Late Time Correlations in Hydrodynamics: Beyond Constitutive Relations

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We investigate the effects of nonlinear stochastic interactions on hydrodynamic response functions. The interactions are parametrized by "stochastic transport coefficients" that are invisible in the classical constitutive relations, but nonetheless affect the late time hydrodynamic correlations. We present a classification scheme for such coefficients that applies beyond the naive stochastic hydrodynamics. Our results indicate that conventional transport coefficients do not provide a universal characterization of long-distance late time behavior of nonequilibrium thermal systems.

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Hydrodynamics is often referred to as the universal low-energy effective description of many-body systems near thermal equilibrium. It is argued that if one waits long enough for all the high-energy "fast" modes to thermalize, the spectrum can effectively be captured by the remaining "slow" modes associated with conserved quantities (such as energy, momentum, and particle number). A hydrodynamic system is characterized by its conservation equations, expressed in terms of conserved densities and their derivatives, known as constitutive relations, with transport coefficients such as viscosities and conductivities entering as free parameters.

It is known that this classical picture of hydrodynamics is incomplete. Hydrodynamic equations can be used to obtain the physically observable retarded correlation functions; see [1]. However, these results can potentially be contaminated by interactions between the slow hydrodynamic modes and a background of fast modes [2,3]. A more complete picture is offered by the formalism of stochastic hydrodynamics, wherein the collective excitations of fast modes are modeled by random small-scale noise [2–6]. The short-ranged stochastic interactions are fine-tuned to reproduce the classical hydrodynamic results at tree level. However, consistently including fluctuation corrections, one finds that, for instance, the two-point correlation function of fluid velocity has long time tails that are not predicted by classical hydrodynamics [7].

This formalism, however, is not exhaustive, as the requirement to reproduce classical hydrodynamics does not uniquely fix the structure of stochastic interactions. Importantly, assuming these random interactions to be Gaussian, as is typically done, still leaves room for ambiguities. Physically, these ambiguities correspond to some high-energy "fast physics," which has been ignored at the classical level, leaking into the low-energy correlation functions via interactions. This would mean that, contrary to what is typically believed, the hydrodynamic transport coefficients do *not* universally characterize the low-energy spectrum of thermal systems.

This stochastic contamination can also be motivated from the fluctuation-dissipation theorems (FDTs) in thermal field theory. All the information in two- and three-point thermal correlation functions in a system can be captured by the respective retarded functions. However, for four- or higher-point correlations, retarded functions are no longer enough [8]. Classical hydrodynamics is only sensitive to retarded correlations of conserved densities and is, consequently, blind to any information that might be contained in nonretarded higher-point correlation functions. These can nonetheless affect the classically observable retarded functions through stochastic interactions. The point of this Letter is to make the above qualitative arguments precise and to explore the limitations of hydrodynamics in describing macroscopic real-time correlations.

To probe these questions effectively, one needs a systematic prescription to include stochastic noise into the hydrodynamic framework. While classically, hydrodynamics is posed as a system of conservation equations, there now exists a complete effective field theory (EFT) framework for hydrodynamics [9–12]; see [13] for a review. The framework ensures that the FDT requirements are nonlinearly realized in the presence of interactions and can be used to investigate stochastic signatures in hydrodynamic response functions.

The EFT methods allow one to systematically evaluate the hydrodynamic correlation functions at low frequency and wave vector. As in any effective theory, the information about the microscopics is encoded in a few parameters, in this case, thermodynamic and transport coefficients.

In particular, there is no need to assume the validity of Boltzmann equation or to make detailed assumptions about the microscopic dynamics. The main assumptions of the EFT are the existence of thermal equilibrium and the finiteness of the correlation length at nonzero temperature.

We argue that the effective action for hydrodynamics admits a set of new stochastic transport coefficients pertaining to the structure of stochastic interactions, which are left unfixed by the classical constitutive relations. These coefficients, nonetheless, show up as corrections to the hydrodynamic response functions. Notably, these new effects are typically suppressed by a factor of k^{d+2}, where d is the number of spatial dimensions, compared to the already known stochastic corrections due to classical transport coefficients. We present a classification scheme for stochastic coefficients based on the effective field theory framework that generically applies beyond naive stochastic hydrodynamics. This suggests that nonuniversal (in the classical sense) stochastic interactions discussed in this Letter are of physical relevance for virtually any macroscopic system to which the notion of local thermodynamic equilibrium is relevant.

