Multiple-Phase Quantum Interferometry: Real and Apparent Gains of Measuring All the Phases Simultaneously

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We characterize operationally meaningful quantum gains in a paradigmatic model of lossless multiple-phase interferometry and stress the insufficiency of the analysis based solely on the concept of quantum Fisher information. We show that the advantage of the optimal simultaneous estimation scheme amounts to a constant factor improvement when compared with schemes where each phase is estimated separately, which is contrary to widely cited results claiming a better precision scaling in terms of the number of phases involved.

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Introduction and summary of results.—Quantum metrology aims at identifying optimal ways of using quantum systems as sensing probes [1–9]. When N quantum probes are used independently, the estimation variance decreases inversely proportional to the number of probes in accordance with the standard quantum limit (SQL). The hallmark of quantum metrology is the potential quadratic scaling improvement over the SQL known as the Heisenberg limit (HL) [10–20].

In *multiple* parameter estimation scenarios, simultaneous estimation of p parameters in a single experiment may additionally provide an improved performance when compared with strategies where each parameter is estimated separately [21–27].

A paradigmatic model to study the potential of multiparameter quantum enhanced metrological protocols is the multiple-phase estimation problem. The goal is to estimate all the relative phase shifts in a multiple-arm interferometer with the best precision possible given a constraint on the total number of photons used; see Fig. 1.

The most common tool to analyze the potential of quantum metrological strategies is the quantum Fisher information (QFI), the inverse of which lower-bounds the variance of any locally unbiased estimator $\tilde{\theta}$ via the famous quantum Cramér-Rao (CR) bound [28–30]. In the single parameter case, the CR bound takes the form

$$\Delta^2\tilde{\theta} \geq \frac{1}{kF(\rho_\theta^n)}, \tag{1}$$

where k is the number of repetitions of an experiment and $F(\rho_\theta^n)$ is the QFI computed on the n-probe output state on which the parameter θ has been encoded. In the case of the standard two-arm optical interferometry, a single relative phase between the two arms is being estimated, and $F(\rho_\theta^n) = n$ for n uncorrelated photons sent into the interferometer, while the maximum value $F(\rho_\theta^n) = n^2$ is obtained for an optimally entangled state of n photons—the $n00n$ state—resulting in the $1/n^2$ Heisenberg scaling (HS) of precision [5,12–19]. In general, this bound is operationally saturable provided one takes the asymptotic limit $k \to \infty$ while keeping the n fixed. Such a case corresponds to an experimental realization where the amount of resources used in a single realization is large but limited and the experiment may be repeated an arbitrary number of times.

However, a fundamental question is what the true HL for precision will be if the total amount of resources $N = n \cdot k$ is restricted and the $N \to \infty$ limit is taken. Since the scaling of precision is quadratic in n and linear in k, the optimal

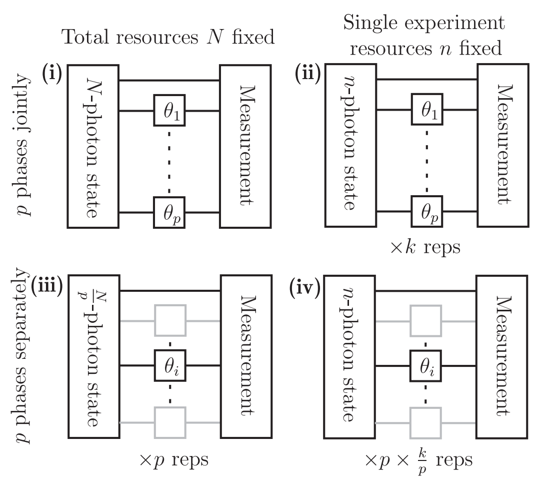

FIG. 1. Multiple-phase estimation schemes where a constraint is imposed on the total resources used (left column) or resources used in a single experiment (right column). The top and bottom rows represent, respectively, the protocols where all the phases are measured jointly or the estimation procedure is repeated for each of the phases separately. The lower bounds for the sum of variances obtainable within each of (i)–(iv) paradigms is presented in Table I.

choice appears to be $n = N$, $k = 1$. However, in this case one cannot use saturability arguments based on the many-repetition scenario and the predictions of the QFI may be misleading with respect to the choice of the optimal probe states as well as the asymptotically achievable precision limit. This becomes clear when the results are contrasted with the ones obtained via the minimax [31], Bayesian [32], or information theoretic approach [33].

