Editors' Suggestion

Unexpected Upper Critical Dimension for Spin Glass Models in a Field Predicted by the Loop Expansion around the Bethe Solution at Zero Temperature

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The spin-glass transition in a field in finite dimension is analyzed directly at zero temperature using a perturbative loop expansion around the Bethe lattice solution. The loop expansion is generated by the M-layer construction whose first diagrams are evaluated numerically and analytically. The generalized Ginzburg criterion reveals that the upper critical dimension below which mean-field theory fails is $D_U \geq 8$, at variance with the classical result $D_U = 6$ yielded by finite-temperature replica field theory. Our expansion around the Bethe lattice has two crucial differences with respect to the classical one. The finite connectivity z of the lattice is directly included from the beginning in the Bethe lattice, while in the classical computation the finite connectivity is obtained through an expansion in $1/z$. Moreover, if one is interested in the zero temperature ($T = 0$) transition, one can directly expand around the $T = 0$ Bethe transition. The expansion directly at $T = 0$ is not possible in the classical framework because the fully connected spin glass does not have a transition at $T = 0$, being in the broken phase for any value of the external field.

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Spin glasses (SG) are the prototype of disordered models. The fully connected (FC) mean-field (MF) version, introduced by Sherrington and Kirkpatrick (SK) in [1], was solved forty years ago [2]. The SK model in a field h undergoes a phase transition from a paramagnetic to a SG phase along the de Almeida-Thouless (dAT) line $h_c(T)$ [3], that diverges for $T \to 0$. At $T = 0$ the SK model is in the SG phase, no matter how strong the external field is.

The solution to the SK model requires the introduction of replicas [2]. To identify the dAT line one can compute the fluctuations around the paramagnetic solution, via the study of the spectrum of the Hessian of the replicated free energy [3]. One can identify three sectors of Hessian eigenvectors, that are called replicon, longitudinal, and anomalous [4,5]. On the dAT line, the replicon eigenvalue becomes critical and stays critical in the whole SG phase, which is thus marginally stable. Below the dAT line the replica symmetry is spontaneously broken and the SK model has an exponential number of pure states, organized in an ultrametric structure. This highly nontrivial solution has been proved to be rigorously exact [6,7].

Beyond MF, things are much less clear. In particular, it is not known whether the finite-dimensional model with external field has a transition to a SG phase. Numerical simulations suggest a positive answer for $D = 4$ [8], but for $D = 3$ the results are inconclusive due to huge finite-size effects and very large equilibration times [9,10]: at the state of the art, it is impossible to decide if a transition exists just based on numerical results.

Usually, in statistical mechanics, the finite-dimensional behavior of models can be deduced using the powerful method of renormalization group (RG) [11]. One can set up a field theory for the order parameter associated with the desired transition, constructing a Lagrangian that is the most general one compatible with the symmetries of the problem. The basic approximation gives the so-called Landau-Ginzburg (LG) theory. It corresponds to the assumption that there are no fluctuations in the field and it is exact for the MF FC model. The next step is to see how the fluctuations, associated with the short-range interactions, modify the MF picture. Performing this task perturbatively leads to a loop-expansion around the LG solution. Looking at when the one-loop correction becomes important, one identifies the upper critical dimension D_U at which the MF theory does not predict the correct critical behavior anymore: this is the so-called generalized Ginzburg criterium. At this point, a perturbative expansion around the MF solution can be constructed, with a small parameter $\epsilon = D_U - D$, to see how the MF transitions are modified at dimension D below D_U.

Unfortunately, this program cannot be carried out so simply for SG models in a field. The MF theory in the high-temperature phase and the first-order perturbative expansion around it were analyzed in different papers [12–17].

Let us stress that the Lagrangian is very complicated: three bare masses, associated with the three sectors, and eight cubic vertices involving the replica fields. Forty years of work were not enough to understand the fate of the SG transition in finite dimension. For $D > D_U^{\mathrm{FC}} = 6$, the MF FC Fixed Point (FP) is stable, however, its basin of attraction shrinks to zero approaching D_U^{FC} from above. The main problem is the absence of a perturbative stable FP below $D = 6$ [12,14]: this lack is not a proof of nonexistence of SG phase in low dimensions and many scenarios have been put forward. Some authors have tried to extract information from the perturbative analysis nonetheless [16,17], possibly including quartic interactions [18] that are known to have a nontrivial role [19]. It could also be possible that a nonperturbative FP exists [20]. Recently, the perturbative expansion was computed up to the second-order [21,22], finding a strong-coupling FP that could in principle be stable at any dimension, even above D_U^{FC}. This new FP is in a way "nonperturbative" as it cannot be reached continuously from the MF FC one just lowering the dimension. However, the perturbative analysis in the strong-coupling regime is uncontrolled: thus the existence and relevance of this new FP cannot be stated just with the methods of Refs. [21,22].

