Editors' Suggestion

Geometric Test for Topological States of Matter

S. Klevtsov[1] and D. Zvonkine[2,3]

[1]*IRMA, Université de Strasbourg, UMR 7501, 7 rue René Descartes, 67084 Strasbourg, France*
[2]*CNRS, Université de Versailles St-Quentin (Paris-Saclay), 78000 Versailles, France*
[3]*Cluster Geometry Laboratory, HSE, 11 Pokrovsky bd., 109028 Moscow, Russia*

(Received 31 May 2021; revised 1 September 2021; accepted 21 December 2021; published 21 January 2022)

We generalize the flux insertion argument due to Laughlin, Niu-Thouless-Tao-Wu, and Avron-Seiler-Zograf to the case of fractional quantum Hall states on a higher-genus surface. We propose this setting as a test to characterize the robustness, or topologicity, of the quantum state of matter and apply our test to the Laughlin states. Laughlin states form a vector bundle, the Laughlin bundle, over the Jacobian—the space of Aharonov-Bohm fluxes through the holes of the surface. The rank of the Laughlin bundle is the degeneracy of Laughlin states or, in the presence of quasiholes, the dimension of the corresponding full many-body Hilbert space; its slope, which is the first Chern class divided by the rank, is the Hall conductance. We compute the rank and all the Chern classes of Laughlin bundles for any genus and any number of quasiholes, settling, in particular, the Wen-Niu conjecture. Then we show that Laughlin bundles with nonlocalized quasiholes are not projectively flat and that the Hall current is precisely quantized only for the states with localized quasiholes. Hence our test distinguishes these states from the full many-body Hilbert space.

DOI: 10.1103/PhysRevLett.128.036602

Introduction.—Topological states describe special phases of strongly correlated quantum matter arising at low temperatures, which exhibit certain remarkable properties, such as precise quantization phenomena in materials with impurities, fractional and non-Abelian statistics, and ground state degeneracy robust under local perturbations. These unusual properties make the topological states suitable for a range of applications from quantum metrology to fault-tolerant quantum computing.

In this Letter, we suggest a concrete criterion that distinguishes between topological and nontopological states of matter. The criterion is of geometric nature and applies to situations where the ground state of the system is degenerate.

Best known examples of topological states of matter include superconductors, spin liquids, quantum Hall states, etc. We focus here on the fractional quantum Hall effect, where explicitly defined trial states are available for investigation, although the criterion is potentially applicable to a broader class of situations.

Projective flatness test for the ground state bundle.—Here we describe our test in a general setting. We consider the situation, when the ground state is degenerate and separated by a gap from the rest of the spectrum, and depends continuously on n classical parameters forming an n-dimensional manifold M. We assume that the degeneracy r of the ground state is constant over the whole parameter manifold. Our ground states thus form a rank-r Hermitian vector bundle V over the parameter space M. The adiabatic theorem applies to determine the Berry connection, whose curvature $\mathcal{R}$ is in general non-Abelian, as, e.g., in Ref. [1]. Then a generic bundle V of quantum states can be assigned a set of cohomology classes, represented by the traces of powers of the Berry curvature $\mathrm{ch}_m(V) = [(-1)^m/(2\pi i)^m m!]\mathrm{tr}\mathcal{R}^m$, called the Chern characters. In particular $\mathrm{ch}_0(V) = r$ is the degeneracy, $\mathrm{ch}_1(V) = c_1(V)$ the first Chern class and ch_i is a $2i$-cohomology class on M, $i = 0, \ldots, [n/2]$ (Chern characters are a more convenient notion than the more familiar Chern classes $c_i(V)$. They are polynomial functions of the latter: $\mathrm{ch}_0(V) = r$, $\mathrm{ch}_1(V) = c_1(V)$, $\mathrm{ch}_2(V) = \frac{1}{2}(c_1^2 - c_2)$, etc. In general, the full Chern class is recovered from the Chern characters by the formula $c(V) = c_0(V) + c_1(V) + \cdots + c_m(V) = \exp \sum_{i\geq 1}(-1)^{i-1}(i-1)!\mathrm{ch}_i(V)$).

