Emergent Kardar-Parisi-Zhang Phase in Quadratically Driven Condensates

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In bosonic gases at thermal equilibrium, an external quadratic drive can induce a Bose-Einstein condensation described by the Ising transition, as a consequence of the explicitly broken U(1) phase rotation symmetry down to $\mathbb{Z}_2$. However, in physical realizations such as exciton polaritons and nonlinear photonic lattices, thermal equilibrium is lost and the state is rather determined by a balance between losses and external drive. A fundamental question is then how nonequilibrium fluctuations affect this transition. Here, we show that in a two-dimensional driven-dissipative Bose system the Ising phase is suppressed and replaced by a nonequilibrium phase featuring Kardar-Parisi-Zhang (KPZ) physics. Its emergence is rooted in a U(1)-symmetry restoration mechanism enabled by the strong fluctuations in reduced dimensionality. Moreover, we show that the presence of the quadratic drive term enhances the visibility of the KPZ scaling, compared to two-dimensional U(1)-symmetric gases, where it has remained so far elusive.

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How the absence of thermal equilibrium affects the properties of matter is one of the fundamental questions of many-body physics, with far-reaching consequences in the engineering of novel materials, the development of quantum technologies, and the understanding of active and living matter. In nonequilibrium systems, the lack of detailed balance can radically modify the collective behaviours typical of equilibrium systems. Accordingly, novel phases can be expected, such as nonreciprocal (or chiral) phases in active matter [1,2], quantum optical platforms [3,4] and ultracold atoms [5], or dissipative time crystals in many-body quantum systems [6–9].

An intriguing aspect concerns the impact of nonequilibrium fluctuations in low spatial dimensions. At equilibrium, the Mermin-Wagner theorem forbids the spontaneous breaking of a continuous symmetry in spatial dimensions $d \leq 2$ for systems with short-ranged interactions. Out of equilibrium, the theorem does not hold: two-dimensional flocks [10] or driven quantum spin chains [11,12] can feature transitions to phases with long-range order. On the converse, the Berezinskii-Kosterlitz-Thouless (BKT) phase transition, expected for equilibrium Bose gases in two spatial dimensions, is erased in their driven-dissipative counterpart and replaced by a disordered phase featuring a Kardar-Parisi-Zhang (KPZ) scaling of the phase fluctuations [13]. Promising candidates to experimentally observe this scaling are exciton-polariton fluids in microcavities [14,15], although the length scales at which its signatures are expected are dramatically larger than the typical system sizes [16–19].

The fate of nonequilibrium systems with discrete symmetries is less explored. At equilibrium, they can exhibit order also in two dimensions. This is the case for the arguably most paradigmatic phase transition, namely, the Ising transition. Among its many incarnations, the Ising phase transition can be realized in bosonic gases in the presence of an externally imprinted pair creation term: in ultracold atoms, this can be induced by coupling to a molecular condensate [20] (see also Ref. [21] for a wire of fermionic atoms), by using a parametric down-conversion scheme in microcavities [22,23], or by Feshbach-like resonances in polariton biexcitons [24,25] or Rydberg polaritons [26]. At equilibrium, the bosons undergo a Bose-Einstein condensation (BEC) transition belonging to the Ising universality class [27,28]. In optical systems, the unavoidable presence of incoherent processes causes a departure from equilibrium. Still, recent numerical analyses showed that these driven-dissipative models can undergo a BEC transition characterized by either the quantum or classical Ising universality class [29–32].

In this Letter, we show that the absence of thermal equilibrium suppresses the Ising phase transition in a two-dimensional, driven-dissipative Bose gas, in favor of an emerging KPZ phase. Our two main results are summarized as follows. First, the long-wavelength description of the quadratically driven Bose gas is given by a driven sine-Gordon equation for the phase degree of freedom. In two spatial dimensions, this dynamics is dominated by the KPZ scaling at long wavelengths, resulting in the suppression of the BKT and Ising phases, present instead at equilibrium. Second, the presence of the quadratic drive reduces the scale at which the KPZ physics sets in, enhancing its visibility in finite-size systems. This holds promise for identifying this physics in two spatial dimensions, where experimental realizations remain so far elusive [33,34].