Stochastic interactions in simple diffusion.—For a concrete realization of these ideas, consider single conserved density $J^t = n(\mu)$, where μ is the associated chemical potential. Classical evolution of n is governed by the diffusion equation $\partial_\mu J^\mu = 0$, with $J^i = -\sigma(\mu)\partial^i\mu$ at leading order in derivatives, with diffusion constant given by $D = \sigma(\mu)/n'(\mu)$. The conductivity $\sigma(\mu)$ is a non-negative classical transport coefficient.

The EFT for diffusion is described by a phase field φ_r and an associated stochastic noise field φ_a [10]; see also [14]. We introduce background gauge fields $\phi_{r,a} = (A_{r,a\mu})$ coupled to the noise and physical current operators $\mathcal{O}_{a,r} = (J^\mu_{a,r})$, respectively. The effective action S of the theory is constructed out of the background gauge-invariant building blocks $\Phi_{r,a} = (B_{r,a\mu} = A_{r,a\mu} + \partial_\mu\varphi_{r,a})$. Connected correlation functions of $\mathcal{O}_{r,a}$ are computed via a path integral over the dynamical fields $\psi = (\varphi_{r,a})$, i.e., [15],

$$G_{\alpha\ldots} = i^{n_a}\left(\frac{-i\delta}{\delta\phi_{\bar\alpha}}\ldots\right)W, \qquad W = \ln\int \mathcal{D}\psi\exp(iS), \quad (1)$$

where $\alpha = r, a$ and $\bar\alpha = a, r$, while n_a is the number of a-type fields on the left. $G_{ra,\ldots,a}$ computes the retarded function, while $G_{rr,\ldots,r}$ computes the symmetric one, with all the remaining combinations in between [16]. The theory is required to satisfy a set of Schwinger-Keldysh (SK) constraints

$$S[\Phi_r,\Phi_a{=}0;\beta]{=}0, \quad S[\Phi_r,-\Phi_a;\beta]{=}-S^*[\Phi_r,\Phi_a;\beta], \quad (2a)$$

$$\mathrm{Im}\, S[\Phi_r,\Phi_a;\beta] \geq 0, \quad (2b)$$

$$S[\Phi_r,\Phi_a;\beta] = S[\Theta\Phi_r,\Theta\Phi_a + i\Theta £_\beta\Phi_r;\Theta\beta]. \quad (2c)$$

Here $£_\beta$ denotes a Lie derivative along the thermal vector $\beta^\mu = 1/T_0\delta^\mu_t$, with T_0 being the constant global temperature, and $\Theta = PT$ represents a discrete spacetime parity transformation. In particular, (2b) implements the inequality in the second law of thermodynamics, while the Kubo-Martin-Schwinger (KMS) symmetry (2c) implements FDTs [17]. The theory also has a local phase shift symmetry

$$\varphi_r(x) \to \varphi_r(x) - \lambda(x), \quad \text{such that } \beta^\mu\partial_\mu\lambda(x) = 0. \quad (3)$$

Given these requirements, at leading order in derivatives, we find the effective Lagrangian [10]

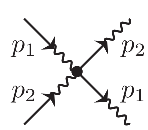
$$\mathcal{L}_1 = n(\mu)B_{at} + iT_0\sigma(\mu)B_{ai}(B^i_a + i£_\beta B^i_r), \quad (4a)$$

where $\mu = B_{rt} = \partial_t\varphi_r + A_{rt}$. Given that σ is non-negative, conditions (2a) and (2b) are trivially satisfied. The second term maps to itself under (2c), while the first term merely generates an additional total derivative term $\partial_t p(\mu)$ such that $p'(\mu) = n(\mu)$. The classical diffusion equation can be recovered upon varying the action with respect to φ_a, restricting to configurations with $\varphi_a = 0$ and setting the background to $A_{r\mu} = \mu_0\delta^t_\mu$, $A_{a\mu} = 0$.