Furthermore, the bundles of the topological states are characterized by their robustness against perturbations. We formalize this condition as the requirement that the adiabatic transport of the quantum states of V along a path γ in the parameter space M is independent of continuous deformations of the path and depends only on its topology, possibly up to a $U(1)$ Berry phase. In more precise terms, we require that the adiabatic transport defines a projectively flat connection on V.

For example, in Fig. 1, the adiabatic transport along the curves γ_1 and γ_2, which can be continuously deformed into each other, would have the holonomies equivalent up to the $U(1)$ phase, while the transport along γ_3 could yield an *a priori* different holonomy.

Now, by a standard result on complex vector bundles [2], if V is projectively flat then its total Chern character $\mathrm{ch}(V) = \sum_{i=0}^{n} \mathrm{ch}_i(V)$ simplifies and is given by

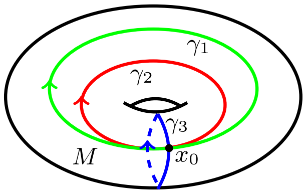

FIG. 1. The adiabatic transport of quantum states in the parameter space M along the curves γ_1, γ_2, γ_3 starting and ending at the same point x_0.

$$\mathrm{ch}(V) = re^{\frac{1}{r}c_1(V)}.$$

In other words, the higher Chern characters of a projectively flat bundle V are essentially powers of its first Chern class:

$$\mathrm{ch}_i(V) = \frac{[c_1(V)]^i}{i!r^{i-1}}, \qquad 1 \le i \le \left\lfloor \frac{n}{2} \right\rfloor. \tag{1}$$

We claim that these relations can be used to test the topologicity for quantum states of matter. Here is the strategy as to how the geometric test can be implemented: (i) Given a parameter space M, and a bundle of quantum states V, compute the non-Abelian Berry curvature $\mathcal{R}$, (ii) take its mth power $\mathcal{R}^m$, (iii) compute its trace $\mathrm{tr}\mathcal{R}^m$, and (iv) check whether $\mathrm{tr}\mathcal{R}^m = (1/m!r^{m-1})(\mathrm{tr}\mathcal{R})^m$ in cohomology. If this relation does not hold, then the quantum states in V are not topological in the sense above.

If $r = 1$, the ground state is nondegenerate and the vector bundle becomes a line bundle, for which any connection is projectively flat. Also, parameter spaces of the real dimension $\dim M \le 3$ do not support higher Chern characters. Therefore the projective flatness test only gives nontrivial results when $r > 1$ and $\dim M \ge 4$. In particular, for Laughlin states the crucial observables, that allow one to distinguish between topological and nontopological states of matter, appear in genus $g \ge 2$.

Laughlin states on Riemann surfaces.—A Laughlin trial state, Ref. [3], describes the fractional quantum Hall effect for filling fractions of the form $1/\beta$, where β is an odd integer. First define an N-particle Laughlin state with p quasiholes on the Riemann sphere ($g = 0$) for any positive integer β as follows:

$$\Psi_L = P(z_1, z_2, \ldots, z_N) \prod_{1 \le n < m \le N} (z_n - z_m)^\beta; \tag{2}$$

here P is a completely symmetric polynomial in N variables $z_1, \ldots, z_N \in \mathbb{C}$, of degree at most p in each z_n. Thus we do not restrict to the fermionic case and consider the bosonic states (even β) as well. Besides the (anti-)symmetry for (odd) even β, the two other defining properties of Ψ_L are the vanishing on the diagonal $\Delta = \cup_{n<m} \{z_n = z_m\}$ to the order β and total degree in each variable z_n being equal to the magnetic flux, $N_\phi = p + \beta(N - 1)$. The vanishing on the diagonal is a simplified model for the Coulomb interactions between particles. The total degree condition ensures that each electron is on the lowest Landau level for the magnetic field with flux N_ϕ.

The full many-body Hilbert space of functions of the form (2) has dimension $\binom{N+p}{p}$, by the number of linearly independent polynomials P. It has a special one-dimensional subspace of states with p quasiholes localized at positions $w_1, w_2, \ldots, w_p$:

$$\Psi_L = \prod_{i=1}^{p} \prod_{n=1}^{N} (z_n - w_i) \prod_{n<m}^{N} (z_n - z_m)^\beta. \tag{3}$$

As long as the positions of the quasiholes are fixed, the states above do not depend on any continuous parameters.