Microscopic model.—We consider a gas of quadratically driven and dissipative bosons, described by the master equation

$$\partial_t\hat{\rho} = -i[\hat{H},\hat{\rho}] + \int_{\mathbf{r}} \sum_n \left[\hat{L}_n \rho \hat{L}_n^\dagger - \frac{1}{2}\{\hat{\rho}, \hat{L}_n^\dagger \hat{L}_n\}\right], \quad (1)$$

with $\hat{\rho}$ the system's density matrix, $\hat{H}$ the Hamiltonian, and $\hat{L}_n = \hat{L}_n(\mathbf{r})$ Lindblad operators. The quadratic drive can be regarded as a process coherently creating or destroying two particles at a given position. The Hamiltonian is thus given by

$$\hat{H} = \int_{\mathbf{r}} \left[\frac{\nabla\hat{\psi}^\dagger\nabla\hat{\psi}}{2m} + \delta\hat{\psi}^\dagger\hat{\psi} + \frac{G}{2}(\hat{\psi}^2 + \hat{\psi}^{\dagger 2}) + \frac{U}{2}\hat{\psi}^{\dagger 2}\hat{\psi}^2\right], \quad (2)$$

with m the mass of the bosons, $\delta > 0$ the detuning between the bosonic and the drive frequency, and $U > 0$ the particle interaction. The quadratic drive has a strength G, and we can set $G > 0$ without loss of generality, by absorbing its phase into a redefinition of the fields. The presence of further incoherent processes, such as single particle losses and pump, as well as two-particle losses, is included via the Lindblad operators $\hat{L}_{1l} = \hat{\psi}$, $\hat{L}_{1p} = \hat{\psi}^\dagger$, and $\hat{L}_{2l} = \hat{\psi}^2$, respectively. The nature of the instabilities arising in the presence of both a strong incoherent pump [35,36] and quadratic drive is an open question that we leave for future investigations. In the following, we assume the single-particle pump to be weaker than single-particle losses.

Since we are interested in the critical properties of this model, we neglect quantum fluctuations, as they are irrelevant compared to the statistical fluctuations induced by the incoherent processes [37,38]. This approximation allows us to treat $\hat{\psi}$ as a stochastic field rather than an operator: its dynamics is accordingly described by the Langevin equation

$$\partial_t\psi = -(-K\nabla^2 + r + u|\psi|^2)\psi - iG\psi^* + \zeta, \quad (3)$$

with K, r, u complex numbers, and ζ a Gaussian, zero-average white noise with correlations $\langle\zeta(\mathbf{r},t)\zeta^*(\mathbf{r}',t')\rangle = 2\sigma\delta(t-t')\delta^{(2)}(\mathbf{r}-\mathbf{r}')$. The imaginary parts of K, r, u (denoted by a "c" subscript) correspond to coherent couplings describing reversible dynamics, while their real parts (denoted by a "d" subscript) correspond to dissipative couplings representing irreversible processes. Moreover, Eq. (3) includes terms which, while zero at the microscopic level, are expected to be generated by coarse graining, e.g., K_d, describing spatial diffusion.

For $G = 0$, Eq. (3) is invariant under the U(1) transformation $\psi \to e^{i\alpha}\psi$, $\psi^* \to e^{-i\alpha}\psi^*$, and it is known as the complex Ginzburg-Landau equation [39,40], or as the driven-dissipative Gross-Pitaevski equation in the context of exciton polaritons [15]. For finite G, Eq. (3) is invariant under the $\mathbb{Z}_2$ transformation $\psi \to -\psi$, $\psi^* \to -\psi^*$, and it is known as the periodically driven complex Gross-Pitaevski equation [40].