While the action (4a) is sufficient to reproduce classical evolution, the formalism does allow for extra terms that are at least quadratic in noise fields and hence leave the classical dynamics untouched. For instance,

$$\mathcal{L}_2 = iT_0^2\vartheta_1(\mu)B_{ai}B_{aj}(£_\beta B^i_r £_\beta B^j_r - \delta^{ij}£_\beta B^k_r £_\beta B_{rk}) + iT_0^2\vartheta_2(\mu)B^i_a B_{ai}(B^j_a + i£_\beta B^j_r)(B_{aj} + i£_\beta B_{rj}), \quad (4b)$$

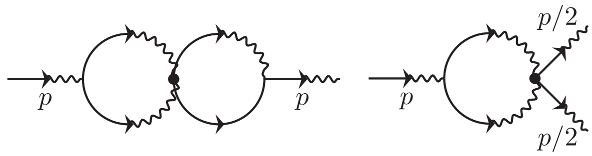
where $\vartheta_{1,2}(\mu)$ are arbitrary stochastic coefficients. These are the most generic such terms that appear at the lowest order in derivatives. Each term here involves at least four fields, so the stochastic coefficients only contribute to four- and higher-point non-fully-retarded correlation function at tree level. For example, denoting "r"-type fields by solid and a type by wavy lines, the partially retarded function G_{rraa} of n receives a tree-level stochastic contribution due to interactions in (4b) (see Supplemental Material [18]),



$$G_{rraa} = \ldots + \frac{2\omega^2 k^4}{(\omega + iDk^2)^4}\times \left(2\vartheta_2\cos^2\theta - \vartheta_1\sin^2\theta\right). \quad (5a)$$

for $p_1 = (\omega, k, 0, 0)$ and $p_2 = (\omega, k\cos\theta, k\sin\theta, 0)$. Here $D = \sigma/\chi$ is the diffusion constant and $\chi = \partial n/\partial\mu$ is the susceptibility. Ellipsis denote further nonstochastic corrections due to terms in Eq. (4a). One can use the retarded functions G_{raaa}, G_{raa}, and G_{ra} to cancel these terms and obtain a Kubo formula for ϑ_1, ϑ_2 using (5a).

Although stochastic coefficients do not contribute to fully retarded correlation functions at tree level, they do show up in the loop corrections, such as



Here $p = (\omega, k, 0, 0)$. Stochastic vertices from (4b) are denoted in bold. We find that the retarded two-point function of n behaves in $k^2 \ll \omega/D$ limit as (see Supplemental Material [18])

$$\frac{\omega}{k^2}\mathrm{Im}G_{ra} = \chi D + \frac{\chi^2\lambda^2 T_0}{32\pi D^{3/2}}\omega^{1/2}k^2 + \cdots - \frac{\lambda^2 T_0(\frac{2}{3}\vartheta_1 + \vartheta_2)}{1024\pi^2 D^4}\omega^2 k^4 + \cdots. \quad (5b)$$

Here $\lambda = \partial D/\partial n$. The leading correction to G_{ra} dictated by the constitutive relations goes as $\omega^{1/2}k^2$ [14], while the leading correction due to stochastic coefficients goes as $\omega^2 k^4$. The retarded three-point function, on the other hand, receives a nonanalytic stochastic correction

$$-\frac{2\omega^2}{k^4}\mathrm{Re}G_{raa} = \chi^2 D\tilde{\lambda} + \cdots + \frac{\lambda(\frac{2}{3}\vartheta_1 + \vartheta_2)}{32\pi D^{5/2}}\omega^{5/2} + \cdots. \quad (5c)$$

Here $\tilde{\lambda} = \partial D/\partial n + D/\chi \partial\chi/\partial n$. The middle ellipsis in (5) denote subleading terms coming from (4a), while the final ellipsis denote terms higher order in k^2. Detailed calculations for finite k^2 are given in the Supplemental Material [18]. These results illustrate that the hydrodynamic correlation functions start to receive higher-derivative corrections that are *not* fixed by the constitutive relations.