The standard way to bring a parameter space into play in QHE is to consider Laughlin states on a Riemann surface Σ of genus $g > 0$ [4–8]. The definition here mimics the one given above for the sphere [9]. Namely, we require the (anti-)symmetry for (odd) even β and vanishing on the diagonal $\Delta = \cup_{n<m} \{z_n = z_m\}$ to the order β. The analog of being a degree-N_ϕ polynomial on a compact Riemann surface is the condition that Ψ_L is a section of a degree-N_ϕ holomorphic line bundle L. Now, the latter come with a natural parameter space: he moduli space of degree-N_ϕ line bundles is the Picard variety $\mathrm{Pic}^{N_\phi}(\Sigma)$ isomorphic to a g-dimensional complex torus. These inequivalent configurations of the magnetic field of flux N_ϕ through the surface are obtained by applying the Aharonov-Bohm solenoid fluxes through the $2g$ cycles on the surface, $\{\phi_a\}_{a=1,\ldots,2g} \in [0, 2\pi]^{2g}$, see Fig. 2.

This is precisely the setting of a higher genus surface considered in the integer QHE case in Ref. [8], and generalizing Laughlin's gauge argument [10] (for the case of a torus see Refs. [5–7]). Following the standard argument of these references, when changing the flux through the cycle b of the surface, $\phi_b = -V_b t$ adiabatically with time t, the Hall current through the cycle a equals V_b times the Hall conductance,

$$I_a = (\sigma_H)_{ab} V_b,$$

which is the first Chern class of the Laughlin bundle over Pic^{N_ϕ} divided by its rank in case of degeneracy,

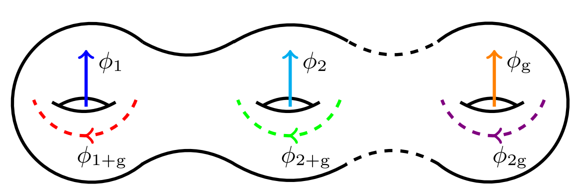

FIG. 2. *AB* phases on the genus-g Riemann surface.

$$\sigma_H = \sum_{a,b} (\sigma_H)_{ab} d\phi_a \wedge d\phi_b = \frac{c_1(V)}{r}. \tag{4}$$

Thus we need to compute the rank and the first Chern class of the Laughlin bundle.

Quantum optimal packing problem.—We begin by computing the rank of the Laughlin bundle, i.e., the degeneracy of states (3) on a genus-g Riemann surface. The space of sections of a holomorphic line bundle S over a complex manifold X is denoted by $H^0(X, S)$ and its dimension by $h^0(X, L)$. The main tool for computing $h^0(X, S)$ is the Hirzebruch-Riemann-Roch formula

$$\sum_{i=0}^{\dim_{\mathbb{C}} X} (-1)^i h^i(X, S) = \int_X e^{c_1(S)} \mathrm{td}(X),$$

where $h^i(X, S)$ are the dimensions of cohomology groups of S, $c_1(S)$ is the first Chern class of S, and $\mathrm{td}(X)$ is the Todd class of X. In the case of Laughlin states, one can use the Kodaira vanishing theorem to show that the higher cohomology groups in $H^i(X, S)$, $i > 0$ vanish and what remains in the left-hand side is just $h^0(L, X)$, which is the degeneracy of ground states:

$$r = \int_X e^{c_1(S)} \mathrm{td}(X). \tag{5}$$

We do not recall the general definition of the Todd class here, but an expression for $\mathrm{td}(X)$ in the case of Laughlin states is given below in Eq. (7). Let us describe X and S in this case.

As a multiparticle wave function, the Laughlin state is naturally a function on the Cartesian product of N copies of the Riemann surface Σ^N. More precisely it is a section of the line bundle $L^{\boxtimes N} = \pi_1^* L \otimes \cdots \otimes \pi_N^* L$ over Σ^N, which is (anti-)symmetric for (odd) even β and vanishes on the diagonal to the order β. Twisting $L^{\boxtimes N}$ by the divisor β times the diagonal $\Delta = \cup_{n<m}^N \{z_n = z_m\}$ we reinterpret a Laughlin state as a completely symmetric section of the bundle $S = L^{\boxtimes N}(-\beta\Delta)$ over the Nth symmetric power of the Riemann surface $X = S^N\Sigma = \Sigma^N / S_N$. Note that $S^N\Sigma$ is a smooth complex manifold: indeed, locally unordered sets $\{z_1, \ldots, z_N\}$ of N complex numbers are parametrized by the coefficients of the polynomial $(z - z_1) \cdots (z - z_N)$.