The use of a single $N00N$ state (note that the use of N instead of n is intentional) is clearly not an operationally meaningful strategy when discussing the HL, as it is not capable of discriminating phases that differ by a multiple of $2\pi/N$. There is clearly a need to sacrifice part of the resources to get rid of the arising ambiguity, and this leads to a π^2 increase in the asymptotically saturable bound, which can be rigorously shown within a Bayesian estimation framework [32,34–39]. Therefore, in order to avoid confusion, we will introduce a clear distinction between the two approaches and refer to them as HS in which the amount of resources used in a single repetition of experiment n is large but finite and the whole experiment may be repeated $k \to \infty$ times: $\Delta^2\tilde{\theta} \propto 1/kn^2$, and HL in which the total amount of resources N is restricted and no repetitions of an experiment are assumed: $\Delta^2\tilde{\theta} \propto 1/N^2$.

The two approaches may only be reconciled provided one is able to properly account for the scaling of the required number of repetitions k with the increasing number of probes n used in a single experiment that guarantees saturability within the HS approach. This is, however, hardly ever possible and typically the issue is simply ignored in the literature.

In the multiparameter case, a rigorous study of the achievable HL is much more challenging and the common approach is to rather work in the HS paradigm where efficiently computable multiparameter generalizations of CR bounds are used [23,27,40–45]. Rigorous analyses of the actual saturable HL are typically restricted to Bayesian framework case studies using some underlying group structure of the problem [46–53]. However, quite surprisingly, the actual analytical form of the asymptotically saturable HL for the paradigmatic multiple-phase estimation problem is missing (see Ref. [54] for a recent numerical attempt to tackle the problem).

In this Letter, we employ an operationally meaningful minimax approach to derive an asymptotically saturable HL for the multiple-phase estimation problem and demonstrate that it manifests a p^3 scaling, with the number of parameters involved, rather than the p^2 that is advocated when following the HS approach [55]. We also clarify apparent gains that may be obtained thanks to simultaneous phase estimation when compared with strategies that estimate all the phases separately. We show that the advantage amounts to a constant factor gain and, contrary to the claims of [55–57], does not lead to a better scaling of precision with the number of parameters involved.

TABLE I. Asymptotically achievable lower bounds on the sum of variances of estimated phases. The main result presented in this Letter is the bolded formula representing the proper multiple-phase Heisenberg limit (HL) and demonstrating the p^3 scaling with the number of estimated phases. All bounds are tight in the asymptotic limit $N \to \infty$ (or $k \to \infty$), while in the case of joint-phases estimation tightness requires an additional $p \to \infty$ limit ($c = 1.89$ yields a universally valid bound, while $c = 2$ yields an asymptotically achievable cost). The SQL column can be regarded as a special case of the Heisenberg scaling (HS) column when $n = 1$ and $k = N$.

$\Delta^2\tilde{\boldsymbol{\theta}} \geq$	Independent probes SQL	Entangled probes $N = kn$ probes HL	Entangled probes n probes, k reps HS
Single phase	$\frac{1}{N}$	$\frac{\pi^2}{N^2}$	$\frac{1}{kn^2}$
p phases jointly	$\frac{p^2}{4N}$	(i) $\boldsymbol{\frac{cp^3}{N^2}}$	(ii) $\frac{p^2}{4kn^2}$
p phases separately	$\frac{p^2}{N}$	(iii) $\frac{\pi^2 p^3}{N^2}$	(iv) $\frac{p^2}{kn^2}$

We explain the apparent contradiction by pointing out the improper use of saturability arguments that are often invoked when following the HS approach. (Table I summarizes the main results presented in this Letter.)

Multiple-phase estimation problem.—Consider a multiple-phase estimation problem as depicted in Fig. 1, where the goal is to estimate the value of p phases $\boldsymbol{\theta} = [\theta_1, \ldots, \theta_p]$, the relative phase delays in the ith arm of an interferometer with respect to the reference arm. For a general n-photon state at the input, the output state with phase information encoded will have the form

$$|\Psi_{\boldsymbol{\theta}}^n\rangle = \sum_{\boldsymbol{m}:\, \sum_{i=1}^p m_i \leq n} c_{\boldsymbol{m}} e^{i\boldsymbol{m}\cdot\boldsymbol{\theta}} |m_1, m_2, \ldots m_p\rangle, \qquad (2)$$