Driven sine-Gordon equation.—A mean-field analysis of Eq. (3) shows that a phase transition is expected for $G > G_c$, predicting the spontaneous breaking of the $\mathbb{Z}_2$ symmetry and the emergence of a condensate. This result is expected to be qualitatively robust in higher spatial dimensions $d > 2$, while in lower dimensions fluctuations can dramatically modify the mean-field result. In passing, we notice that for negative detunings $\delta < 0$ the mean-field solution may exhibit a bistable behavior, as predicted in Ref. [41]. In the following, we will restrict ourselves to the case $\delta \geq 0$, where no bistability is expected. The analysis of the $\delta < 0$ case is left for future work.

To assess the effect of fluctuations, we proceed in the spirit of the hydrodynamic theory for quasicondensates [13,42], and we represent the bosonic complex field as $\psi(\mathbf{r},t) = \chi(\mathbf{r},t)e^{i\theta(\mathbf{r},t)}$, with χ and θ real fields associated with density and phase fluctuations. By assuming that a condensate exists, with a density determined by the saddle-point equations, the dynamics is dominated by configurations of χ around that value. The density field χ is gapped and can therefore be eliminated adiabatically from the dynamics (see the Supplemental Material Sec. S1 [43] for the derivation of the driven sine-Gordon equation). This results in the following effective equation for the phase:

$$\eta\partial_t\theta = \gamma\nabla^2\theta - 2g\sin(2\theta) + \frac{\lambda}{2}(\nabla\theta)^2 + F + \xi, \quad (4)$$

with ξ a zero-average Gaussian white noise with correlations $\langle\xi(\mathbf{r},t)\xi(\mathbf{r}',t')\rangle = 2D\delta^{(2)}(\mathbf{r}-\mathbf{r}')\delta(t-t')$. The microscopic values of the six parameters $\eta, \gamma, g, \lambda, F, D$ are given by

$$\begin{aligned} &\eta = 1, && \gamma = K_d + \tfrac{u_c}{u_d}K_c, \\ &g = \tfrac{G}{2}\sqrt{1 + \tfrac{u_c^2}{u_d^2}}, && \lambda = 2\left(-K_c + \tfrac{u_c}{u_d}K_d\right), \\ &F = -r_c + \tfrac{u_c}{u_d}r_d, && D = \tfrac{\sigma}{2\chi_0^2}\left(1 + \tfrac{u_c^2}{u_d^2}\right). \end{aligned} \quad (5)$$

The $\mathbb{Z}_2$ symmetry appears in Eq. (4) as an invariance under the transformation $\theta \to \theta + m\pi$, for all odd integers m. The properties of the phase θ derived from the solutions of Eq. (4) can be directly translated into the correlations of the original complex fields ψ, ψ^* via

$$\langle\psi(\mathbf{r})\rangle \approx \chi_0 e^{i\theta_0} e^{-\frac{1}{2}\langle\theta(\mathbf{r})^2\rangle}, \quad (6a)$$

$$\langle\psi(\mathbf{r})\psi^*(0)\rangle \approx \chi_0^2 e^{\langle\theta(\mathbf{r})\theta(0)\rangle - \langle\theta(\mathbf{r})^2\rangle}, \quad (6b)$$

with θ_0 the saddle point value of θ. The previous relations are obtained by neglecting the fluctuations of χ, and retaining only the leading terms in the cumulant expansion of $\langle e^{i(\theta(\mathbf{r})-\theta(0))}\rangle$.