For $N \geq 2g - 1$, there is a particularly useful description of the Nth symmetric power $X = S^N\Sigma$ of a Riemann surface as a holomorphic bundle of projective spaces $\mathbb{P}^{N-g}$ over the Picard group $\mathrm{Pic}^N(\Sigma)$ of the surface, [11] (Sec. 3.a), see Fig. 3 (The Picard group is a g-dimensional complex torus isomorphic to the Jacobian of Σ, but the isomorphism is not canonical. We use Pic^N rather than the Jacobian because it makes the map $X \to \mathrm{Pic}^N(\Sigma)$ canonical and also because it allows us to distinguish Pic^N from *another* complex torus responsible for the AB fluxes, which will appear in the next section). $\mathrm{Pic}^N(\Sigma)$ carries a natural (1,1)-cohomology class Θ, Poincaré dual to the theta divisor, and X carries another cohomology class ξ, dual to the divisor of configurations of N points where at least one point coincides with a fixed point on Σ. The class ξ restricts to the hyperplane class in each fiber $\mathbb{P}^{N-g}$.

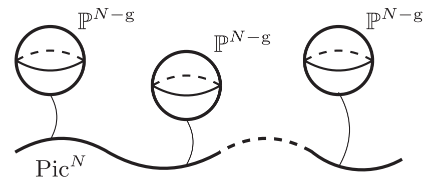

FIG. 3. Representation of the Nth symmetric power of the Riemann surface as projective spaces $\mathbb{P}^{N-g}$ fibered over the Picard variety.

We adopt the notation Θ_{conf} for the class Θ in $\mathrm{Pic}^N(\Sigma)$, arising from the configurations of N points on the surface in order to distinguish it from another class Θ in $\mathrm{Pic}^{N_\phi}(\Sigma)$ on the space of AB fluxes. The first Chern class of the line bundle $S = L^{\boxtimes N}(-\beta\Delta)$ over $X = S^N\Sigma$ then equals

$$c_1(S) = \beta\Theta_{\mathrm{conf}} + p\xi, \tag{6}$$

where $p = N_\phi - \beta(N + g - 1)$, and all three classes Θ_{conf}, ξ, and $c_1(S)$ lie in $H^2(X, \mathbb{Z})$. Further, the Todd class of X reads

$$\mathrm{td}(X) = (\mathrm{td}\xi)^{N-g-1} \exp\left(\Theta_{\mathrm{conf}} \frac{\mathrm{td}\xi - 1 - \xi}{\xi}\right), \tag{7}$$

where $\mathrm{td}\xi = [\xi/(1 - e^{-\xi})]$. This is a mixed degree even cohomology class spanning all even degrees from 0 to $2\dim_{\mathbb{C}} X$. Plugging this into Eq. (5) we arrive at the following formula for the dimension of the vector space of Laughlin states, $r = r(N, \beta, p, g)$, and consequently for the rank of the Laughlin bundle:

$$r = \sum_{k=0}^{g} \binom{g}{k} \binom{N - g + p}{k - g + p} \cdot \beta^k, \tag{8}$$

with the convention $\binom{a}{b} = 0$ if $b < 0$. Since both $c_1(S)$ and $\mathrm{td}(X)$ are expressed in terms of ξ and Θ_{conf} and that the intersection numbers of these two classes are known, deducing the formula for the rank r is a purely combinatorial problem, but it is not entirely trivial; actually, the computation involves the Lagrange inversion theorem [12].

