Ghosts without Runaway Instabilities

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We present a simple class of mechanical models where a canonical degree of freedom interacts with another one with a negative kinetic term, i.e., with a ghost. We prove analytically that the classical motion of the system is completely stable for all initial conditions, notwithstanding that the conserved Hamiltonian is unbounded from below and above. This is fully supported by numerical computations. Systems with negative kinetic terms often appear in modern cosmology, quantum gravity, and high energy physics and are usually deemed as unstable. Our result demonstrates that for mechanical systems this common lore can be too naive and that living with ghosts can be stable.

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There are various reasons to be interested in field theories, or their simpler mechanical counterparts, where 1 degree of freedom has a ghostly nature, i.e., a negative kinetic term, while the other degrees of freedom have the usual positive-definite kinetic terms. Following the pioneer work of Ostrogradsky [1] it has been shown that such a situation appears generically in the Hamiltonian formulation of theories with higher derivative interactions [2,3] (see also, e.g., Ref. [4,5]), while such interactions have in turn interesting properties to regulate field theories in the UV [2,6–8]. A ghost copy of the standard model was also invoked in attempts to solve the cosmological constant problem [9–11]. More recently, ghostly theories have also emerged in cosmology to describe nonstandard dynamics of the universe including bouncing cosmologies (see, e.g., Ref. [12]) and dark energy with a phantom equation of state [13] which is still allowed by the latest cosmological data (e.g., Ref. [14]). Related proposals can also address the Hubble tension problem (see, e.g., Ref. [15]). Sometimes such nonstandard dynamics can be obtained in ghostfree theories (see, e.g., Ref. [16]) where, however, the Hamiltonians are necessarily unbounded from below [17]. This implies a potential vulnerability to nonlinear instabilities.

The standard lore is of course that theories with ghosts and/or energies unbounded from below are unstable and as such problematic, even though various authors have advocated differently [6,8,18–24]. The instability inherent to ghostly models, usually dramatic at the quantum field theory level (see, e.g., Ref. [25]), is already seen at the classical level in the associated Hamiltonian motion. More specifically it can be linked to interactions between a ghostly degree of freedom and one of positive energy, as one isolated ghost would be stable. There again it is usually believed that *any* such interactions would generically lead to catastrophic trajectories with divergences or runaway instabilities already at the classical level.

However, there exist scarce studies indicating that this could be more subtle. Indeed, the well-known Kolmogorov-Arnold-Moser (KAM) theorem (see, e.g., Ref. [26]) opens the way to stable motions in systems with a specific ghost-positive energy degree of freedom interaction and a restricted set of initial conditions, so-called islands of stability [27]. An analytic study of such a situation has been carried out [28], showing that, for a specific model, there exist stable motions in the vicinity of one particular point in phase space. Some numerical works have reached similar conclusions in a restricted set of models [22,27,29–31]. On the other hand, Refs. [32,33] found numerically, for very specific models, stable motions for all the investigated initial conditions. However, all these findings are either based on numerical integrations and, as such, are not fully conclusive (as they cannot cover all the Hamiltonian trajectories) and/or only yield islands of stability, and not global stability, all this being true in addition at best for a very restricted set of interactions.

Here we propose a new look at these issues and present a large set of models, with a ghost in interaction with a positive energy degree of freedom, which have stable classical motions for all initial values. This stability is proven analytically.

The model considered here is defined by a particular interaction potential $V_I(x,y)$ between a normal harmonic oscillator x and a ghost oscillator y of the same frequency. Hence, the total Hamiltonian is

$$H = \frac{1}{2}(p_x^2 + x^2) - \frac{1}{2}(p_y^2 + y^2) + V_I(x,y), \tag{1}$$

with $V_I(x,y)$ given by

$$V_I(x,y) = \lambda[(x^2 - y^2 - 1)^2 + 4x^2]^{-1/2}, \tag{2}$$

where λ is the coupling constant. The model is well defined for all values of x and y. Indeed, the expression under the square root in Eq. (2) is positive definite so that the interaction potential V_I is always smooth and finite. The total potential energy $V_{\rm tot}$ (see Fig. 1) and the total Hamiltonian are unbounded from above and from below. However, the interaction potential V_I is bounded as

