Operational Theories in Phase Space: Toy Model for the Harmonic Oscillator

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We show how to construct general probabilistic theories that contain an energy observable dependent on position and momentum. The construction is in accordance with classical and quantum theory and allows for physical predictions, such as the probability distribution for position, momentum, and energy. We demonstrate the construction by formulating a toy model for the harmonic oscillator that is neither classical nor quantum. The model features a discrete energy spectrum, a ground state with sharp position and momentum, an eigenstate with a nonpositive Wigner function as well as a state that has tunneling properties. The toy model demonstrates that operational theories can be a viable alternative approach for formulating physical theories.

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Introduction.—Various ideas have been proposed to generalize quantum theory. For example, quaternionic [1] and non-Hermitian [2,3] reformulations of quantum theory were investigated, as well as a more general, operational, approach to physical theories. While the former are closely enough related to quantum theory to allow for experimental tests [4], the operational approaches studied to date are bound to black-box-like, device-independent, and finite-dimensional prototheories, which do not describe actual physical systems, but rather describe information-theoretic effects, like, for example, bounds on violations of Bell inequalities [5–7]. The operational approaches we have in mind here are built on the assumption that convexity represents mixtures of states and are collected under the term general probabilistic theories [8]. Within this framework, for example, thermodynamics [9,10], different notions of entropy [11,12], dynamics [13,14], and recently even the operational consequences of gravitational effects [15] were investigated.

The goal of this Letter is to offer a bridge between operational theories in the above sense and extensions of quantum theory mentioned earlier. We accomplish our goal by constructing an operational theory where the energy is linked to position and momentum, analogically to classical and quantum theory. Thus we are able to formulate a toy model for the most archetypal of all physical systems—the harmonic oscillator. Our construction demonstrates that there are alternatives to quantization and operator formalism for building physical theories and that one can use the operational approach for such a construction.

Generalized theories with continuous position and momentum were investigated before [16]. In these theories, states are described by pseudoprobability densities, that is by real valued and possibly nonpositive functions $\rho(q,p)$ such that $\int_{\mathbb{R}} \rho(q,p)dp$ is the probability density for the random variable $\tilde{q}$ corresponding to a position measurement and similarly for momentum. For a function $f(q,p)$, one obtains the mean value of f via $\langle \tilde{f} \rangle_\rho = \int_{\mathbb{R}^2} f(q,p)\rho(q,p)dqdp$. This approach is consistent with classical theory and with the Wigner function formalism of quantum theory [17–24]. Hence, we can compute the mean value $\langle \tilde{H} \rangle_\rho$ of the energy, but we cannot compute the probability distribution of the energy, or, equivalently, we cannot compute its higher moments $\langle \tilde{H}^k \rangle_\rho$. But it is crucial for any theory to allow us to compute the probability distribution of $\tilde{H}$ since using the probability distribution we can, for example, determine whether a state is an eigenstate of the energy observable and, even more important, we can determine the spectrum of the energy observable. This is no small feat, as predicting the spectrum of the hydrogen atom was one of the first results of quantum theory and to this day finding the energy spectrum of various Hamiltonians is an important problem.

The first naïve solution would be to compute $\langle \tilde{H}^2 \rangle_\rho$ as the mean value of $H^2(q,p)$, but even in quantum theory we have $\langle \tilde{H}^2 \rangle_\rho \neq \int_{\mathbb{R}^2} H^2(q,p)\rho(q,p)dqdp$, see Ref. [25]. One can even show that $\langle \tilde{H}^k \rangle_\rho = \int_{\mathbb{R}^2} H^k(q,p)\rho(q,p)dqdp$ only holds in classical theory [26]. Another naïve solution would be to treat energy as an independent variable ϵ and to have pseudoprobability densities of the form $\rho(q,p,\epsilon)$. But then energy is not linked to position and momentum and so we do not follow this approach.