It follows from Eq. (8) that there are no Laughlin states for $p < 0$, in other words, for a given filling fraction $1/\beta$ and magnetic flux N_ϕ, the configuration of $N = N_{\max}$ particles, where

$$N_{\max} = \left\lfloor \frac{N_\phi}{\beta} \right\rfloor + 1 - g \tag{9}$$

is optimally packed in the sense that no extra particles can be added on the lowest Landau level. Moreover, the state with $N = N_{\max}$ when N_ϕ is divisible by β, is incompressible, and this result demonstrates the Wen-Zee shift formula Ref. [13]. In the latter case, the degeneracy of the Laughlin states

$$r|_{p=0} = \beta^g$$

is purely topological, that is, independent of N. Thus our result establishes the topological degeneracy of Haldane-Rezayi and Wen-Niu [4,14]. The explicit expression for the Laughlin states in the optimally packed configurations we refer to Ref. [9].

For the case $p > 0$, formula (8) computes the dimension of the full many-body Hilbert spaces of Laughlin states, corresponding to suboptimally packed configurations, generalizing the $\binom{N+p}{p}$ degeneracy of Eq. (2) on the sphere [15], [Eq. (3)] and [16] [Eq. (6)] on the torus.

Now, the case of $p > 0$ quasiholes localized at fixed points $w_1, \ldots, w_p$ corresponds to a subspace of the full many-body Hilbert space analogous to the one in Eq. (3), but also degenerate for $g > 0$. The same calculation goes through in this case with the replacement of the original line bundle L by the line bundle $L(-w_1 - \cdots - w_p)$. Thus plugging $N_\phi \to N_\phi - p$ we obtain that the dimension of this subspace again equals β^g.

Chern characters of the Laughlin bundle—Turning on the solenoid AB fluxes, see Fig. 2, brings in the parameter space $M = \mathrm{Pic}^{N_\phi}(\Sigma)$ of real dimension $2g$: the number of independent fluxes through cycles on the surface. In this setting, the Hall conductance σ_H for IQHE was computed in Ref. [8] as the first Chern class on M. Here we generalize this result to the FQHE.

Again we define Laughlin states as sections of a line bundle $S = L^{\boxtimes N}(-\beta\Delta)$, but now over the product space $M \times X$, where once more, $X = S^N\Sigma$ is viewed as a $\mathbb{P}^{N-g}$ bundle over the Picard variety $\mathrm{Pic}^N(\Sigma)$. Since as manifolds both $M = \mathrm{Pic}^{N_\phi}(\Sigma)$ and $\mathrm{Pic}^N(\Sigma)$ are isomorphic to the same $2g$-dimensional torus, we distinguish them in what follows by putting prime on the object related to the latter.

In order to describe how the AB fluxes couple to the electronic states, we makes use of the canonical basis of one-cycles on Σ, which are g pairs of simple loops (γ_a, γ_{g+a}) for each handle of the surface. Let (α_a, β_a) be the corresponding dual basis of harmonic one-forms on Σ. Then insertion of AB fluxes ϕ_a, ϕ_b leads to the change of the one-particle $U(1)$ electromagnetic connection $\nabla_z \to \nabla_z + \sum_{a=1}^g (\phi_a \alpha_a + \phi_{a+g}\beta_a)$, [8], [Eq. (3)]. Taking the trivial connection along $\mathrm{Pic}^{N_\phi}(\Sigma)$, $\nabla_\phi = \sum_a (d\phi_a \partial_{\phi_a} + d\phi_{a+g}\partial_{\phi_{a+g}})$ and summing over all N particles, we arrive at the expression for the first Chern class generalizing Eq. (6),

$$c_1(S) = \beta\Theta_{\mathrm{conf}} + p\xi + \sum_{a=1}^{g} (d\phi_a \wedge d\phi'_{a+g} + d\phi'_a \wedge d\phi_{a+g}),$$

where the novel term is a two-from with one component along $M = \mathrm{Pic}^{N_\phi}(\Sigma)$ and the other, primed, component along $\mathrm{Pic}^N(\Sigma)$.