We note that the ϑ_1 contribution to the effective action (4b) is quadratic in the noise field. Gaussian noise can be captured by the conventional stochastic model, wherein one introduces a random microscopic term r^i in the flux $J^i = -\sigma(\mu)\partial^i\mu + r^i$. Correlation functions are obtained by path integrating over r^i weighted by a Gaussian factor $\exp(-1/4\int d^4x r^i r^j \lambda_{ij}(\mu))$ [19]. Imposing FDT constrains the form of the coefficient λ_{ij} in terms of hydrodynamic transport coefficients. At leading order in derivatives, FDT uniquely fixes $\lambda_{ij} = \delta_{ij}/(T_0\sigma(\mu))$. However, this uniqueness is violated by higher-derivative corrections pertaining to stochastic coefficients, such as $\vartheta_{1,2}$, that are *not* fixed by FDTs. For example, the ϑ_1-term from (4b) appears as $\lambda_{ij} \sim -\vartheta_1(\mu)/(T_0\sigma(\mu))^2(\partial_i\mu\partial_j\mu - \delta_{ij}\partial_k\mu\partial^k\mu)$.

Stochastic interactions in hydrodynamics.—The discussion of stochastic interactions can be extended to full relativistic hydrodynamics. In addition to the phase pair φ_r, φ_a associated with density fluctuations, the theory also contains the Lagrangian coordinates $\sigma^{A=0,\ldots,3}$ of the fluid elements and respective noise X_a^μ as fundamental fields associated with energy-momentum fluctuations [10,20]. We take σ^0 to define the local rest frame associated with the global thermal state. The thermal vector β^μ is no longer a constant, but is given by $\beta^\mu = 1/T_0 \partial x^\mu(\sigma(x))/\partial\sigma^0(x)$. Introducing background fields $\phi_{r,a} = (g_{r,a\mu\nu}, A_{r,a\mu})$ coupled to noise and physical conserved operators $\mathcal{O}_{a,r} = (T^{\mu\nu}_{a,r}, J^\mu_{a,r})$, respectively, the correlation functions can be computed by (1), with the path integral over $\psi = (\varphi_{r,a}, \sigma^A, X^\mu_a)$. The building blocks for the respective effective action S, besides β^μ, are (see [13])

$$B_{r\mu} = A_{r\mu} + \partial_\mu\varphi_r, \qquad B_{a\mu} = A_{a\mu} + \partial_\mu\varphi_a + \pounds_{X_a}A_{r\mu},$$
$$G_{r\mu\nu} = g_{r\mu\nu}, \qquad G_{a\mu\nu} = g_{a\mu\nu} + \pounds_{X_a}g_{r\mu\nu}. \quad (6)$$

Denoting $\Phi_{r,a} = (G_{r,a\mu\nu}, B_{r,a\mu})$, the SK constraints and phase shift symmetry are still given by (2) and (3).

Expressing $S = \int d^4x\sqrt{-g_r}\mathcal{L}$, up to leading order in derivatives, the effective action for relativistic hydrodynamics satisfying these requirements is given as

$$\begin{aligned}\mathcal{L}_1 = {}& \frac{1}{2}(\epsilon u^\mu u^\nu + p\Delta^{\mu\nu})G_{a\mu\nu} + n u^\mu B_{a\mu} \\ & + \frac{iT}{4}\left[2\eta\Delta^{\mu\rho}\Delta^{\nu\sigma} + \left(\zeta - \frac{2}{d}\eta\right)\Delta^{\mu\nu}\Delta^{\rho\sigma}\right] \\ & \times G_{a\mu\nu}(G_{a\mu\nu} + i\pounds_\beta G_{r\mu\nu}) \\ & + iT\sigma\Delta^{\mu\nu}B_{a\mu}(B_{a\nu} + i\pounds_\beta B_{r\nu}),\end{aligned} \quad (7a)$$

where $\Delta^{\mu\nu} = g_r^{\mu\nu} + u^\mu u^\nu$. Velocity u^μ (with $u^\mu u_\mu = -1$), temperature T, and chemical potential μ are defined via $u^\mu/T = \beta^\mu$ and $\mu/T = \beta^\mu B_{r\mu}$. Energy density ϵ, pressure p, number density n, viscosities η and ζ, and conductivity σ are functions of T and μ. They satisfy $dp = sdT + nd\mu$ and $\epsilon + p = Ts + \mu n$ for entropy density s. Condition (2b) requires η, ζ, and σ to be non-negative. Classical conservation equations of hydrodynamics are obtained by varying the action with respect to X_a^μ, φ_a in a configuration with $X_a^\mu = \varphi_a = 0$ and setting the background to $g_{r\mu\nu} = \eta_{\mu\nu}, A_{r\mu} = \mu_0\delta^t_\mu, g_{a\mu\nu} = A_{a\mu} = 0$.

