. Shenker, and D. Stanford, arXiv:1806.06840.
[26] P. Saad, arXiv:1910.10311.
[27] A. Almheiri, N. Engelhardt, D. Marolf, and H. Maxfield, J. High Energy Phys. 12 (2019) 063.
[28] N. Engelhardt, S. Fischetti, and A. Maloney, Phys. Rev. D **103**, 046021 (2021).
[29] G. Penington, S. H. Shenker, D. Stanford, and Z. Yang, arXiv:1911.11977.
[30] A. Almheiri, T. Hartman, J. Maldacena, E. Shaghoulian, and A. Tajdini, J. High Energy Phys. 05 (2020) 013.
[31] M. V. Raamsdonk, J. High Energy Phys. 12 (2021) 156.
[32] A. M. García-García and V. Godet, Phys. Rev. D **103**, 046014 (2021).
[33] A. M. García-García and S. Zacarías, Phys. Rev. Research **2**, 043310 (2020).
[34] S. B. Giddings and G. J. Turiaci, J. High Energy Phys. 09 (2020) 194.
[35] P. Saad, S. H. Shenker, and D. Stanford, arXiv:1903.11115.
[36] C. V. Johnson and F. Rosso, J. High Energy Phys. 04 (2021) 030.
[37] C. V. Johnson, arXiv:2008.13120.
[38] C. V. Johnson, arXiv:2104.02733.
[39] C. V. Johnson, Phys. Rev. D **103**, 046013 (2021).
[40] K. Okuyama and K. Sakai, J. High Energy Phys. 08 (2020) 126.
[41] K. Okuyama, J. High Energy Phys. 12 (2020) 080.
[42] K. Okuyama, J. High Energy Phys. 03 (2021) 073.
[43] J. Maldacena and X.-L. Qi, arXiv:1804.00491.
[44] A. Almheiri, T. Hartman, J. Maldacena, E. Shaghoulian, and A. Tajdini, Rev. Mod. Phys. **93**, 035002 (2021).
[45] D. N. Page, Phys. Rev. Lett. **71**, 3743 (1993).
[46] I. Aref'eva, M. Khramtsov, M. Tikhanovskaya, and I. Volovich, J. High Energy Phys. 07 (2019) 113.
[47] H. Wang, D. Bagrets, A. L. Chudnovskiy, and A. Kamenev, J. High Energy Phys. 09 (2019) 057.
[48] J. French and S. Wong, Phys. Lett. **33B**, 449 (1970).
[49] O. Bohigas and J. Flores, Phys. Lett. **34B**, 261 (1971).
[50] S. Sachdev and J. Ye, Phys. Rev. Lett. **70**, 3339 (1993).
[51] A. Kitaev, A simple model of quantum holography, KITP strings seminar and Entanglement 2015 program, 12 February, 7 April, and 27 May 2015, http://online.kitp.ucsb.edu/online/entangled15/.
[52] Y.-Z. You, A. W. W. Ludwig, and C. Xu, Phys. Rev. B **95**, 115150 (2017).
[53] A. M. García-García and J. J. M. Verbaarschot, Phys. Rev. D **94**, 126010 (2016).
[54] J. S. Cotler, G. Gur-Ari, M. Hanada, J. Polchinski, P. Saad, S. H. Shenker, D. Stanford, A. Streicher, and M. Tezuka, J. High Energy Phys. 05 (2017) 118; 09 (2018) 002(E).
[55] J. Ginibre, J. Math. Phys. (N.Y.) **6**, 440 (1965).
[56] Y. V. Fyodorov, B. A. Khoruzhenko, and H.-J. Sommers, Phys. Rev. Lett. **79**, 557 (1997).
[57] A. M. García-García, T. Nosaka, D. Rosa, and J. J. M. Verbaarschot, Phys. Rev. D **100**, 026002 (2019).
[58] J. Maldacena and A. Milekhin, J. High Energy Phys. 04 (2021) 258.
[59] A. M. García-García, Y. Jia, D. Rosa, and J. J. M. Verbaarschot (to be published).
[60] S. Plugge, E. Lantagne-Hurtubise, and M. Franz, Phys. Rev. Lett. **124**, 221601 (2020).
[61] O. Bohigas, M. J. Giannoni, and C. Schmit, Phys. Rev. Lett. **52**, 1 (1984).