We solve this problem by introducing phase space spectral measures. With these measures we achieve our goal to obtain the probability distribution of $\tilde{H}$ and to describe the energy spectrum of a general system in a similar way to using the spectral measure of an operator in

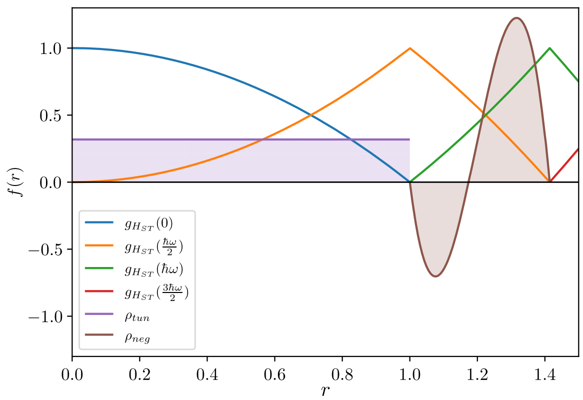

FIG. 1. Phase space spectral measure and phase space states for the sawtooth oscillator as a function of $r = \sqrt{(p^2/\hbar m\omega) + (m\omega q^2/\hbar)}$ in units of $\hbar$. The functions of the phase space spectral measure for the energies $E_0 = 0$, $E_1 = (\hbar\omega/2)$, $E_2 = \hbar\omega$, and $E_3 = (3\hbar\omega/2)$ are shown in blue, orange, green, and red, respectively. The "tunneling" state ρ_{tun} and the eigenstate ρ_{neg} for the energy E_2 are shown in purple and brown, respectively, with the area under the functions filled.

quantum theory. In fact one can express both quantum and classical theories using phase space spectral measures, that is, our general construction includes both theories as special cases. Thus one can also use our construction as a groundwork for finding axioms that would uniquely specify quantum theory among other theories, which would generalize the known results for finite-dimensional systems [27–30].

As a demonstration of the generality of our construction, we present the toy model of the sawtooth oscillator; it is an infinite-dimensional general probabilistic theory with continuous position and momentum and with the energy observable expressed using position and momentum, as in classical and quantum theory. But the sawtooth oscillator is neither classical, nor quantum, nor any transitional form of classical and quantum oscillators. The sawtooth oscillator is characterized by the sawtooth-shaped phase space spectral measure $g_{H_{\mathrm{ST}}}$ and it has an eigenstate ρ_{neg} that is given by a nonpositive pseudoprobability density. We also show that the model exhibits tunneling properties for an appropriately chosen state ρ_{tun}. Both states, as well as the spectral measure are depicted in Fig. 1.

Phase space spectral measures in operational theories.—We work with a general operational theory on phase space where states are given as pseudoprobability densities $\rho(q,p)$ and observables are real-valued phase-space functions $A(q,p)$. The states are hence real-valued phase space functions with normalization $\int_{\mathbb{R}^2} \rho(q,p)dqdp = 1$. Importantly, the function $\rho(q,p)$ is not required to be a probability density and hence may attain negative values for some regions in phase space. The mean value of the random variable $\tilde{A}$ associated to the outcomes of a measurement of the observable A is given by the phase-space integral

$$\langle \tilde{A} \rangle_\rho = \int_{\mathbb{R}^2} A(q,p)\rho(q,p)dqdp. \tag{1}$$

As outlined in the introduction, this formalism is not yet sufficient to describe the probability distribution of $\tilde{A}$. To enable a full probabilistic description, we require that each observable A has associated a phase space spectral measure g_A with the property that the probability for $\tilde{A}$ to attain a value in the set $I \subset \mathbb{R}$ is given by

$$\mathbb{P}(\tilde{A} \in I) = \int_{\mathbb{R}^2} g_A(I;q,p)\rho(q,p)dqdp. \tag{2}$$