Alternatively, the use of real-space RG methods is the natural choice if we are looking for nonperturbative FP in finite dimensions. The ensemble RG (ERG)[23] and the Migdal-Kadanoff (MK) RG [24] were applied to the SG in a field: for high enough dimensions ($D \gtrsim 8$) a critical FP at $T = 0$ was found, different from the MF FC one. We keep in mind that in the FC SK model there is no transition at $T = 0$ due to the diverging connectivity, an unrealistic feature that is not present in finite-dimensional models. However, the MK and the ERG flows are obtained after some crude approximations, as usually done when using nonperturbative RG, that are not exact. Thus they can provide useful indications, but cannot offer a definite answer to the problem.

Recently, a new loop expansion around the MF Bethe solution has been proposed in [25]. SG models in a field can be solved on the Bethe lattice (BL) and the finite connectivity allows for local fluctuations of the order parameter. This is an important feature shared with finite-dimensional systems. The loop expansion around the Bethe solution is obtained via the M-layer construction [25]. One introduces M copies of the original finite-dimensional lattice and generates a new lattice through a local random rewiring of the links. For large M the resulting M-layer lattice looks locally like a BL (and thus all observables tend to their MF BL values with small $1/M$ corrections), while at large distances the lattice retains its finite-dimensional character. This has important consequences for critical behavior: close to the MF critical point the system displays MF critical behavior until the correlation length reaches a size where the finite-dimensional nature of the model is dominant and the correct non-MF exponents are observed due to universality. The $1/M$ expansion (for $M = 1$ one recovers the original model) takes the form of a diagrammatic loops expansion with appropriate rules [25] and it is very useful to study critical phenomena. Similarly to field-theoretical loops expansion, one can apply the Ginzburg criterion and identify the upper critical dimension D_U where the corrections alter the MF behavior. For $D < D_U$ the expansion can then be used to obtain the critical exponents through standard RG treatments.

The expansion around the BL solution has the same advantages as standard field-theoretical loop expansions, but has a larger range of applicability, as it can be used for any problem that displays a continuous phase transition on the BL. Moreover, while in the classical expansion finite connectivity z is obtained as a result of a $1/z$ expansion around infinite connectivity, in the expansion around Bethe lattice z is finite and fixed from the beginning, introducing fewer artifacts. Recent applications of the BL expansion include the random field Ising model (RFIM) at zero temperature [26], the bootstrap percolation [27], and the glass crossover [28]. It has also been applied to the SG in a field in the limit of high connectivity for $T > 0$ [29], showing that in such a limit the expansion is completely equivalent to the standard expansion around the MF FC solution [12,14]. This is in agreement with the fact discussed in Ref. [25] that the $1/M$ expansion and the standard field theoretical expansion are completely equivalent if the physics of the model on the BL is like the one on the FC lattice.

In this Letter, we study the M-layer BL expansion of the SG in a field directly at $T = 0$ from both the paramagnetic and the SG phase. We are particularly interested in doing the computation directly at $T = 0$ because we know that, in this particular situation, degeneracy of eigenvalues can lead to different physics with respect to finite temperature, as happens, for example, for the RFIM. However, the classical expansion cannot be used directly at $T = 0$, because there is no transition in the SK model at $T = 0$: the system is in the broken phase for any value of the field. Things are different in the Bethe lattice, for which the dAT line ends at a finite critical field h_c at $T = 0$. Direct expansion around this point is thus possible. We show that finite connectivity and zero temperature lead to a critical behavior different from the one of the replicated field theory expansion at finite temperature. In particular, the generalized Ginzburg criterion leads to an upper critical dimension $D_U \geq 8$.

To be concrete, we consider the model Hamiltonian

$$H = -\sum_{(ij)\in E} J_{ij}\sigma_i\sigma_j - h\sum_i \sigma_i, \quad (1)$$

where the spins take the values $\sigma_i = \pm 1$, h is a constant external field [30], and the quenched couplings J_{ij} have a

Gaussian distribution with $\bar{J}=0$, $\overline{J^2}=(1/z-1)$, z being the (fixed) connectivity of the model. The first sum is over the set of edges E of a D-dimensional lattice.