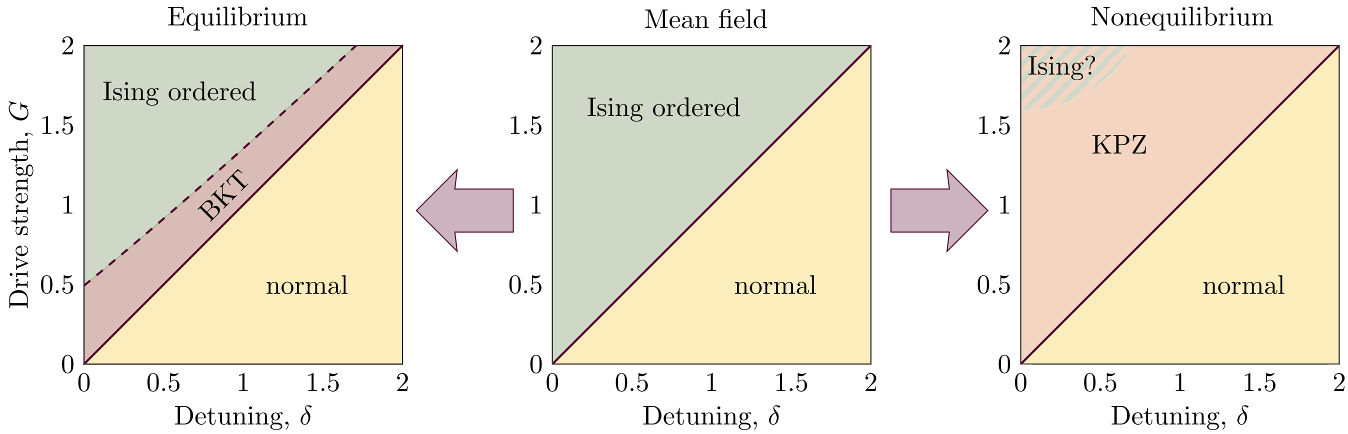

FIG. 1. Center: mean-field phase diagram of a quadratically driven, open condensate, as a function of the imprinted pairing strength G and the detuning δ. Left: equilibrium phase diagram (see Sec. S2 in [43] for the derivation). Fluctuations give rise to an intermediate phase featuring BKT scaling. Right: nonequilibrium phase diagram of model (4). The Ising phase is replaced by a phase with KPZ scaling. A residual Ising phase (striped region) may persist in the nonperturbative regime of large G, inaccessible to our method.

A first insight into the solution of Eq. (4) can be gained by considering two limiting cases and only then the general scenario:

(i) KPZ limit: For $g = 0$, the equation possesses a U(1) symmetry, realized by the invariance under the transformation $\theta \to \theta + \alpha$, with α any real number, and the drift term F can be removed by a gauge transformation $\theta \to \theta + Ft/\eta$. Equation (4) thus reduces to the pristine KPZ equation [44]. In two spatial dimensions, the massless, KPZ-like fluctuations of the phase were shown to erase the BKT phase usually expected in equilibrium Bose gases, and replace it with a disordered phase [13].

(ii) Equilibrium limit: Thermal equilibrium is achieved when the condition $K_c/K_d = r_c/r_d = u_c/u_d$ is satisfied [13,45], which entails the validity of the fluctuation-dissipation theorem, or, more generally, the presence of the associated thermal symmetry of the Keldysh action [46]. In this case, $\lambda = 0$ and $F = 0$, and Eq. (4) reduces to the relaxational dynamics of a sine-Gordon field, whose renormalization was first studied in relation to the roughening transition of crystal surfaces [47].

This model predicts two phases, depending on the relevance of the sine term. In the first phase, the field θ is massive, which is signaled by g being relevant in the RG sense. The value of $\langle\theta(\mathbf{r})^2\rangle$ is then infrared-convergent, while $\langle\theta(\mathbf{r})\theta(0)\rangle$ decays exponentially. Accordingly, Eqs. (6) predict a finite order parameter $\langle\psi\rangle$ and long-range order, indicating that the system lies in the ordered phase with a spontaneously broken $\mathbb{Z}_2$ symmetry. In the second phase, g is irrelevant in the RG sense, and θ becomes massless. Accordingly, $\langle\theta(\mathbf{r})^2\rangle$ is infinitely large as a consequence of the infrared divergence, while $\langle\theta(\mathbf{r})\theta(0)\rangle - \langle\theta(\mathbf{r})^2\rangle$ grows logarithmically, implying an algebraic decay of $\langle\psi^*(\mathbf{r})\psi(0)\rangle$. This suggests that long-range order is suppressed, and the condensed phase is replaced by a BKT phase with quasi-long-range order. This is the usual case for two-dimensional Bose gases with U(1) symmetry [i.e., $G = 0$ in Eq. (2)].