Consequently, this measure is a pseudoprobability measure at each phase space point, $g_A(q,p): \mathbb{R} \supset I \mapsto g_A(I;q,p)$. That is, $g_A(\emptyset;q,p) = 0$, $g_A(\mathbb{R};q,p) = 1$, and $g_A(q,p)$ is countably additive on disjoint sets, meaning that for a countable collection $(I_n)_n$ of mutually disjoint subsets of $\mathbb{R}$, we have $g_A(\cup_n I_n;q,p) = \sum_n g_A(I_n;q,p)$. We have $\mathbb{P}_\rho(\tilde{A} \in \mathbb{R}) = 1$ from the normalization of g_A and ρ and for the mean value of $\tilde{A}$ we get

$$\begin{aligned} \langle \tilde{A} \rangle_\rho &= \int_{\mathbb{R}} a\mathbb{P}_\rho(\tilde{A} = a)da, \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}^2} a g_A(I;q,p)\rho(q,p)dqdpda. \end{aligned} \tag{3}$$

To make this equation coincide with Eq. (1) we impose the consistency condition

$$A(q,p) = \int_{\mathbb{R}} a g_A(a;q,p)da. \tag{4}$$

In order for a given ρ to be a phase-space density, we require that the probability densities of position and momentum are given as the marginals $\int_{\mathbb{R}} \rho(q,p)dp$ and $\int_{\mathbb{R}} \rho(q,p)dp$, respectively. This allows us to identify the phase space spectral measure of the position observable q as

$$g_q(I;q,p) = \int_I \delta(q-\xi)d\xi, \tag{5}$$

and analogously for the momentum observable p.

At this point we mention a generic way to construct a phase space spectral measure: let $(t_n)_n$, $t_n: \mathbb{R} \to \mathbb{R}$, be a family of functions and $(a_n)_n$ a corresponding family of eigenvalues such that $\sum_n t_n(x) = 1$ and $\sum_n a_n t_n(x) = x$. Then a phase space spectral measure for A is given by

$$g_A(I;q,p) = \sum_{\tau a_n \in I} t_n[A(q,p)/\tau], \tag{6}$$

where τ is a constant with the same units as A.

Phase space spectral measures in quantum theory.—We illustrate now how phase-space spectral measures are obtained in the phase-space formulation of quantum theory. We start with a short review of the Wigner-Weyl formalism, for a full review see Ref. [25]. One defines the Weyl transform of a self-adjoint operator $\hat{A}$ as

$$\hat{A}^W(q,p) = \int_{\mathbb{R}} e^{-\frac{i}{\hbar}p\cdot y}\left\langle q+\frac{y}{2}\middle|\hat{A}\middle|q-\frac{y}{2}\right\rangle dy, \qquad (7)$$

where $|x\rangle$ denotes the formal eigenvector of the position operator $\hat{q}$ with eigenvalue x. The density operator is represented in phase space using the Wigner transform

$$\rho^W(q,p) = h^{-1}\int_{\mathbb{R}} e^{-\frac{i}{\hbar}p\cdot y}\left\langle q+\frac{y}{2}\middle|\rho\middle|q-\frac{y}{2}\right\rangle dy. \qquad (8)$$

Note that both Eqs. (7) and (8) describe the same transformation up to the factor of h^{-1} and both transformations yield real-valued functions. The Wigner transform of a state is normalized and can attain negative values, but its marginals are the probability density of position and momentum. In the general case of an arbitrary observable $\hat{A}$, its mean value is obtained as

$$\int_{\mathbb{R}^2}\hat{A}^W(q,p)\rho^W(q,p)dqdp = \mathrm{tr}(\hat{A}\rho) = \langle\tilde{A}\rangle_\rho, \qquad (9)$$

where $\tilde{A}$ again denotes the random variable corresponding to a measurement of $\hat{A}$.