Approaching the transition from the paramagnetic side the order parameter is zero and we analyze, as usual, the behavior of spin correlations. Following Ref. [25], a generic correlation or response function $G(x)$ between two points at distance x on the original lattice is given at leading order in $1/M$ by

$$G(x)=\frac{1}{M}\sum_{L=1}^{\infty}\mathcal{N}(x,L)G^{\mathrm{BL}}(L), \tag{2}$$

where $\mathcal{N}(x,L)$ is the number of nonbacktracking paths of length L connecting the two points at distance x on the original lattice $(M=1)$ and $G^{\mathrm{BL}}(L)$ is the analyzed correlation function between two spins at distance L on a BL with connectivity $z=2D$. While $\mathcal{N}(x,L)$ is known [25]

$$\mathcal{N}(x,L)\propto(2D-1)^L\exp\left[-x^2/(4L)\right]L^{-D/2}, \tag{3}$$

the crucial model-dependent quantity to be computed is $G^{\mathrm{BL}}(L)$. Working at $T=0$ it is worth focusing on the response function R_{ij} defined for the spin-glass model via the following procedure: being $\boldsymbol{\sigma}^\star$ the ground state (GS) configuration; compute the new GS under the constraint $\sigma_i=-\sigma_i^\star$; if also σ_j flips, then $R_{ij}=1$, otherwise $R_{ij}=0$. One can show [31] that the average response function on the BL can be computed exactly by applying L times an integral operator. Consequently, its behavior at large L is given by

$$R^{\mathrm{BL}}(L)\propto\lambda^L, \tag{4}$$

where λ is the largest eigenvalue of the integral operator [more details on λ can be found in Supplemental Material (SM) [31]]. It goes to $\lambda_c=(1/2D-1)$ at the critical point of the BL, such that the total response diverges and the paramagnetic solution is no longer stable [35]. Inserting Eqs. (4) and (3) into Eq. (2), we obtain for the Fourier transform of the response function in the small momentum region:

$$R(p)\propto\frac{1}{M}\sum_{L=1,\infty}[\lambda(2D-1)]^L\exp(-Lp^2)$$
$$\simeq\frac{1}{M}\int_0^\infty dL\exp\left[-L(p^2+\tau)\right]=\frac{1}{M}\frac{1}{p^2+\tau}, \tag{5}$$

with $\tau\equiv-\log[\lambda(2D-1)]$. Note that $\tau\to0$ when $\lambda\to\lambda_c$: at leading order the response has the form of the bare propagator in a field theory and becomes critical at the BL critical point.

Let us now look at the $1/M^2$ correction to the bare propagator. According to Ref. [25], this is given by the sum of the contributions coming from all the paths that connect the two points on the original lattice containing just one topological loop. The contribution of a specific topological diagram in Fourier space is

$$\tilde{G}_{\mathrm{loop}}(p)=\frac{1}{M^2}\sum_{\vec{L}}\mathcal{N}(p,\vec{L})G^{\mathrm{BL}}_{\mathrm{loop}}(\vec{L}), \tag{6}$$

where $\vec{L}$ is a vector containing the lengths of each line in the topological diagram and the factor $\mathcal{N}(p,\vec{L})$ accounts for the number of such topological diagrams on the original regular lattice with $M=1$. The term $G^{\mathrm{BL}}_{\mathrm{loop}}(\vec{L})$ is again the only term depending on the model: it is the so-called line-connected value [25] that the observable takes on a BL in which the analyzed topological loop has been manually inserted. The term "line connected" means that one should add the value of the observable evaluated on each of the subgraphs that are obtained from the original structure by sequentially removing its lines times a factor -1 for each line removed.

Let us point out two crucial differences between this expansion and the standard expansion around LG theory: (i) the latter has just cubic vertices, while in the BL expansion vertices of all degrees can be present; (ii) the diagrams of the BL expansion have a clear physical meaning while the Feynman diagrams of the standard expansion are just a smart way to compute the desired corrections.

At one loop we consider the two diagrams shown in Fig. 1. The left one has a quartic vertex, for this reason it is not included in the standard cubic theory. We compute $G^{\mathrm{BL}}_{\mathrm{loop}}(\vec{L})$ on this diagram with the same tools as for the 0-loop term (all the details in the SM). The resulting contribution to the response function coming from this quartic loop is $R^{\mathrm{BL}}_{4-\mathrm{loop}}(\vec{L})\propto L_A\lambda^{\Sigma(\vec{L})}$, where $\Sigma(\vec{L})$ is the sum of all L's, i.e., $\Sigma(\vec{L})=L_A+L_I+L_O$ in this diagram,