$$0 < V_I(x,y)\lambda^{-1} \le 1. \tag{3}$$

Expanding the total potential energy in x and y around the origin yields

$$V_{\rm tot} = \frac{\omega_x^2}{2}x^2 - \frac{\omega_y^2}{2}y^2 + \lambda(x^4 + 4y^2x^2 + y^4) + \cdots, \tag{4}$$

where the frequencies are corrected by the interaction as

$$\omega_x^2 = 1 - 2\lambda, \quad \text{and} \quad \omega_y^2 = 1 + 2\lambda, \tag{5}$$

and where we neglected an irrelevant constant and terms of total power higher than 4. Thus, for $|\lambda| < 1/2$, both oscillators are linearly stable around the origin. (For $\lambda > 1/2$, the system exhibits tachyonic instability around the origin, but is locally stable around the saddles of the total potential at $(x,y) = [\pm(\sqrt{2\lambda} - 1)^{1/2}, 0]$. For $\lambda < -1/2$, on the hand, the system is locally stable around saddles at $(x,y) = [0, \pm(\sqrt{-2\lambda} - 1)^{1/2}]$.) However, we will not here restrict ourselves to motions staying in the vicinity of some point in phase space, and which could possibly be described by perturbation theory. Rather we will prove a stability theorem valid for all initial conditions and Hamiltonian motions. To that end it is crucial to note that the model defined by Eqs. (1) and (2) is integrable: in addition to the conserved Hamiltonian H, it has a first integral

$$C = K^2 + (p_x^2 + x^2) - (x^2 - y^2 - 1)V_I(x,y), \tag{6}$$

where $K = p_y x + p_x y$ is the momentum of hyperbolic rotations (boosts) in the (x,y) plane. One can explicitly check that the above quantity C is conserved by the Hamiltonian motion. However, we also note that the above model can be obtained from a class of integrable models obtained by Darboux in 1901 [34] with 2 positive energy degrees of freedom x and $\tilde{y}$, using the complex canonical transformations

$$y = i\tilde{y}, \quad \text{and} \quad p_y = -i\tilde{p}_y, \tag{7}$$

which in our case not only preserve the Hamiltonian motion but also keep both H and C real. It is useful to introduce the sum of absolute values of the energies of both oscillators and the square of K

$$\Sigma = (p_y x + p_x y)^2 + \frac{1}{2}(p_x^2 + x^2) + \frac{1}{2}(p_y^2 + y^2). \tag{8}$$

Each term in Σ is manifestly non-negative and corresponds to a first integral of the system [Eq. (1)] without interaction, i.e., for $\lambda = 0$. The distance from a point in the phase space $\xi = (x, y, p_x, p_y)$ to the origin (the Euclidean norm of the state) is always bounded: $|\xi|^2 \le 2\Sigma$. The following difference $\mathcal{E}$ between two integrals of motion is useful:

$$\mathcal{E} = C - H = \Sigma + (y^2 - x^2)V_I(x,y). \tag{9}$$

The second term in the last equality is bounded in the stripe

$$-|\lambda| \le (y^2 - x^2)V_I(x,y) \le |\lambda|. \tag{10}$$

As a consequence, one gets that at all times

$$\Sigma - |\lambda| \le \mathcal{E} \le \Sigma + |\lambda|. \tag{11}$$

Applying this inequality at two different times t_a and t_b, and using the conservation of $\mathcal{E}$ we get that (the index a, b referring to the corresponding time)