Summarizing, for a two-dimensional equilibrium gas, three phases are expected: a normal fluid with short-range correlations (corresponding to the mean-field solution without condensate), a BKT phase with quasi-long-range order, and a $\mathbb{Z}_2$-symmetry-broken phase with long-range order. The corresponding phase diagram in terms of G and δ is reported in Fig. 1, (see Sec. S2 [43] for derivation). Analogous phases have been obtained for the axial next-nearest neighbor Ising (ANNNI) model [48,49] and the *XYZ* spin chain in transverse field [50,51], which share the same effective dimensionality and $\mathbb{Z}_2$ symmetry with the present model.

(iii) Full problem: In Eq. (4), the KPZ fluctuations wash out the sine-Gordon physics, thus destabilizing the phases predicted at thermal equilibrium. The renormalization analysis of this equation was first performed in Refs. [52,53] to study the effect of nonlinearities on the roughening transition of crystal surfaces. There, it was shown that the KPZ physics dominates over large distances. We will show that this has dramatic implications for driven-dissipative Bose gases, as the equilibrium ordered and BKT phases are destabilized by nonequilibrium fluctuations, and replaced by a phase with short-range order, see Fig. 1.

Absence of long-range order.—The long-wavelength physics of Eq. (4) can be conveniently studied using a perturbative renormalization group approach. The idea consists in treating g and λ perturbatively around the Gaussian model, and in deriving an effective long-wavelength theory by progressively integrating out high-energy modes. The couplings of the long-wavelength model are then expressed by a set of flow equations. We will consider two RG schemes, derived in Refs. [53] and [54], and discussed in Sec. S3 [43], which includes Ref. [55]. The equations are expressed in terms of the dimensionless

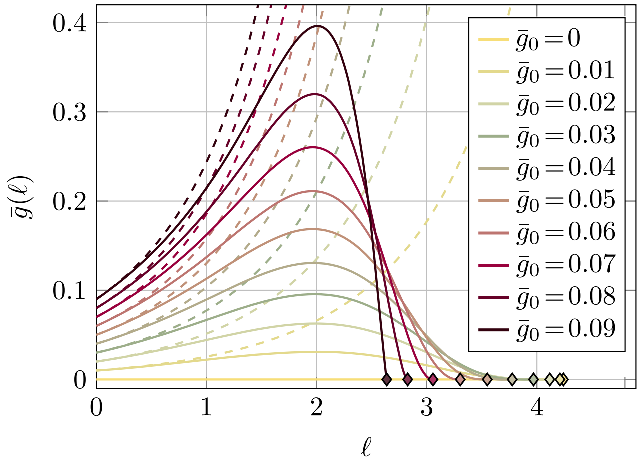

FIG. 2. Flow of $\bar{g}(\ell)$ for different initial values of $\bar{g}_0$. Parameters for the solid curves: $\gamma_0 = 0.3$, $T_0 = 1$, $\lambda_0 = 0.4$, $\eta_0 = 1$, and $\bar{F}_0 = 0$. Parameters for the dashed curves: $\gamma_0 = 0.3$, $T_0 = 1$, $\lambda_0 = 0.0$, $\eta_0 = 1$, and $\bar{F}_0 = 0$. The diamond symbols denote the onset of a divergence in the RG flow ℓ^*.