with $\boldsymbol{m} = [m_1, \ldots, m_p]$, where m_i represents the number of photons in the ith "signal" arm, while the remaining $n - \sum_{i=1}^p m_i$ photons are sent through the reference arm. A general quantum measurement is then performed in order to extract information on the encoded phases, mathematically specified by a set of positive operators, $\{M_x\}$, $\sum_x M_x = \mathbb{1}$, in which x labels a measurement outcome observed with probability $p_{\boldsymbol{\theta}}(x) = \langle\Psi_{\boldsymbol{\theta}}^n|M_x|\Psi_{\boldsymbol{\theta}}^n\rangle$. The measurement outcomes are then fed into an estimator function that yields the inferred values of the phases. In scenario (i), the estimator $\tilde{\boldsymbol{\theta}}(x)$ is a function of just a single measurement outcome (in this case $n = N$ as all resources are used in a single shot); in (ii) the experiment is repeated k times and the estimator is a function of all k measurement outcomes $\tilde{\boldsymbol{\theta}}(x^1, \ldots, x^k)$; in (iii) p separate protocols involving p different states (each containing N/p photons) and different measurements are performed, yielding measurement outcomes $x_1, \ldots, x_p$, where each outcome x_i feeds the estimator of the ith phase

$\tilde{\theta}_i(x_i)$; finally, in (iv) each of the p separate protocols is repeated k/p times, yielding in total $p \times (k/p) = k$ measurement outcomes and resulting in p separate estimators of each phase $\tilde{\theta}_i(x_i^1, \ldots, x_i^{k/p})$, $i \in \{1, \ldots, p\}$.

Irrespective of which scenario is considered, the figure of merit to be minimized is the sum of squared errors of estimated phases:

$$\Delta^2 \tilde{\boldsymbol{\theta}} = \int d\boldsymbol{x}\, p_{\boldsymbol{\theta}}(\boldsymbol{x})(\tilde{\boldsymbol{\theta}}(\boldsymbol{x}) - \boldsymbol{\theta})^2, \tag{3}$$

where $\int d\boldsymbol{x}$ formally represents integration over all (possibly continuous) measurement outcomes and $(\tilde{\boldsymbol{\theta}}(\boldsymbol{x}) - \boldsymbol{\theta})^2 = \sum_{i=1}^{p} (\tilde{\theta}_i(\boldsymbol{x}) - \theta_i)^2$. As this is a pointwise figure of merit (calculated at a given $\boldsymbol{\theta}$), in order to make the minimization task meaningful, one needs to impose additional constraints on the estimator function because otherwise a trivial solution $\tilde{\boldsymbol{\theta}}(\boldsymbol{x}) = \boldsymbol{\theta}$ yields zero cost.

The most commonly used constraint is the locally unbiasedness condition, which is also the key assumption behind the derivation of the CR-type bounds [28–30,45,57]. This assumption itself may not be sufficient to obtain operationally saturable bounds, as in principle the region where the use of the local-unbiased estimator makes sense may shrink when taking the asymptotic limit $N \to \infty$ [31–33].

Alternatively, one may follow the so-called "minimax" approach and define a region Θ inside which the true value of $\boldsymbol{\theta}$ is guaranteed to be and then consider the estimator that gives the best results in the most pessimistic scenario, i.e., which minimizes the cost maximized over all $\boldsymbol{\theta} \in \Theta$:

$$\Delta^2 \tilde{\boldsymbol{\theta}}_{\mathrm{minimax}} \equiv \inf_{M_{\boldsymbol{x}}, \tilde{\boldsymbol{\theta}}(\boldsymbol{x})} \sup_{\boldsymbol{\theta} \in \Theta} \int d\boldsymbol{x}\, p_{\boldsymbol{\theta}}(\boldsymbol{x})(\tilde{\boldsymbol{\theta}}(\boldsymbol{x}) - \boldsymbol{\theta})^2. \tag{4}$$

The advantage of the approach is that Θ is fixed while taking the asymptotic limit $N \to \infty$ and hence no region shrinking issues arise. We now proceed to derive an asymptotically saturable lower bound on the above cost in the most fundamental scenario (i) and then contrast it with scenarios (ii), (iii), and (iv).