The Laughlin bundle V is a rank-r vector bundle over $M = \mathrm{Pic}^{N_\phi}(\Sigma)$, whose fibers are vector spaces of Laughlin states. To compute its Chern characters we apply the Grothendieck-Riemann-Roch theorem,

$$\mathrm{ch}(V) = \int_X e^{c_1(S)} \mathrm{td}(X),$$

and the result of the integration over the fibers X in the product $X \times M$ is the Chern character of V on M. Performing the integration we obtain

$$\mathrm{ch}_m(V) = \sum_{k=m}^{g} \binom{g-m}{k-m} \binom{N-g+p}{k-g+p} \beta^{k-m} \frac{\Theta_{\mathrm{flux}}^m}{m!}. \tag{10}$$

As a consistency check, for $m = 0$ we do recover Eq. (8) for the rank. The theta class Θ_{flux} can be represented as $\Theta_{\mathrm{flux}} = \sum_{a=1}^g d\phi_a \wedge d\phi_{a+g}$.

For $p = 0$ (no quasiholes), only last term on the rhs of Eq. (10) remains, and the total Chern character becomes

$$\mathrm{ch}(V) = r e^{\frac{1}{\beta}\Theta_{\mathrm{flux}}}, \qquad r = \beta^g. \tag{11}$$

In particular, the first Chern class is given by

$$c_1(V) = \beta^{g-1}\Theta_{\mathrm{flux}}, \tag{12}$$

and Eq. (11) is consistent with Eq. (1) for projectively flat bundles, confirming the topological nature of FQHE states. Equation (11) stands unchanged for the case of p localized quasiholes, where the degeneracy is still β^g, as discussed in the end of Sec. 4.

However, when we have $p \geq 1$ nonlocalized quasiholes, the relation (1) does not hold as soon as $g \geq 2$. We can conclude that in this case the Laughlin bundle is definitely not projectively flat.

We also note that Eq. (10) remains valid for any space of ground states topologically equivalent to the Laughlin states, i.e., which can be obtained from the latter by a continuous deformation, preserving the ground state degeneracy and the gap. Under a deformation like that the vector bundle of ground states changes continuously and its characteristic classes remain the same.

Hall conductance and the projective flatness test.—Following Refs. [5–7] we now consider the charge transport on our surface for $g \geq 1$ [8]. If the Laughlin state is completely filled (9) or else if the p quasiholes are completely localized, the topological contribution to the Hall current reads $I_a = \sigma_{a,a+g} V_{a+g}$. Thus changing the AB flux through the cycle $a+g$ of the surface induces the Hall current in the dual cycle a controlled by the precisely quantized Hall conductance two-form, given by Eq. (4),

$$\sigma_H = \frac{1}{\beta} \Theta_{\text{flux}}.$$

We stress that this equation remains exact for any number of particles N. Next we increase the magnetic flux N_ϕ until we find ourselves in the situation when there are more quasiholes than the number of impurities that can localize them. For p nonlocalized fluxes of the magnetic field, we are in the setting of the full many-body Hilbert space (2), whose dimension, i.e., the rank of the corresponding Laughlin bundle, is given by Eq. (8). Its first Chern class reads

$$c_1(V) = \sum_{k=1}^{g} \binom{g-1}{k-1} \binom{N-g+p}{k-g+p} \beta^{k-1} \Theta_{\text{flux}}.$$

These states no longer pass the projective flatness test, and one of the immediate most striking consequences is that the Hall current is no longer precisely quantized. Indeed, taking the large N asymptotics of the rank and the first Chern class, while keeping p and g fixed, we arrive at the following asymptotic expression for the Hall conductance:

$$\sigma_H = \left[\frac{1}{\beta} - \frac{p}{\beta^2 g N} + \mathcal{O}(1/N^2)\right] \Theta_{\text{flux}}. \tag{13}$$

We see that as the number p of quasiholes increases, the Hall conductance starts to decrease from its precise quantized value $1/\beta$. This is consistent with the fact that increasing p corresponds to increasing the flux of the magnetic field. Equation (13) generalizes the FQHE result, see, e.g., Ref. [17] [Eq. (4.72)] for the $g=1$ case.

Discussion.—The Hall conductance is the trace of the curvature of the adiabatic Berry connection $M = \text{Pic}^{N_\phi}(\Sigma)$ and our computation only determines the cohomology class represented by this two-form, but not form itself. The L^2 adiabatic connection requires a computation of N-fold L^2-normalization integrals, see, e.g., Ref. [18]. In the IQHE [8,19] the adiabatic curvature is indeed exponentially close to the one in Eq. (12) for large N. We expect the same effect to hold for $\beta > 1$, see, e.g., Refs. [18,20,21] for the case of the torus, although this point definitely deserves further investigation.