In order to obtain the phase space spectral measure of an observable we use that the probability of observing a value in a given set of values I is given by $\mathbb{P}_\rho(\tilde{A}\in I) = \mathrm{tr}(\rho\Pi_I^A)$, where Π_I^A is the spectral measure of $\hat{A}$, that is, $\hat{A} = \int_{\mathbb{R}} a\Pi_a^A da$. This probability $\mathrm{tr}(\rho\Pi_I^A)$ can also be seen as the mean value of the operator Π_I^A and so we can use Eq. (9) to express this mean value in terms of functions on phase space; the Weyl transform of Π_I^A yields the phase space spectral measure $g_A(I;q,p)$, that is, $g_A(I;q,p) = \int_I(\Pi_a^A)^W(q,p)da$.

To illustrate this construction, assume that $\hat{A}$ has a discrete and nondegenerate spectrum with eigenvalues a_n and ρ_n^W the Wigner functions of the corresponding eigenstates. Then

$$g_A(I;q,p) = \sum_{a_n\in I} h\rho_n^W(q,p). \qquad (10)$$

Note that one can also define phase space spectral measures in classical theory, see the Supplemental Material [31].

Time-evolution and positivity in operational theories.—For the purpose of this Letter, we assume that the time evolution in an operational theory is given by the Liouville equation $\dot{\rho} = \{H,\rho\}$, with $\{f,g\} = (\partial f/\partial q)(\partial g/\partial p) - (\partial f/\partial p)(\partial g/\partial q)$ the Poisson bracket and where $H(q,p)$ is the energy observable. In general one can also use other possible Hamiltonians as generators of other translations, but for simplicity we will consider only the time translations.

Note, that in the Wigner-Weyl formalism, the Poisson bracket is replaced by the Moyal bracket [19,24]. The Moyal bracket contains quantum corrections of the order $\hbar^2$ and higher, but it is equal to the Poisson bracket for simple Hamiltonians, such as for the harmonic oscillator. Yet, in a general theory, one could imagine a different dynamical equation, but we choose here an equation that reproduces the situation for the quantum and classical harmonic oscillator.

In order to get a consistent theory we must require that all observable probabilities are positive. Thus if $A(q,p)$ is an observable in our theory with phase space spectral measure $g_A(I;q,p)$, then we must have

$$\mathbb{P}_\rho(\tilde{A}\in I) \geq 0, \quad \text{for all } I\subset\mathbb{R}. \qquad (11)$$

Naïvely one would say that any pseudoprobability density $\rho(q,p)$ that satisfies positivity for all observables should be a valid state. This is not the case, because we also have to require that the time evolution preserves the positivity. This condition is nontrivial, see the Supplemental Material [31] for an example.

Note that the positivity conditions for the position and momentum observables during time evolution are closely related to any linear combination of position and momentum observables being a well-defined observable. This property holds in both classical and quantum theory and is used as a defining property for Wigner representations on discrete phase spaces [32–36].

We define the set of states using the no-restriction hypothesis [37,38] as the largest set of pseudoprobability densities that satisfies the positivity condition for all (future) times. Since the left-hand side of Eq. (11) and the time evolution are linear in ρ, it follows that the set of states is convex. Thus the resulting theory is a general probabilistic theory and the dimension of the theory is infinite, because already the spectral measures for q and p contain infinitely many linearly independent functions, for example, $g_q(I_n;q,p)$ where I_n is any open interval $(n,n+1)$.

The sawtooth oscillator.—The quantum and classical harmonic oscillator have both the same phase space representation $\hat{H}^W = H_c = (p^2/2m) + (m\omega^2/2)q^2$ for the energy observable and in both models the time evolution is given by the Liouville equation, see the Supplemental Material [31] for a short review of the Wigner-Weyl formalism for the quantum harmonic oscillator. The significant difference between both models lies in the phase space spectral measure and the corresponding eigenstates for the energy observable.