$$\Sigma_a - 2|\lambda| \le \Sigma_b \le \Sigma_a + 2|\lambda|. \tag{12}$$

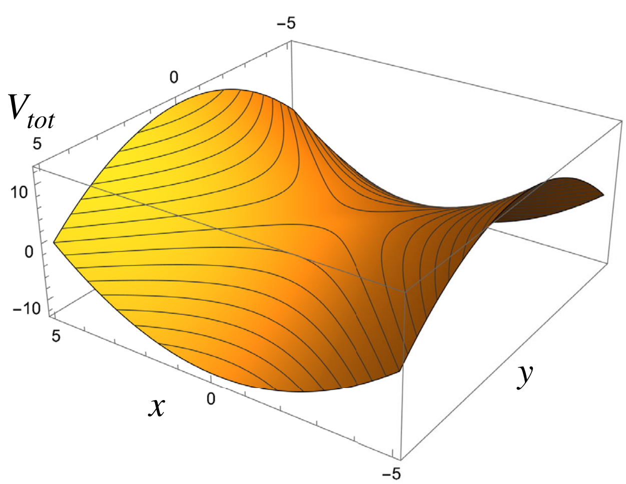

FIG. 1. Total potential energy is plotted for coupling constant $\lambda = 1/3$ in the interaction potential [Eq. (2)].

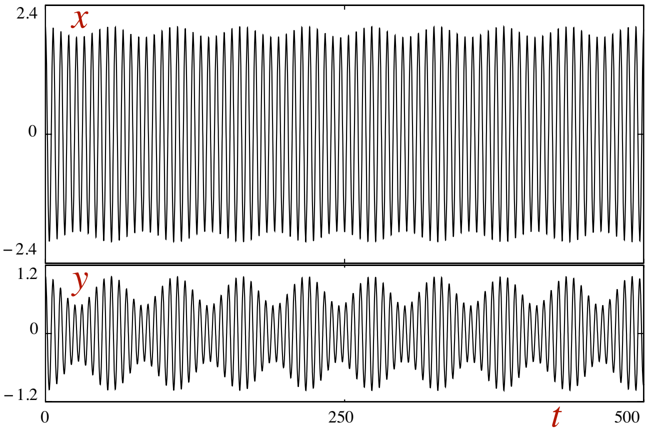

FIG. 2. The plot of $x(t)$ and $y(t)$.

In particular, the last inequality implies that

$$|\xi_b|^2 \leq |\xi_a|^2 + 2K_a^2 + 4|\lambda|, \tag{13}$$

using $2\Sigma = 2K^2 + |\xi|^2 \geq |\xi|^2$. Hence, the motion is confined inside of a sphere whose radius is fixed by the (initial) data at t_a. This completes the proof that the motion of our system is always bounded and shows no runaway for all values of the initial data and coupling constant λ. We stress that this also holds for parameters yielding "tachyonic" negative ω_x^2 or/and ω_y^2 corresponding to an unstable origin.

We now show the result of numerical integration of the system in Eqs. (1) and (2) with $\lambda = 1/3$. The Hamiltonian equation of motion is directly solved numerically from the initial time $t = 0$ to the final time $t = 500$ with the initial condition $[x(0), y(0), p_x(0), p_y(0)] = (2, 1, 0, 0)$. Numerical errors in H and C remain of order 10^{-13}. Figure 2 shows the behaviors of $x(t)$ and $y(t)$. Each of them stably oscillates with some modulation induced by the interaction between them. Figures 3 and 4 show the projection of the trajectory onto the xy and yp_y planes, respectively; the color represents the value of t.

Note that the trajectories close to the origin can also be analyzed more precisely. First of all, from the first inequality in Eq. (12), one obtains

$$|\xi_a|^2 - 2K_b^2 - 4|\lambda| \leq |\xi_b|^2. \tag{14}$$

Close to the origin of the phase space, K^2 is higher order than $|\xi|^2$ and can be neglected. Thus, in this case, we conclude that, for $4|\lambda| < |\xi|^2 \ll 1$, the trajectories are located in the spherical shell which is $4|\lambda|/|\xi_a|$ thin:

$$|\xi_a|^2 - 4|\lambda| \lesssim |\xi_b|^2 \lesssim |\xi_a|^2 + 4|\lambda|,$$

where we omitted $\mathcal{O}(|\xi|^4)$ terms. Close to the origin one can use Eq. (3) to refine Eq. (11), so that for $\lambda(y^2 - x^2) < 0$