Similar to (4b), the full hydrodynamic action can also be modified with arbitrary stochastic terms based on the symmetries of the theory. For instance, we have

$$\begin{aligned}\mathcal{L}_2 = {}& iT^2\vartheta_1(\Delta^{\mu\rho}\Delta^{\nu\sigma} - \Delta^{\mu\nu}\Delta^{\rho\sigma})\pounds_\beta B_{r\rho}\pounds_\beta B_{r\sigma}B_{a\mu}B_{a\nu} \\ & + iT^2\vartheta_2\Delta^{\mu\nu}\Delta^{\rho\sigma}B_{a\mu}B_{a\nu}(B_{a\rho} + i\pounds_\beta B_{r\rho})(B_{a\sigma} + i\pounds_\beta B_{r\sigma}) \\ & + iT^2\vartheta_3(\Delta^{\mu\mu'}\Delta^{\nu\nu'}\Delta^{\rho\rho'}\Delta^{\sigma\sigma'} - \Delta^{\mu\rho}\Delta^{\nu\sigma}\Delta^{\mu'\rho'}\Delta^{\nu'\sigma'}) \\ & \times G_{a\mu\nu}G_{a\rho\sigma}\pounds_\beta G_{r\mu'\nu'}\pounds_\beta G_{r\rho'\sigma'} \\ & + iT^2\vartheta_4\Delta^{\mu\nu}\Delta^{\mu'\nu'}\Delta^{\rho\sigma}\Delta^{\rho'\sigma'}G_{a\mu\nu}G_{a\mu'\nu'} \\ & \times (G_{a\rho\sigma} + i\pounds_\beta G_{r\rho\sigma})(G_{a\rho'\sigma'} + i\pounds_\beta G_{r\rho'\sigma'}) + \cdots,\end{aligned} \quad (7b)$$

with ϑ_i being a few stochastic coefficients; we do not perform the exhaustive counting exercise here.

Contributions from stochastic interactions in (7b) to hydrodynamic response functions can be computed similar to (5). We note, however, that nonstochastic interactions in the simplified diffusion model only set in at one-derivative order as opposed to full nonlinear hydrodynamics where momentum or velocity fluctuations in (7a) lead to ideal-order interactions; see [19]. Since part of the derivative suppression of stochastic signatures in (5b) and (5b) arises from nonstochastic vertices, we expect this suppression to be relaxed in full hydrodynamics. Note, however, that their relative suppression compared to diagrams with only nonstochastic vertices will remain unchanged.

The stochastic coefficients ϑ_i also arise in the context of nonrelativistic (Galilean) hydrodynamics, in complete analogy with its relativistic counterpart. The quantitative details can be worked out along the lines of [21].

KMS blocks.—The effective Lagrangian for a generic thermal system can be organized into "KMS blocks." The nth KMS block $\mathcal{L}_n$ contains the most generic terms involving n number of a fields allowed by symmetries, plus specific terms with higher number of a fields required to satisfy KMS-FDT requirements. Classical dynamics of the system, and tree-level fully retarded correlation functions $G_{ra,\ldots,a}$, are completely characterized by $\mathcal{L}_1$. Higher KMS blocks $\mathcal{L}_{n>1}$ contain stochastic interactions that contribute to tree-level non-fully-retarded correlators $G_{r,\ldots,ra,\ldots,a}$ involving at least n instances of r-type operators.

This decomposition is not unique; we can always redefine a KMS block with terms from higher KMS blocks. Such ambiguity in $\mathcal{L}_1$ is precisely the nonuniversality of classical hydrodynamics.