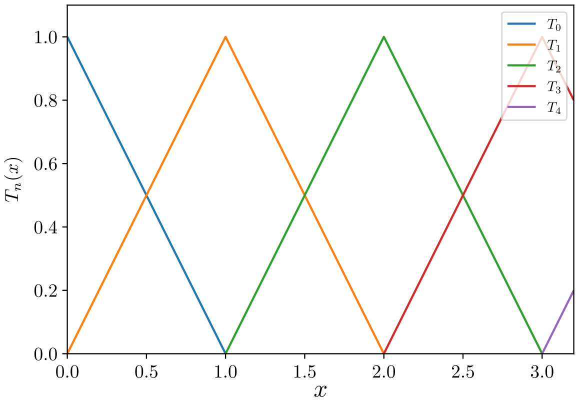

FIG. 2. The functions T_n used in the construction of the phase space spectral measure for the energy observable of the sawtooth oscillator.

We now introduce the sawtooth oscillator as a toy model which follows the same principles but is neither classical nor quantum. That is, the energy coincides with the classical case, $H_{\mathrm{ST}} = H_c$ and position and momentum have the phase space spectral measures given as in Eq. (5). Our sawtooth model is defined according to Eq. (6) with the family of functions $t_n = T_n$, $n \geq 0$, the values $a_n = n$ and $\tau = \frac{1}{2}\hbar\omega$. The function T_n are depicted in Fig. 2 and defined in the Supplemental Material [31]. Hence, our phase space spectral measure reads

$$g_{H_{\mathrm{ST}}}(I; q, p) = \sum_{n=0}^{\infty} \left(\int_I \delta\left[\hbar\omega \frac{n}{2} - \epsilon\right] d\epsilon \right) T_n(r^2), \quad (12)$$

where $r^2 = 2H_c(q,p)/\hbar\omega$. By construction, the energy spectrum of the sawtooth oscillator is discrete, with energies $\hbar\omega(n/2)$ for $n = 0, 1, 2, \ldots$ From $\sum_n T_n(x) = 1$ it follows that the normalization condition is satisfied, $g_{H_{\mathrm{ST}}}(\mathbb{R}; q, p) = \sum_{n=0}^{\infty} T_n(r^2) = 1$ and furthermore, due to $\sum_n n T_n(x) = x$, Eq. (4) is also satisfied,

$$H_{\mathrm{ST}}(q,p) = \sum_{n=0}^{\infty} \hbar\omega \frac{n}{2} T_n(r^2) = \frac{1}{2}\hbar\omega r^2 = H_c(q,p). \quad (13)$$

We mention that, following our general construction scheme, one can in principle apply similar sawtooth constructions to other systems, for example the hydrogen atom with $H_{\mathrm{H}}(q,p) = (|p|^2/2m) - (\kappa/|q|)$. By extending T_0 and T_1 to negative x, $T_0(x) = 1 - x$, and $T_1(x) = x$, we only have to adapt the constant τ, for example, $\tau = -m\kappa^2/2\hbar^2$. Then, Eq. (6) yields a phase space spectral measure for H_{H} with spectrum $0, \tau, 2\tau, \ldots$ However, this construction is rather naïve, for example, the spectrum does not have a lower bound and the construction completely ignores the role of the angular momentum.

Returning to the sawtooth oscillator, it is different from the classical and quantum case as can be demonstrated by considering possible states in the model. First, we consider the state $\rho_0(q,p) = \delta(q)\delta(p)$. Then we have $\mathbb{P}_{\rho_0}(\tilde{H} = 0) = \langle g_{H_{\mathrm{ST}}}(0), \rho_0 \rangle = 1$ and so ρ_0 is a time-invariant eigenstate of the sawtooth oscillator corresponding to zero energy. This state is completely localized in the phase space, that is, the preparation uncertainty of both position and momentum is jointly zero and hence is at variance with quantum theory. Second, the sawtooth oscillator is not classical: for the nonpositive pseudoprobability density ρ_{neg} depicted in Fig. 1 and defined in the Supplemental Material [31], one can show that ρ_{neg} is an eigenstate of the sawtooth oscillator corresponding to the energy $\hbar\omega$. The nonpositivity of ρ_{neg} implies that we cannot jointly measure the position and momentum of ρ_{neg}, hence the state demonstrates nonclassicality in the toy model. We also define the "tunneling" state ρ_{tun}, see Fig. 1 and the Supplemental Material [31]. This state is not an eigenstate of the sawtooth oscillator, it has probability $\frac{1}{2}$ for $\tilde{H} = 0$ and $\tilde{H} = (\hbar\omega/2)$. We discuss now in which sense this state has tunneling behavior.