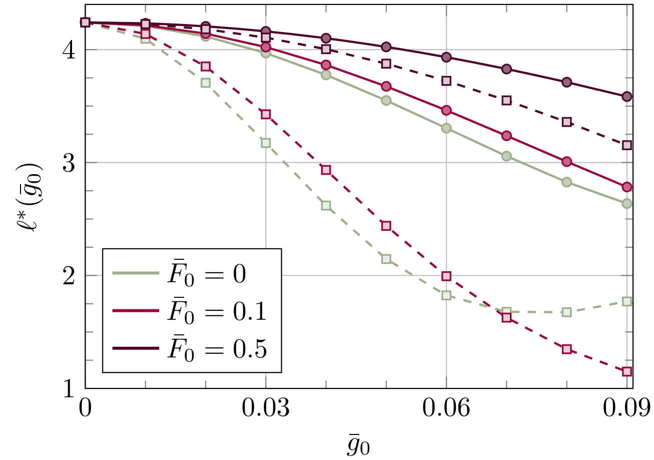

FIG. 3. RG scale for the KPZ crossover as a function of the sine-Gordon nonlinearity $\bar{g}_0$, for different values of $\bar{F}_0$. The solid and dashed lines correspond to the RG schemes derived in Ref. [53] and [54] (cf. Sec. S3 [43]). Parameters: $\gamma_0 = 0.3$, $T_0 = 1$, $\lambda_0 = 0.4$, $\eta_0 = 1$.

quantities $\bar{g} \equiv g/\Lambda^2$, and $\bar{F} \equiv F/\Lambda^2$. Before proceeding to a more detailed analysis, we discuss the qualitative solution of the RG equations.

If the system is in thermal equilibrium, the equations reduce to the ones for the relaxational sine-Gordon model of Ref. [47]. If $\bar{g} = 0$, instead, the equations reduce to the ones for KPZ [56]: the noise level D and the effective temperature T flow to infinity, indicating the relevance of the KPZ scaling. Finally, if both $\bar{g}_0$ and λ_0 are finite, the KPZ nonlinearity λ dominates over the sine-Gordon one $\bar{g}$, which eventually renormalizes to zero. Typical flows of $\bar{g}$ are shown in Fig. 2: for $\lambda = 0$ and $\bar{F} = 0$, $\bar{g}(\ell)$ grows indefinitely (dashed curves), signalling that the field θ is gapped. For finite initial values of λ_0 or $\bar{F}_0$, however, $\bar{g}$ flows back to zero, indicating the irrelevance of the sine-Gordon term. The diamond symbols denote the onset of a divergence in the RG flow (see below).

At long wavelengths, the phase correlations are then expected to be captured by the KPZ exponents, i.e., $\langle\theta(\mathbf{r})\theta(0)\rangle - \langle\theta(\mathbf{r})^2\rangle \sim -|\mathbf{r}|^{2\chi}$, with $\chi \approx 0.38$ [57,58]. Moreover, $\langle\theta(\mathbf{r})^2\rangle$ diverges due to long-wavelength fluctuations. Accordingly, by replacing these values in Eqs. (6), we find that complex fields are short-range correlated via a stretched exponential, implying that no phase transition takes place. Whether the ordered phase is completely removed or survives for large values of the two-particle drive (corresponding to large values of g) cannot be determined from our analysis, as the RG analysis is not valid for nonperturbative values of g. Finally, here we neglected the presence of topological excitations, such as vortices and antivortices. Whether such defects play a role on length scales below or only above the KPZ length scale is a nonuniversal question, which depends on the microscopic details of the experimental system. In the former case, the KPZ phase would be replaced by a disordered phase; in the latter, the KPZ physics will be visible in experiments [59–64]. The impact of the $\mathbb{Z}_2$ symmetry on these excitations is left for future work.