$$\Sigma + \lambda(y^2 - x^2) \leq \mathcal{E} \leq \Sigma, \tag{15}$$

while for $\lambda(y^2 - x^2) \geq 0$ the limits flip and yield $\mathcal{E} > 0$ (except at the origin). However,

$$\Sigma + \lambda(y^2 - x^2) = K^2 + \frac{1}{2}(p_x^2 + p_y^2 + \omega_x^2 x^2 + \omega_y^2 y^2), \tag{16}$$

where ω_x and ω_y are given by Eq. (5). For $|\lambda| < 1/2$ both ω_x^2 and ω_y^2 are positive, which results in $\mathcal{E} > 0$, except at the origin where $\mathcal{E} = 0$. Thus, $\mathcal{E}$ being in addition conserved, satisfies all requirements of a Lyapunov function and guarantees the stability of the origin for $|\lambda| < 1/2$.

A stable (and integrable) motion such as the one discovered here can of course also be observed for a system of a ghost noninteracting with a positive energy degree of freedom. Hence a legitimate question is whether one could transform the model considered here into such a system, i.e., find a canonical transformation which kills all the interactions. One can show, at least order by order, that this is not possible. To that end one can use a theorem given in Ref. [26], which in the present context amounts to stating

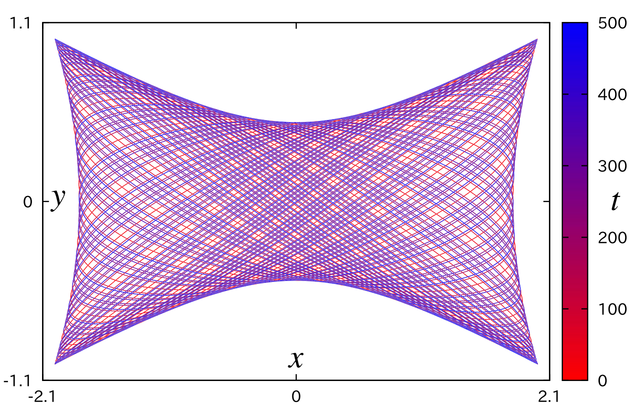

FIG. 3. The projection of the trajectory onto the xy plane. The color represents the value of t.

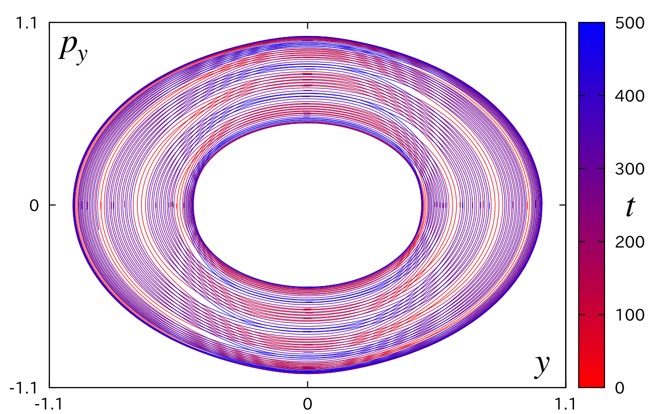

FIG. 4. The projection of the trajectory onto the yp_y plane. The color represents the value of t.

that, using the new variables defined by $z_x = p_x + ix$ and $\bar{z}_x = p_x - ix$, any interaction of the form

$$z_x^{\alpha_x} \bar{z}_x^{\beta_x} z_y^{\alpha_y} \bar{z}_y^{\beta_y} \tag{17}$$

can be removed by a suitable canonical transformation except in the cases where $\alpha_x = \beta_x$ and simultaneously $\alpha_y = \beta_y$. This holds true in the so-called nonresonant case, which includes the case considered here where the ratio $\omega_x/\omega_y = \sqrt{(1-2\lambda)/(1+2\lambda)}$ is generically irrational. In our case, it is easy to see that each term x^4, x^2y^2 and y^4 appearing at order 4 in the expansion of the potential Eq. (4) contains one and only one monomial which cannot be removed, respectively, given by the distinct monomials $z_x^2\bar{z}_x^2$, $z_x\bar{z}_x z_y \bar{z}_y$, and $z_y^2\bar{z}_y^2$. Hence, we conclude that it is not possible to fully remove the quartic interaction of our model via a canonical transformation that keeps the quadratic part of the Hamiltonian.