Quantum tunneling in the sawtooth oscillator.—We use the definition of tunneling presented in Ref. [39]. Usually one says that a quantum particle is tunneling, if the particle crosses a potential barrier that is higher than the energy of the particle. This definition is not applicable to the harmonic oscillator because the potential is not in the form of a barrier. But in the standard scenario, if the particle is able tunnel through a potential barrier, there must be a nonzero probability of observing the particle inside the potential barrier, that is, the wave function of the particle must penetrate into the barrier. One can generalize this statement as follows. The probability of observing a particle in the region inside the barrier is higher than the probability of the particle having energy higher than the energy of the barrier. In this sense a state ρ has tunneling behavior if there is some threshold α such that

$$\mathbb{P}_\rho[V(\tilde{q}) > \alpha] > \mathbb{P}_\rho[\tilde{H} > \alpha], \quad (14)$$

where $V(q)$ is the potential energy and $\tilde{H}$ is again the random variable corresponding to the total energy. According to the classical intuition, one would expect that the potential energy is upper bounded by the total energy of the particle. This does not have to be the case in general, even in the standard formulation of tunneling the particle has nonzero probability of being localized inside the barrier, which is a region where the potential energy is higher than the total energy of the particle. Equation (14) formalizes this using the respective probabilities. When this definition is applied to the quantum harmonic oscillator one finds [39] that the wave function of the ground state has tunneling behavior, which in this case means that the wave function of the ground state spreads more than its energy would allow according to classical intuition.

Returning to the sawtooth oscillator, for the state ρ_0, we have $\mathbb{P}_{\rho_0}[\tilde{H} = 0] = 1$ but $\mathbb{P}_{\rho_0}[V(\tilde{q}) > 0] = 0$ and therefore ρ_0 has no tunneling behavior. Also, for the nonclassical eigenstate ρ_{neg} we find $\mathbb{P}_{\rho_{\mathrm{neg}}}[\tilde{H} = \hbar\omega] = 1$ but $\mathbb{P}_{\rho_{\mathrm{neg}}}[V(q) > \hbar\omega] = 0$. It follows that the eigenstate ρ_{neg} with nonzero energy does not exhibits tunneling behavior, contrary to the quantum case.

In contrast, the state ρ_{tun} has a positive probability density and so one may expect that it must behave according to classical intuition, but this is not true. Because the phase space spectral measure $g_{H_{\mathrm{ST}}}$ is different from the classical one, positivity of ρ_{tun} does not imply classicality. For $0 < \alpha < (\hbar\omega/2)$ we have $\mathbb{P}_{\rho_{\mathrm{tun}}}[\tilde{H} > \alpha] = \frac{1}{2}$. Since $V(q) = \frac{1}{2}m\omega^2 q^2$, it follows that $\mathbb{P}_{\rho_{\mathrm{tun}}}[V(\tilde{q}) > 0] = 1 - \mathbb{P}_{\rho_{\mathrm{tun}}}[\tilde{q}^2 = 0] = 1$ and so for a sufficiently small $\alpha > 0$ we must have $\mathbb{P}_{\rho_{\mathrm{tun}}}[V(\tilde{q}) > \alpha] > \frac{1}{2}$, see the Supplemental Material [31] for details. Hence ρ_{tun} exhibits tunneling behavior for a sufficiently small threshold α.