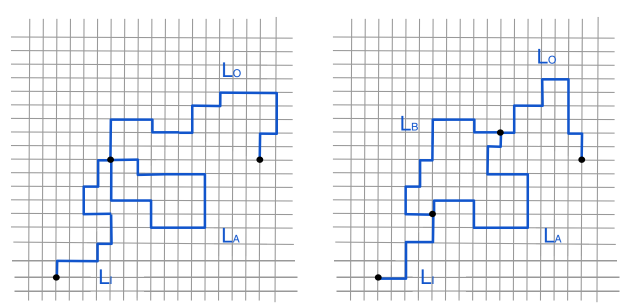

FIG. 1. One loop topological diagrams relevant for the first order correction around the BL: the "quartic loop" on the left has a vertex with four lines, while the "cubic loop" on the right has only vertices with three lines.

and λ is the same eigenvalue on the BL as in the previous discussion. The cubic loop (on the right in Fig. 1) has cubic vertices and is already present in the LG theory. Its behavior should be analyzed when L_A and L_B are large, because we checked that when one of the two internal legs is short, the diagram reduces to the quartic loop. For large L_A and L_B, we obtain $R^{\mathrm{BL}}_{3-\mathrm{loop}}(\vec{L}) \propto (L_A L_B / L_A + L_B)\lambda^{\Sigma(\vec{L})}$, with $\Sigma(\vec{L}) = L_A + L_B + L_I + L_O$.

The term $\mathcal{N}(p, \vec{L})$ has already been computed [26] and it reads, respectively, for the quartic and cubic loops

$$\mathcal{N}(p, \vec{L}) \propto \frac{(2D-1)^{\Sigma(\vec{L})}}{L_A{}^{D/2}} e^{-(L_I+L_O)p^2}, \tag{7}$$

$$\mathcal{N}(p, \vec{L}) \propto \frac{(2D-1)^{\Sigma(\vec{L})}}{(L_A+L_B)^{D/2}} e^{-(L_I+L_O+\frac{L_A L_B}{L_A+L_B})p^2}. \tag{8}$$

Inserting the above expression and $R^{\mathrm{BL}}_{3-\mathrm{loop}}(\vec{L})$ in Eq. (6), we obtain the correction to the response given by the cubic loop. In order to apply the Ginzburg criterion, it is more convenient to consider the inverse susceptibility

$$[MR(p)]^{-1} = \tau + p^2 + \frac{c}{M}\sum_{L_A,L_B} \frac{L_A L_B}{(L_A+L_B)^{D/2+1}} e^{-L_A\tau - L_B\tau - \frac{L_A L_B}{L_A+L_B}p^2},$$

that can be rewritten as

$$[MR(p)]^{-1} = A(\tau - \tau_c) + Bp^2 + O(p^4), \quad \text{with}$$

$$\tau_c = \frac{c}{M}\sum_{L_A,L_B} \frac{L_A L_B}{(L_A+L_B)^{D/2+1}}, \tag{9}$$

$$A = 1 - \frac{c}{M}\sum_{L_A,L_B} \frac{L_A L_B}{(L_A+L_B)^{D/2}}, \tag{10}$$

$$B = 1 - \frac{c}{M}\sum_{L_A,L_B} \frac{L_A^2 L_B^2}{(L_A+L_B)^{D/2+2}}. \tag{11}$$

We see that for large but finite M, the M-layer lattice has the same critical behavior of the BL ($M = \infty$), with small $O(1/M)$ shifts of the critical temperature and of the constants A and B. However, the above sums over L_A and L_B are divergent, respectively, for $D \le 6$, $D \le 8$ and $D \le 8$ and thus the Ginzburg criterion tells us that the critical exponents cannot be those of the Gaussian theory below $D = 8$. The same argument applied to the quartic loop would give a critical dimension equal to 6 (the diagram indeed appears in the computation of the connected correlation of the RFIM [26]) and allows to neglect the quartic loop with respect to the cubic one. We also checked that the generalized Ginzburg criterion coming from the replica symmetry breaking (RSB) phase predicts an upper critical dimension $D_U \ge 8$, in perfect agreement with the computation in the symmetric phase [37].