Condition (2a) implies that the $\mathcal{L}$ can be arranged in a power series in Φ_a starting from the linear term. We start with a parametrization (see Supplemental Material [18])

$$\mathcal{L} = \mathcal{D}_1(\Phi_a) + i\sum_{n=1}^{\infty}\mathcal{D}_{2n}(\underbrace{\Phi_a,\ldots,}_{\times n}\underbrace{\Phi_a + i\pounds_\beta\Phi_r,\ldots}_{\times n}) + \sum_{n=1}^{\infty}\mathcal{D}_{2n+1}\left(\Phi_a + \frac{i}{2}\pounds_\beta\Phi_r,\underbrace{\Phi_a,\ldots,}_{\times n}\underbrace{\Phi_a + i\pounds_\beta\Phi_r,\ldots}_{\times n}\right). \tag{8}$$

Here $\mathcal{D}_m$ are a set of totally symmetric real multilinear differential operators, allowing m arguments, made out of Φ_r and β^μ. Here $\times n$ denotes n identical arguments. Note that only $\mathcal{D}_{1,2,3}$ from (8) can contribute to the classical constitutive relations. For instance, the diffusive Lagrangian (4) corresponds to the choice

$$\begin{aligned}
\mathcal{D}_1(X) &= n(\mu)X_t,\\
\mathcal{D}_2(X,Y) &= T_0\sigma(\mu)X_iY^i + T_0^2\left[\vartheta_1(\mu) + \frac{2}{3}\vartheta_2(\mu)\right]\\
&\quad\times(\pounds_\beta B_r^i\pounds_\beta B_r^j - \delta^{ij}\pounds_\beta B_r^k\pounds_\beta B_{rk})X_iY_j,\\
\mathcal{D}_4(W,X,Y,Z) &= T_0^2\vartheta_2(\mu)\delta^{(ij}\delta^{kl)}W_iX_jY_kZ_l,
\end{aligned} \tag{9}$$

for arbitrary vectors W_μ, X_μ, Y_μ, and Z_μ. Recall that $\mu = B_{rt}$ and $\Phi_{r,a} = (B_{r,a\mu})$. Requiring (8) to respect (2b) and (2c) (up to a total derivative), we find

$$\mathcal{D}_1(\pounds_\beta\Phi_r) = \nabla_\mu\mathcal{N}_0^\mu, \quad \mathcal{D}_m \text{ are } \Theta \text{ even}, \quad \mathcal{D}_2|_{\text{0-der}} \geq 0, \tag{10}$$

for some vector $\mathcal{N}_0^\mu$. Note that $\pounds_\beta\Phi_r$ is Θ odd, therefore the contribution of $\mathcal{D}_m$ to $\mathcal{L}$ is generically *not* Θ even.

Generically, $\mathcal{D}_n$ contain all structures allowed by symmetries at a given derivative order. We call the contribution of each such structure to $\mathcal{L}$ a KMS group, as it is independently invariant under the KMS transformation. The nth KMS block can be defined as the set of all KMS groups wherein the least nonzero power of Φ_a fields is n. We say "at least" because there can be groups in $\mathcal{D}_n$ that identically vanish (up to a total derivative) when one or more of their arguments are $\pounds_\beta\Phi_r$, e.g., $\vartheta_{1,2}$ contribution in $\mathcal{D}_2$ in (9), and are pushed to higher KMS blocks. To account for this subtlety, we can decompose $\mathcal{D}_n = \sum_{m=0}^{n}\mathcal{D}_{n,m}$, where $\mathcal{D}_{n,m}$ is the component of $\mathcal{D}_n$ with m of its arguments projected transverse to $\pounds_\beta\Phi_r$.

Plugging this decomposition into (8), we can work out the first KMS block explicitly as

$$\mathcal{L}_1 = \mathcal{D}_{1,1} + i\mathcal{D}_{2,1} + i\mathcal{D}_{2,0} + \mathcal{D}_{3,0}, \tag{11}$$

which completely characterizes classical hydrodynamics. The arguments of $\mathcal{D}_{n,m}$ here are the same as $\mathcal{D}_n$ in (8). The former two terms correspond to Θ-even and Θ-odd "adiabatic" transport, while the latter two correspond to Θ-odd and Θ-even "dissipative" transport. From our examples in (4b) and (7b), $\epsilon, p, n \in \mathcal{D}_{1,1}$ and $\eta, \zeta, \sigma \in \mathcal{D}_{2,0}$; other contributions show up at higher order in derivatives.