Derivation of the multiple-phase HL.—Below we present a sketch of the proof. For a more formal derivation, in Sec. A of the Supplemental Material [58]. First, we consider an extension of the model by replacing discrete variables $m_i \in \{0, 1, \ldots, N\}$ with continuous ones $m_i/N \to \mu_i \in [0, 1]$ and the sums with the respective integrals. Note that such an extension may only decrease the minimal achievable cost, as the discrete model may always be arbitrarily well-approximated as a special case of the continuous model. The probe state is now characterized by a p-dimensional wave function $f(\boldsymbol{\mu})$:

$$|\Psi^N_{f,\boldsymbol{\theta}}\rangle = \int_{\forall\ \mu_i \geq 0, \sum_i \mu_i \leq 1} d\boldsymbol{\mu}\, f(\boldsymbol{\mu}) e^{iN\boldsymbol{\mu}\boldsymbol{\theta}} |\mu_1, \mu_2, ..\mu_p\rangle. \tag{5}$$

Next, as we argue in detail in the SM [58], the asymptotic bound for any finite region Θ is equivalent, up to the leading $1/N^2$ order, to the cost when the region is unbounded $\Theta = \mathbb{R}^p$. In the latter case, the problem is covariant with respect to the translation group and the optimal measurement can be restricted to the class of covariant measurements [29] (thanks to the generalization of the Hunt-Stein lemma [60,61] for noncompact groups [52,62,63]) and without loss of generality may be chosen to be the momentum projection measurement $M_{\tilde{\boldsymbol{\theta}}} = |\chi_{\tilde{\boldsymbol{\theta}}}\rangle\langle\chi_{\tilde{\boldsymbol{\theta}}}|$, where

$$|\chi_{\tilde{\boldsymbol{\theta}}}\rangle = \frac{1}{\sqrt{(2\pi/N)^p}} \int d\boldsymbol{\mu}\, e^{iN\boldsymbol{\mu}\tilde{\boldsymbol{\theta}}} |\boldsymbol{\mu}\rangle. \tag{6}$$

Note that we have implicitly replaced the measurement outcomes x with the actual estimated values $\tilde{\boldsymbol{\theta}}$. Minimization of the resulting lower bound over the probe-state wave function $f(\boldsymbol{\theta})$ leads to the following lower bound on the cost $\Delta^2 \tilde{\boldsymbol{\theta}}$:

$$\min_f \int_{\mathbb{R}^p} d\tilde{\boldsymbol{\theta}} |\langle\chi_{\tilde{\boldsymbol{\theta}}}|\Psi^N_{f,\boldsymbol{\theta}}\rangle|^2 (\tilde{\boldsymbol{\theta}} - \boldsymbol{\theta})^2 = \frac{1}{N^2} \min_f \int_{\mathbb{R}^p} d\tilde{\boldsymbol{\theta}} |\hat{f}(\tilde{\boldsymbol{\theta}})|^2 \tilde{\boldsymbol{\theta}}^2, \tag{7}$$

where $\hat{f}$ is the Fourier transform of f and we dropped the irrelevant dependence on $\boldsymbol{\theta}$. Going back to the $\boldsymbol{\mu}$ representation, the minimization problem takes the following form:

$$\frac{1}{N^2} \min_f \int_{\forall\ \mu_i \geq 0, \sum_i \mu_i \leq 1} d\boldsymbol{\mu}\, f^*(\boldsymbol{\mu}) \left(\sum_{k=1}^{p} -\partial^2_{\mu_k} \right) f(\boldsymbol{\mu}),$$

$$\text{with} \quad \int_{\forall\ \mu_i \geq 0, \sum_i \mu_i \leq 1} d\boldsymbol{\mu} |f(\boldsymbol{\mu})|^2 = 1,$$

$$f(\boldsymbol{\mu}) = 0 \quad \text{for } \boldsymbol{\mu} \text{ on the boundary } \left(\mu_i = 0 \vee \sum_i \mu_i = 1 \right). \tag{8}$$

This problem is therefore equivalent to identifying the ground state energy of a quantum particle in a p-dimensional simplex-shaped infinite potential well, which in general has no known analytical solution (apart from specific cases [68–70]). Still, it may be easily bounded from below in following way.