Conclusions.—The main conceptual result of this Letter is that the constructed phase space spectral measures are key to an operational approach for working with energy and other observables that depend on position and momentum. Using this result, we have essentially formulated all possible theories of the harmonic oscillator, up to introducing exotic dynamics beyond what we observe in classical and quantum theory. One can clearly extend the construction to the case of multiple particles by adding additional pairs of variables q_i, p_i. Using this approach, one can construct a field theory with exotic properties, since the sawtooth oscillator has a discrete energy spectrum, but the ground state has zero energy. In general, the phase space spectral measure of the energy does not have to have an equidistant distribution of the energy levels, therefore one can obtain a field theory without well-defined photons, which may give rise to the prediction of new physical effects.

As already pointed out, one can also formulate theory of hydrogen atom analogically to the sawtooth oscillator, but the results are not fully satisfying. Apart from the aforementioned problems with the energy spectrum and lack of angular momentum, one also has to consider the time evolution, since in quantum mechanics the time evolution of the hydrogen atom is no longer given by the Liouville equation. Hence one needs to find some generalization of the Schrödinger equation and some other phase space spectral measure with more physical spectrum, but then one also needs to verify whether these choices are consistent and whether they produce an operational theory with satisfactory physical properties, for example the phase space spectral measure should be stationary. Further discussion of these problems is left for future work.

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[1] S. Adler, *Quaternionic Quantum Mechanics and Quantum Fields*, International Series of Monographs on Physics (Oxford University Press, New York, 1995).

[2] F. Bagarello, J. Gazeau, F. Szafraniec, and M. Znojil, *Non-Selfadjoint Operators in Quantum Physics: Mathematical Aspects* (Wiley, New York, 2015).

[3] C. M. Bender, D. C. Brody, and M. P. Müller, Hamiltonian for the Zeros of the Riemann Zeta Function, Phys. Rev. Lett. **118**, 130201 (2017).

[4] L. M. Procopio, L. A. Rozema, Z. J. Wong, D. R. Hamel, K. O'Brien, X. Zhang, B. Dakić, and P. Walther, Single-photon test of hyper-complex quantum theories using a metamaterial, Nat. Commun. **8**, 15044 (2017).

[5] H. S. Poh, S. K. Joshi, A. Cerè, A. Cabello, and C. Kurtsiefer, Approaching Tsirelson's Bound in a Photon Pair Experiment, Phys. Rev. Lett. **115**, 180408 (2015).

[6] M. D. Mazurek, M. F. Pusey, K. J. Resch, and R. W. Spekkens, Experimentally bounding deviations from quantum theory in the landscape of generalized probabilistic theories, PRX Quantum **2**, 020302 (2021).

[7] M. Weilenmann and R. Colbeck, Self-Testing of Physical Theories, or, Is Quantum Theory Optimal with Respect to Some Information-Processing Task?, Phys. Rev. Lett. **125**, 060406 (2020).

[8] M. P. Müller, Probabilistic theories and reconstructions of quantum theory, SciPost Phys. Lect. Notes 28 (2021).

[9] G. Chiribella and C. M. Scandolo, Entanglement and thermodynamics in general probabilistic theories, New J. Phys. **17**, 103027 (2015).

[10] M. Krumm, H. Barnum, J. Barrett, and M. P. Müller, Thermodynamics and the structure of quantum theory, New J. Phys. **19**, 043025 (2017).

[11] G. Kimura, J. Ishiguro, and M. Fukui, Entropies in general probabilistic theories and their application to the Holevo bound, Phys. Rev. A **94**, 042113 (2016).

[12] H. Barnum, J. Barrett, L. O. Clark, M. Leifer, R. Spekkens, N. Stepanik, A. Wilce, and R. Wilke, Entropy and information causality in general probabilistic theories, New J. Phys. **14**, 129401 (2012).

[13] D. Gross, M. P. Müller, R. Colbeck, and O. C. O. Dahlsten, All Reversible Dynamics in Maximally Nonlocal Theories are Trivial, Phys. Rev. Lett. **104**, 080402 (2010).