We have seen that the bundles of quantum states passing the projective flatness test do turn out to be sufficiently robust and thus warrant the label of topological states of matter. It would be interesting to apply our test to other FQHE states [22–26] and other parameter spaces [27–34], such as the moduli spaces of complex structures on Σ, where projective flatness has been conjectured to hold for some of the states in FQHE [35]. In the case of the parameter spaces being the space of positions of the quasiholes, our test can be applied to the question of topological braiding [36–38]. Another interesting application of the higher Chern classes is that they can potentially be used as novel indices to distinguish between phases of quantum matter [39]. We further note that the importance of the projective flatness of the quantum bundles for the consistency of the general quantization procedure was emphasized in Ref. [40], where its relevance for conformal field theories has also been discussed.

We thank Y. Avron for careful reading and detailed comments on the manuscript, P. Wiegmann for useful discussions, and anonymous referees for useful remarks. The work of S. K. was partly supported by the Initiative d'excellence program and the University of Strasbourg Institute for Advanced Study Fellowship of the University of Strasbourg, and the ANR-20-CE40-0017 grant. The work of D. Z. was partly supported by the ANR-18-CE40-0009 ENUMGEOM grant.

[1] F. Wilczek and A. Zee, Appearance of Gauge Structure in Simple Dynamical Systems, Phys. Rev. Lett. **52**, 2111 (1984).
[2] S. Kobayashi, *Differential Geometry of Complex Vector Bundles* (Princeton University Press, Princeton, NJ, 2014).
[3] R. B. Laughlin, Anomalous Quantum Hall Effect: An Incompressible Quantum Fluid with Fractionally Charged Excitations, Phys. Rev. Lett. **50**, 1395 (1983).
[4] F. D. M. Haldane and E. H. Rezayi, Periodic Laughlin-Jastrow wave functions for the fractional quantized Hall effect, Phys. Rev. B **31**, 2529 (1985).
[5] Q. Niu, D. J. Thouless, and Y.-S. Wu, Quantized Hall conductance as a topological invariant, Phys. Rev. B **31**, 3372 (1985).
[6] R. Tao and Y.-S. Wu, Gauge invariance and fractional quantum Hall effect, Phys. Rev. B **30**, 1097 (1984).
[7] J. E. Avron and R. Seiler, Quantization of the Hall Conductance for General, Multiparticle Schrödinger Hamiltonians, Phys. Rev. Lett. **54**, 259 (1985).
[8] J. E. Avron, R. Seiler, and P. G. Zograf, Adiabatic Quantum Transport: Quantization and Fluctuations, Phys. Rev. Lett. **73**, 3255 (1994).
[9] S. Klevtsov, Laughlin states on higher genus Riemann surfaces, Commun. Math. Phys. **367**, 837 (2019).
[10] R. B. Laughlin, Quantized Hall conductivity in two dimensions, Phys. Rev. B **23**, 5632 (1981).
[11] R. C. Gunning, *Lectures on Riemann Surfaces: Jacobi Varieties* (Princeton University Press, Princeton, NJ, 2015).
[12] S. Klevtsov and D. Zvonkine (in preparation).
[13] X. G. Wen and A. Zee, Shift and Spin Vector: New Topological Quantum Numbers for the Hall Fluids, Phys. Rev. Lett. **69**, 953 (1992).

[14] X. G. Wen and Q. Niu, Ground-state degeneracy of the fractional quantum Hall states in the presence of a random potential and on high-genus Riemann surfaces, Phys. Rev. B **41**, 9377 (1990).

[15] F. D. M. Haldane, "Fractional Statistics" in Arbitrary Dimensions: A Generalization of the Pauli Principle, Phys. Rev. Lett. **67**, 937 (1991).

[16] A. Fayyazuddin and D. Li, Haldane's Fractional Statistics and the Lowest Landau Level on a Torus, Phys. Rev. Lett. **76**, 1707 (1996).

[17] D. Yoshioka, *The Quantum Hall Effect*, Springer Series in Solid-State Sciences (Springer, Berlin, 2002).