Since the problem enjoys an inherent symmetry with respect to permuting the p "phase" arms (the reference arm is distinguished by the choice of the cost function), and the total number of photons in p "phase" arms is $\leq N$, the expectation value of the number of photons in each single "phase arm" is $\leq N/p$. Now, we will neglect the fact that the distribution of photons in each single arm comes from the multiarm distribution of N photons and keep only the constraint on the photon expectation value. Such a constraint is a weaker one than the original one, but as it refers just to a single "phase" mode, it allows for an effective separation of variables. This allows us to lower-bound the total cost by p times the minimal single-phase estimation cost given the mean number of photons in the mode N/p:

$$\Delta^2\tilde{\boldsymbol{\theta}} \geq p \times \frac{1}{N^2} \min_g \int_0^\infty d\mu g^*(\mu)\left(-\frac{\partial^2}{\partial\mu^2}\right)g(\mu) \quad (9)$$

with constraints

$$g(0)=0, \quad \int_0^\infty d\mu|g(\mu)|^2=1, \quad \int_0^\infty d\mu|g(\mu)|^2 N\mu = N/p. \quad (10)$$

The single mode problem may be solved using the standard Lagrange multiplier method, and we find the solution $g(\mu)$ to be the Airy function $\mathrm{Ai}(\cdot)$ (see also [71,72], where the same solution appeared in a single-phase estimation context), leading to the final bound

$$\Delta^2\tilde{\boldsymbol{\theta}} \geq \frac{p^3}{N^2}\frac{4|A_0|^3}{27} \approx \frac{1.89p^3}{N^2}, \quad (11)$$

where $A_0 \approx -2.34$ is the first zero of $\mathrm{Ai}(\cdot)$. The most important feature of the bound is the p^3 scaling. This bound is valid, even if one considers that the most general adaptive strategy with arbitrary large ancilla is allowed [58].

Note that an analog reasoning could not be performed to bound the QFI as the QFI may be arbitrary large when only the constraint on the *mean* (and not the maximal) number of photons in the sensing arm is imposed and leads to some operationally unjustified claims of sub-Heisenberg estimation strategies [73,74], as discussed in [75].

Comparison of different approaches.—When following the (ii) approach and minimizing the trace of the inverse of the QFI matrix of the output state, one obtains the following bound on the total cost arising from the application of the multiparameter version of the CR bound [55] (see also Ref. [23] for justification of fundamental optimality):

$$\Delta^2\tilde{\boldsymbol{\theta}} \geq \frac{1}{k}\min_{|\Psi^n\rangle}\mathrm{Tr}[F^{-1}(|\Psi^n_{\boldsymbol{\theta}}\rangle)] = \frac{(1+\sqrt{p})^2 p}{4kn^2} \overset{p\gg1}{\approx} \frac{p^2}{4kn^2}, \quad (12)$$

where the optimal input state has the form

$$|\Psi^n\rangle = \beta|n,0,\ldots,0\rangle + \alpha(|0,n,\ldots,0\rangle + \cdots |0,0,\ldots,n\rangle) \quad (13)$$

with $\alpha = 1/\sqrt{p+\sqrt{p}}$, $\beta = 1/\sqrt{1+\sqrt{p}}$. The most visible difference between the two approaches is the scaling of the cost with the number of parameters estimated, p^3 in (i) vs p^2 in (ii). In order to avoid contradiction, this implies that when considering k repetitions in the (ii) scenario, the actual number of repetitions required to saturate the CR bound will in fact need to increase at least linearly with p. This fact lies at the heart of the discrepancy between the claims of Ref. [55] and ours. Interestingly, when considering the Gaussian states only, the QFI based study [76] yields results qualitatively equivalent to ours (p^3 cost scaling for both joint and separate strategies), which should be attributed to the fact that the saturability of the CR-type bounds in Gaussian models is guaranteed already at the single shot level without invoking the multiple repetition argument [29,45].

In scenario (iii), one separately sends N/p photons states into the ith and the reference arm in order to measure a given θ_i phase, using the optimal state for sensing a single completely unknown phase [34–36]:

$$|\Psi_i^{N/p}\rangle = \sqrt{\frac{2}{N/p+2}}\sum_{m=0}^{N/p}\sin\left[\frac{(m+1)\pi}{N/p+2}\right]|N/p-m\rangle_0|m\rangle_i, \quad (14)$$

where $|N/p-m\rangle_0|m\rangle_i$ denotes a state where m photons is sent into the ith arm and $N/p-m$ into the reference arm. The resulting bound on the total variance is therefore lower-bounded by p times the single-phase estimation variance [34–36]:

$$\Delta^2\tilde{\boldsymbol{\theta}} \overset{N/p\gg1}{\gtrsim} p \times \frac{\pi^2}{(N/p)^2} = \frac{p^3\pi^2}{N^2}. \quad (15)$$