To go below the upper critical dimension we rescale lengths as $L = x/\tau$ and momenta as $p^2 = k^2\tau$, obtaining

$$[MR(p)]^{-1}/\tau = 1 + k^2 + \frac{c\tau^{D/2-4}}{M} \times \int_{\tau/\Lambda}^{\infty} dx_A \int_{\tau/\Lambda}^{\infty} dx_B \frac{x_A x_B e^{-x_A - x_B - \frac{x_A x_B}{x_A + x_B}k^2}}{(x_A+x_B)^{D/2+1}}. \tag{12}$$

The above expression shows that loop corrections are not negligible for $D < 8$ when $\tau \to 0$. Indeed, for $D < 8$ the integral would be divergent at short distances if not for the lattice cutoff Λ. One should check if, by standard mass, field, and coupling constant renormalization, the above 1-loop diagrams and higher-order diagrams as well can be made finite in the limit $\Lambda \to \infty$. Then the critical exponents can be computed by standard methods [38–40] provided an $O(\epsilon)$ nontrivial FP of the β function can be identified (at variance with the $T > 0$ case [12]): this program is currently under way. An interesting question is if this putative zero-temperature FP describes also the $T > 0$ physics, i.e., if the temperature is an irrelevant operator in the Wilson RG sense. We already mentioned that the expansion around the BL was applied to the SG in a field for $T > 0$ and in the limit of large z in Ref. [29]. Even if we take the limit $T \to 0$ of that expansion, the 1-loop correction results to be of the standard form (the detailed computation is in the SM). Finite connectivity is thus a crucial ingredient in the computation, and the limits $z \to \infty$ and $T \to 0$ cannot be exchanged. This is a clear indication that for SG models the expansion around the FC model cannot describe the behavior of finite-dimensional systems.

We emphasize that the FP we have found in this work by expanding around the BL is different from the finite temperature MF FC one even for $D > 8$. Indeed when $T > 0$ one can demonstrate that the critical behavior of all the possible correlation functions is the same (mainly because they all receive a critical contribution by the only critical eigenvalue, the replicon [29,41]). However, if the relevant FP is a $T = 0$ one, different correlation functions could decay differently (this effect is linked to the degeneracy of the three eigenvalues that become all critical when $T \to 0$), so one should look at them all. This is what happens in the RFIM, whose physics is governed by a $T = 0$ FP and whose correlation function associated with disorder fluctuations decays more slowly than the one associated with thermal fluctuations [41]. The same behavior is predicted by the MK RG of Ref. [24] for the SG in a field. We leave the analysis of the disorder correlation function to future work.

We just looked to the first order correction in the BL expansion. Going beyond this computation is really hard:

the second-order terms are already much involved in the standard expansion (see Ref. [21]), in the BL expansion one should consider in addition also diagrams with quartic vertices. A simple evaluation of the diagrams with power counting method is not possible because exact cancellation could happen and a quantitative computation is needed. The possibility that two-loop diagrams (or even higher order diagrams) diverge at a dimension $D > 8$ cannot be excluded, even if it would be quite unexpected, having never been observed in any known model. For this reason the best that we could say is that $D_U \geq 8$. The identification of simple rules for the computation of Feynman diagrams is one of our planned next steps.

A final remark on the value $D_U = 8$: in Ref. [24] the upper critical dimension was found to be $D \simeq 8$ with the MK RG method, while for $D < 8$ no stable SG phase was found. It is thus important to numerically address the problem of the identification of D_U. Unfortunately, numerical simulations cannot be performed directly on hypercubic lattices of such high D. For this reason the perfect candidates are the one-dimensional long-range (LR) models. There exist two versions of these. (i) The first is a fully connected version, where the variance of the couplings between any two spins decays with their mutual distance r as a power law $r^{-\sigma}$ [42–46]. When $\sigma \to 0$, one recovers the MF SK model. (ii) The second is a diluted version, introduced to reduce the simulation time, where all couplings are $O(1)$ and present with a probability decaying as $r^{-\sigma}$ [47–50]. When $\sigma \to 0$, one recovers the BL model.

These models are proxies for short range (SR) models in higher dimensions: in both cases changing σ is equivalent to change the dimension d of the corresponding SR model. Relations that link σ and d have been studied in detail [51,52]. The numerical investigation of LR models was focused on the existence of a transition for the SG in external field below D_U, for effective SR dimensions $d \simeq 3, 4, 5$ [48,49,51], while only data in $d \simeq 10, 20$ were collected in the assumed MF region [48,49,53]. One should then check which is D_U with and without field: Is there a signature in the LR models that D_U changes from 6 to 8 when an external field is added? One could look at the critical exponent ν as a function of d that should have a kink exactly at D_U in LR models [23]. Moreover, one could look at which kind of finite size scaling (MF or non-MF) leads to a better collapse of numerical data at different sizes, depending on d [53]. We expect that D_U is different for the FC and the diluted version of the LR models. In the past, the given explanation for their equivalence was based on standard field theoretical analysis [42]. However, if the BL expansion gives different prediction with respect to the FC standard expansion, FC and the diluted version of LR models should display differences, because of their different limits when $\sigma \to 0$.

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