We note further that the model considered here and defined by the Hamiltonian [Eq. (1)] can easily, at least locally, be rewritten as a higher derivative theory for a single degree of freedom q. To that end, one inverts the Ostrogradsky procedure and ends up with an equivalent Lagrangian L given by (where a dot means a time derivative)

$$L(q, \ddot{q}) = (\ddot{q} + q)[2p_2 + (2p_2)^{-1}], \tag{18}$$

where $p_2 \equiv p_2(q, \ddot{q})$ is a solution of the equation

$$(\ddot{q} + q)\sqrt{2q^2 + 1} = -2\lambda p_2 (2p_2^2 + 1)^{-3/2} \tag{19}$$

and is the Legendre conjugate variable to $\ddot{q}$ in the Lagrangian $L(q, \ddot{q})$ (i.e., one has $p_2 = \partial L/\partial \ddot{q}$).

Last, we underline that the above model [Eq. (1)] is not unique. It is part of a larger family of models with the Hamiltonian $H = p_x^2/2 - p_y^2/2 + V$, where

$$V = aU + bW + cUW, \tag{20}$$

where

$$U = d - x^2 + y^2, \qquad W = (U^2 + 4dx^2)^{-1/2}, \tag{21}$$

and a, b, c, d are arbitrary constants satisfying $a < 0$, $c \leq 0$, and $d > 0$. These models are all integrable, with a motion whose stable nature can be proven analytically along the line above, with a Hamiltonian unbounded below and above and a ghost coupling to a positive energy degree of freedom [35].

We have presented an example of classical models where a subsystem with positive energy unbounded from above interacts with another subsystem with negative energy unbounded from below. Yet the dynamics is such that the negative energy is locked and cannot be exploited to further increase the positive energy of the other subsystem. Hence, there are no runaway solutions in the whole phase space. Moreover, we have shown the Lyapunov stability of the origin in the model in Eq. (1) (for $|\lambda| < 1/2$), while our numerical investigations indicate that such a stability exists more widely on phase space [35]. Note also that as the system is integrable, the KAM theorem should allow for the existence of "islands of stability" for a large class of nonintegrable interactions around the considered models. We stress however that the integrability, which plays an important role in our proof, does not *per se* guarantee the absence of runaway solutions [35]. It would be very interesting to understand the quantum mechanical description of such systems and their generalization to continuum number of degrees of freedom. Some investigations along these lines have been made in models with higher derivative equations of motion opening up the possibility to define in some cases a sensible associated quantum theory [2,36]. A crucial property of the system proposed in this Letter is that the motion is bounded for all initial data. Thus, a wave function cannot probe any instability contrary to the systems which are only stable around one point. We leave a detailed investigation of these issues for a future work. Incidentally, the work reported here also points out the existence of a large set of integrable models where ghosts interact with a positive energy degree of freedom. These ghostly models can be obtained via known integrable models with 2 positive energy degrees of freedom and a complex canonical transformation of the form in Eq. (7) [35]. To our knowledge, the only previously discussed example of such an "integrable ghost" (with a total of 2 degrees of freedom) is a very specific model given in Ref. [36] obtained from a supersymmetric field theory (see also Ref. [37]).

It seems that invoking interaction with ghosts may be rather innocent, at least in some cases and at the classical level. Thus, in these cases, it is not stability which precludes the existence of ghosts, and it is not unusual that quantization improves the stability of a given model rather than deteriorates it. This is famously the case, e.g., for the Kepler problem. One then expects to be able to find such stable interacting ghosts in some natural systems, perhaps as some low-energy modes or collective coordinates within a sensible field theory in a wider context than the one mentioned in the introduction. It is known that unstable IR ghosts can appear in a simple field theory system such as a massless canonical scalar field minimally coupled to general relativity [38]. Why not stable ones?

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