The first nontrivial stochastic corrections to classical hydrodynamics come from the second KMS block

$$\mathcal{L}_2 = i\mathcal{D}_{2,2} + \mathcal{D}_{3,1} + i\mathcal{D}_{4,0} + \mathcal{D}_{3,2} + i\mathcal{D}_{4,1} + \mathcal{D}_{5,0}. \tag{12}$$

At this point, we are unable to ascertain any physical distinction between various contributions. In our examples, $(\vartheta_1 + \frac{2}{3}\vartheta_2), \vartheta_3 \in \mathcal{D}_{2,2}$, while $\vartheta_2, \vartheta_4 \in \mathcal{D}_{4,0}$. The details are in the Supplemental Material [18]. Higher KMS blocks can be worked out in a similar manner.

To count derivative ordering in hydrodynamics, we use the canonical scheme from [10], where $\Phi_r \sim \mathcal{O}(\partial^0)$ and $\Phi_a, \pounds_\beta\Phi_r \sim \mathcal{O}(\partial^1)$. The contribution of $\mathcal{D}_{n,m}$ to $\mathcal{L}$, and to the hydrodynamic observables, is typically suppressed with

$\mathcal{O}(\partial^{n+m-1})$ compared to the ideal-order thermodynamic terms in $\mathcal{D}_{1,1}$. Consequently, effects of stochastic KMS blocks $\mathcal{L}_{n>1}$ are suppressed with $\mathcal{O}(\partial^{2n-1})$ compared to $\mathcal{L}_1$, in addition to any loop suppression. Therefore, the first nonuniversal stochastic signatures creep into hydrodynamics at third derivative order. To account for loop suppression, using dimensional analysis (see Supplemental Material [18]), one can argue that $\delta_{\mathcal{B}}\Phi_r, \Phi_a \sim k^{d/2+1}$ in momentum space. Consequently, statistical interactions terms in $\mathcal{L}_n$ are typically suppressed by $k^{(d+2)(n-1)}$ compared to kinetic terms in $\mathcal{L}_1$.

Outlook.—Hydrodynamics is an effective theory and an immensely successful one at that. However, like any effective theory, it has a limited scope of applicability. It posits that the low-energy dynamics of a many-body thermal system can be effectively captured by the long-range transport properties of its conserved charges. While it is certainly true to a leading approximation, short-range stochastic interactions must be included in the framework to consistently describe interactions between hydrodynamic modes. In this Letter, we investigated the sensitivity of hydrodynamics to the choice of stochastic interactions.

We used the EFT framework of hydrodynamics developed recently [10] and identified new stochastic transport coefficients characterizing short-range information. These coefficients do not enter the classical constitutive relations, but nonetheless affect retarded correlation functions in the hydrodynamic regime at subleading orders in derivatives through loop interactions. We explicitly derived the stochastic signatures in two- and three-point retarded functions for diffusive fluctuations in $(3+1)$ dimensions. In particular, we found the stochastic correction to three-point function to be nonanalytic in frequency at one-loop order. It is worth noting that these results are distinct from the usual "long time tails," as the effects we are describing are characterized by entirely new transport coefficients that are invisible in classical constitutive relations. Finally, we classified the general structure of stochastic interactions through KMS blocks. Classical physics is completely characterized by the first KMS block, while the higher KMS blocks characterize a plethora of stochastic coefficients.

We conclude that the sensitivity of hydrodynamic observables to short-range stochastic information signifies a breakdown of the celebrated universality of hydrodynamics in describing long-range correlations. More generally, these nonuniversal stochastic effects will arise as subleading corrections in any macroscopic system out of thermal equilibrium. It would be interesting to find physical systems where the signatures of stochastic coefficients are enhanced enough to overcome the loop suppression. As we already discussed in the Letter, part of this could be achieved by revisiting the computation in the presence of momentum modes in relativistic or nonrelativistic hydrodynamics. The stochastic signatures are also enhanced in lower spatial dimensions. Careful consideration of stochastic signatures can also serve as precision tests for the effective action formalism of hydrodynamics itself, that it indeed describes the late time physics of thermalizing systems. We plan to return to these questions in the future.

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