[18] S. Klevtsov, Geometry and large N limits in Laughlin states, Travaux Math. **24**, 63 (2016), https://wwwfr.uni.lu/recherche/fstm/dmath/travaux_mathematiques.

[19] S. Klevtsov, X. Ma, G. Marinescu, and P. Wiegmann, Quantum Hall effect and Quillen metric, Commun. Math. Phys. **349**, 819 (2017).

[20] E. Keski-Vakkuri and X. G. Wen, The ground state structure and modular transformations of fractional quantum Hall states on a torus, Int. J. Mod. Phys. B **07**, 4227 (1993).

[21] R. Varnhagen, Topology and fractional quantum Hall effect, Nucl. Phys. **B443**, 501 (1995).

[22] G. Moore and N. Read, Nonabelions in the fractional quantum Hall effect, Nucl. Phys. **B360**, 362 (1991).

[23] N. Read and E. Rezayi, Quasiholes and fermionic zero modes of paired fractional quantum Hall states: The mechanism for non-Abelian statistics, Phys. Rev. B **54**, 16864 (1996).

[24] B. A. Bernevig and F. D. M. Haldane, Model Fractional Quantum Hall States and Jack Polynomials, Phys. Rev. Lett. **100**, 246802 (2008).

[25] J. K. Jain, Composite-Fermion Approach for the Fractional Quantum Hall Effect, Phys. Rev. Lett. **63**, 199 (1989).

[26] T. H. Hansson, M. Hermanns, S. H. Simon, and S. F. Viefers, Quantum Hall physics: Hierarchies and conformal field theory techniques, Rev. Mod. Phys. **89**, 025005 (2017).

[27] J. E. Avron, R. Seiler, and P. G. Zograf, Viscosity of Quantum Hall Fluids, Phys. Rev. Lett. **75**, 697 (1995).

[28] P. Lévay, Berry phases for Landau Hamiltonians on deformed tori, J. Math. Phys. (N.Y.) **36**, 2792 (1995).

[29] I. V. Tokatly and G. Vignale, Lorentz shear modulus of a two-dimensional electron gas at high magnetic field, Phys. Rev. B **76**, 161305(R) (2007).

[30] S. Klevtsov and P. Wiegmann, Geometric Adiabatic Transport in Quantum Hall States, Phys. Rev. Lett. **115**, 086801 (2015).

[31] B. Bradlyn and N. Read, Topological central charge from Berry curvature: Gravitational anomalies in trial wave functions for topological phases, Phys. Rev. B **91**, 165306 (2015).

[32] T. Can, Y. H. Chiu, M. Laskin, and P. Wiegmann, Emergent Conformal Symmetry and Geometric Transport Properties of Quantum Hall States on Singular Surfaces, Phys. Rev. Lett. **117**, 266803 (2016).

[33] M. Barkeshli, C.-M. Jian, and X.-L. Qi, Twist defects and projective non-Abelian braiding statistics, Phys. Rev. B **87**, 045130 (2013).

[34] A. Gromov, Geometric defects in quantum Hall states, Phys. Rev. B **94**, 085116 (2016).

[35] N. Read, Non-Abelian adiabatic statistics and Hall viscosity in quantum Hall states and $p_x + \mathrm{i}p_y$ paired superfluids, Phys. Rev. B **79**, 045308 (2009).

[36] B. I. Halperin, Statistics of Quasiparticles and the Hierarchy of Fractional Quantized Hall States, Phys. Rev. Lett. **52**, 1583 (1984).

[37] D. Arovas, J. R. Schrieffer, and F. Wilczek, Fractional Statistics and the Quantum Hall Effect, Phys. Rev. Lett. **53**, 722 (1984).

[38] P. Bonderson, V. Gurarie, and C. Nayak, Plasma analogy and non-Abelian statistics for Ising-type quantum Hall states, Phys. Rev. B **83**, 075303 (2011).

[39] X.-G. Wen, Colloquium: Zoo of quantum-topological phases of matter, Rev. Mod. Phys. **89**, 041004 (2017).

[40] S. Axelrod, S. Della Pietra, and E. Witten, Geometric quantization of Chern-Simons gauge theory, J. Diff. Geom. **33**, 787 (1991).