[14] D. Branford, O. C. O. Dahlsten, and A. J. P. Garner, On defining the Hamiltonian beyond quantum theory, Found. Phys. **48**, 982 (2018).

[15] T. D. Galley, F. Giacomini, and J. H. Selby, A no-go theorem on the nature of the gravitational field beyond quantum theory, arXiv:2012.01441.

[16] R. W. Spekkens, Quasi-quantization: Classical statistical theories with an epistemic restriction, Fundam. Theor. Phys. **181**, 83 (2014).

[17] E. Wigner, On the quantum correction for thermodynamic equilibrium, Phys. Rev. **40**, 749 (1932).

[18] H. Groenewold, On the principles of elementary quantum mechanics, Physica (Amsterdam) **12**, 405 (1946).

[19] J. E. Moyal, Quantum mechanics as a statistical theory, Math. Proc. Cambridge Philos. Soc. **45**, 99 (1949).

[20] G. Rosen, Mathematical formalism for probabilistic dynamical theories, J. Franklin Inst. **279**, 457 (1965).

[21] L. Cohen, Generalized phase-space distribution functions, J. Math. Phys. (N.Y.) **7**, 781 (1966).

[22] H. Bergeron, From classical to quantum mechanics: "How to translate physical ideas into mathematical language", J. Math. Phys. (N.Y.) **42**, 3983 (2001).

[23] M. Hillery, R. O'Connell, M. Scully, and E. Wigner, Distribution functions in physics: Fundamentals, Phys. Rep. **106**, 121 (1984).

[24] D. B. Fairlie, The formulation of quantum mechanics in terms of phase space functions, Math. Proc. Cambridge Philos. Soc. **60**, 581 (1964).

[25] W. B. Case, Wigner functions and Weyl transforms for pedestrians, Am. J. Phys. **76**, 937 (2008).

[26] M. Kleinmann and M. Plávala, Analysis of operational theories in phase space (to be published).

[27] L. Hardy, Quantum theory from five reasonable axioms, arXiv:quant-ph/0101012.

[28] L. Masanes and M. P. Müller, A derivation of quantum theory from physical requirements, New J. Phys. **13**, 063001 (2011).

[29] G. Chiribella, G. M. D'Ariano, and P. Perinotti, Informational derivation of quantum theory, Phys. Rev. A **84**, 012311 (2011).

[30] A. Wilce, A royal road to quantum theory (or thereabouts), Entropy **20**, 227 (2018).

[31] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.128.040405 for more details.

[32] W. K. Wootters, A Wigner-function formulation of finite-state quantum mechanics, Ann. Phys. (N.Y.) **176**, 1 (1987).

[33] K. S. Gibbons, M. J. Hoffman, and W. K. Wootters, Discrete phase space based on finite fields, Phys. Rev. A **70**, 062101 (2004).

[34] D. Gross, Hudson's theorem for finite-dimensional quantum systems, J. Math. Phys. (N.Y.) **47**, 122107 (2006).

[35] J. B. DeBrota and B. C. Stacey, Discrete Wigner functions from informationally complete quantum measurements, Phys. Rev. A **102**, 032221 (2020).

[36] R. Schwonnek and R. F. Werner, The Wigner distribution of n arbitrary observables, J. Math. Phys. (N.Y.) **61**, 082103 (2020).

[37] G. Chiribella, G. M. D'Ariano, and P. Perinotti, Probabilistic theories with purification, Phys. Rev. A **81**, 062348 (2010).

[38] S. N. Filippov, S. Gudder, T. Heinosaari, and L. Leppäjärvi, Operational restrictions in general probabilistic theories, Found. Phys. **50**, 850 (2020).

[39] Y. L. Lin and O. C. O. Dahlsten, Necessity of negative Wigner function for tunneling, Phys. Rev. A **102**, 062210 (2020).