Enhancement of KPZ physics.—An essential question concerns the visibility of the predicted 2D KPZ physics in experimental systems or numerical simulations with limited size. In fact, the length scale L^* above which the KPZ physics becomes visible is usually very large, and can exceed the accessible systems' size: this is the case for, e.g., the roughening transition in crystal surfaces [65], and for exciton polaritons in two-dimensional microcavities [13,18,66–68]. Here we show that the presence of a sine-Gordon nonlinearity can actually lower L^*, thus enhancing the visibility of the 2D KPZ.

The value of L^* can be extracted from the solution of the flow equations [69]. To illustrate this, it is convenient to first focus on the pure KPZ case of Eqs. S5 in [43], i.e., $\bar{g} = 0$. Here, the relevant RG equation is the one for the effective temperature T in Eq. S5d [43] with γ and λ constant under the RG flow. $T(\ell)$ features a divergence for finite values of the flow parameter ℓ, namely, $\ell^* = 8\pi\gamma^3/(T_0\lambda^2)$, with T_0 the initial value of T. The value of ℓ^* determines therefore the physical length scale above which the KPZ scaling is visible via $L^* = \xi_0 e^{\ell^*}$, with ξ_0 some microscopic length scale. As L^* is exponentially sensitive to the value of ℓ^*, finding conditions to minimize ℓ^* is crucial to observe the KPZ physics. For finite values of $\bar{g}$, ℓ^* cannot be determined analytically, but it can be extracted from the divergence of the numerical solutions. We computed ℓ^* for different values of $\bar{g}_0$ and $\bar{F}_0$: the results are reported in Fig. 3. Since ℓ^* is not a universal quantity, we extracted its value using two different RG schemes (cf. S3), finding the same qualitative behavior.

Our results indicate that ℓ^* generically decreases as a function of $\bar{g}_0$. The decrease can be optimized by varying the value of $\bar{F}$, which, corresponding to the laser detuning [cf. Eq. (6)], is an experimentally tunable parameter. The value of ℓ^* can be reduced by up to a factor 4 upon reaching $\bar{g}_0 \sim 0.1$, indicating that L^* can be reduced by four orders of magnitude compared to the case with $\bar{g}_0 = 0$. This implies a dramatic improvement of the visibility of the KPZ scaling in two-dimensional driven-dissipative gases, where it has so far remained elusive. As an example, in exciton-polariton fluids in the optical parametric oscillator regime, the KPZ length scale was predicted to be $\sim 10^3$ μm in the bad-cavity regime [66], which is one order of magnitude larger than the typical size in current experiments [70–72]. We estimated the KPZ length scale for current exciton-polariton experiments [43]. We considered typical experimental values for GaAs/AlAs microcavities [73], namely, polariton mass $m_P = 10^{-4} m_e$ (with m_e the vacuum electron mass), single-particle losses $\gamma_1 = 0.4$ meV, polariton-polariton interaction $U = 1.5\ \mu\text{eV}\,\mu\text{m}^2$ (see, e.g., Ref. [74]), and a pump-spot size of $L = 50\ \mu$m. At resonance ($\delta = 0$) and for a drive strength $G \approx 5$ meV, we find $L^*/L \approx 0.83$, indicating that the KPZ physics is expected to be visible in current experimental setups.

Outlook.—We showed that, in two-dimensional quadratically driven Bose gases, the absence of thermal equilibrium leads to an emerging phase characterized by KPZ scaling. Correspondingly, the BKT and Ising phases expected at thermal equilibrium are suppressed. Moreover, we discovered that the presence of a quadratic drive may shrink the length scale at which the KPZ physics occurs, thus enhancing its visibility in systems with finite size. Our results open novel perspectives for the detection of non-equilibrium phases of matter in experimental platforms, in particular exciton polaritons in microcavities and nonlinear photonic lattices. There, a quadratic drive can serve as a tool to enhance the nonequilibrium nature of driven-dissipative condensates and may provide the necessary assist to experimentally access the unexplored physics of the 2D KPZ equation.

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