We see the same scaling as in the joint-phase estimation protocol (i), which implies that the largest possible gain coming from joint-phase estimation amounts to a constant factor $\leq \pi^2/1.89$. In order to show that the gain over the separate strategy is indeed achievable, we need to find a state wave function $f(\boldsymbol{\mu})$ that manifests an advantage over the separate strategy when plugged into the joint estimation cost formula Eq. (8). We propose a simple ansatz for the structure of the state that satisfies the boundary conditions:

$$f(\boldsymbol{\mu}) \propto \left(\prod_{i=1}^p \mu_i\right)^\alpha \left(1-\sum_{i=1}^p \mu_i\right)^\beta. \quad (16)$$

The minimal cost

$$\Delta^2\tilde{\boldsymbol{\theta}} = \frac{p(1+2\sqrt{p})^2\sqrt{p}(4p+2\sqrt{p}-1)}{(8\sqrt{p}-4)N^2} \overset{p\gg1}{\approx} \frac{2p^3}{N^2} \quad (17)$$

is obtained for $\alpha = 3/2$, $\beta = \sqrt{p}$. For large p, the cost approaches closely the fundamental bound (2 vs 1.89 coefficient), demonstrating that the $\pi^2/2 \approx 4.93$ advantage of joint-phase estimation over separate strategies is achievable. Note that although this result was obtained for the problem with continuous variables μ_i, it may be arbitrarily well approximated within the original discrete model Eq. (2) with increasing N (in the same spirit as discussed in [67] for the single parameter case). See Sec. B of the Supplemental Material [58] for the details of computation, more discussion on the structure of the state, and a numerical investigation of the convergence of the discrete model to the continuous one when N is being increased.

Finally, the optimal strategy in (iv) is to use subsequently p $n00n$ states:

$$|\Psi_i^n\rangle = \frac{1}{\sqrt{2}}(|0\rangle_0|n\rangle_i + |n\rangle_0|0\rangle_i), \tag{18}$$

where each state is designed to sense the ith phase optimally. Since the total number of repetitions is k, each phase will be sensed k/p times and hence the final cost resulting from the application of the CR bound reads

$$\Delta^2\tilde{\boldsymbol{\theta}} \geq p \times \frac{1}{k/p} \times \frac{1}{n^2} = \frac{p^2}{kn^2}. \tag{19}$$

Comparing this result with Eq. (12), we see that joint-phase estimation offers again just a constant factor improvement over separate strategies. This result is different from the claims of [55], where a scaling improvement (p^2 vs p^3) was claimed. Indeed, if n/p photons were used in a single-phase estimation experiment instead of considering k/p uses of the n-photon state, one would obtain the bound on the cost in the form $\Delta^2\tilde{\theta} \geq p \times 1/k \times 1/(n/p)^2 = p^3/kn^2$. This latter calculation, however, does not reflect the cost of the optimal separate strategy in the framework in which there is some fixed number of photons n used in a single experiment and the experiment is repeated k times. The optimal strategy is captured by the former reasoning, leading to the p^2 scaling, as we can always regard this strategy as an equivalent k repetitions of an experiment using a mixed state $\rho^n = (1/p)\sum_{i=1}^p |\Psi_i^n\rangle\langle\Psi_i^n|$, where the factor $(1/p)$ in formula for ρ^n expresses the fact that in each repetition we measure only one parameter with equal probability for each of them. Effectively, k/p repetition for each parameter is performed with n resources each time.

Conclusions and discussion.—In this Letter, we have clarified the relation between formulas for the optimal cost in multiple-phase interferometry obtained within different paradigms involving either fixing the total resources used or the resources used in a single experiment. Doing so, we have shown that within both paradigms joint-phase estimation leads to at most a constant factor improvement over the optimal separate strategies. This constant factor improvement may be attributed to the fact that in the limit of many phases being sensed, the number of photons that need to be sent into the reference arm becomes negligible compared with the total number of photons used, whereas in the separate strategy it effectively consumes half of the resources available [58]. This claim remains valid also in the lossy optical interferometry case (where, however, HS does not occur) as shown in [77].

Note that similar issues regarding the apparent scaling advantage of joint vs separate parameter estimation may arise in other multiparameter estimation problems [22,23] and in order to arrive at operationally meaningful conclusions, one should avoid implicit switching between the (i)–(iv) paradigms and be aware of nontrivial saturability issues when following the QFI based approach.

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