Duality between Weak and Strong Interactions in Quantum Gases

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In one-dimensional quantum gases there is a well known "duality" between hard core bosons and noninteracting fermions. However, at the field theory level, no exact duality connecting strongly interacting bosons to weakly interacting fermions is known. Here we propose a solution to this long-standing problem. Our derivation relies on regularizing the only pointlike interaction between fermions in one dimension that induces a discontinuity in the wave function proportional to its derivative. In contrast to all known regularizations our potential is weak for small interaction strengths. Crucially, this allows one to apply standard methods of diagrammatic perturbation theory to strongly interacting bosons. As a first application we compute the finite temperature spectral function of the Cheon-Shigehara model, the fermionic model dual to the celebrated Lieb-Liniger model.

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Duality is an important concept in physics that refers to alternative descriptions of the same physical situation. This is particularly useful in cases where duality relates strongly interacting theories to weakly interacting ones. A prototypical example is the Bose-Fermi mapping in $1+1$ dimensional field theories with pointlike interactions [1–3] which relate the Lieb-Liniger (LL) [4] and Cheon-Shigehara (CS) [3] models. Given that the LL model of strongly interacting bosons has been at the heart of numerous experimental discoveries over the last two decades, see, e.g., Refs. [5–14], one would expect the Bose-Fermi duality to provide a very useful tool for understanding the observed behaviors. However, the CS form of the fermionic interaction potential does not allow for a perturbative analysis and masks the fact that there exists a weakly coupled regime corresponding to strong interactions between the LL bosons. There have been previous attempts to reformulate the CS interaction in order to clearly exhibit this weakly coupled regime, but these either violate the nonperturbative duality [15,16] (even though they allow for first order perturbative calculations [17,18] as well as Hartree-Fock and random-phase approximations and low-energy effective field theories [19–25]) or cannot be formulated in second quantization [26]. In the following we present a nonperturbative reformulation of the CS model that makes the existence of a weak coupling regime manifest and allows the full machinery of many-particle perturbation theory to be applied.

At the heart of our approach is the much studied problem of pointlike interactions in quantum mechanics. It is well known that bosons interacting in one dimension via a pointlike potential have a wave function that is continuous when two bosons coincide, but with a discontinuous derivative [27]. For two particles the relative motion is described by the textbook Schrödinger equation

$$\phi''(x) + k^2\phi(x) = 2\gamma\delta(x)\phi(x), \tag{1}$$

with γ a coupling parameter. Integrating over a small interval $-\epsilon < x < \epsilon$ around zero one obtains the condition $\phi'(0^+) - \phi'(0^-) = 2\gamma\phi(0)$. However, this manipulation should be considered as heuristic, since the product of a distribution with a nonsmooth function is ill defined. The proper way of analyzing (1) and deriving the discontinuity of $\phi'(x)$ at zero is by *regularizing* the δ distribution, namely, by replacing it with a smooth function $\delta_a(x)$ satisfying $\int \delta_a(x)dx = 1$ and $\lim_{a\to 0}\delta_a(x) = 0$ for $x \neq 0$. Then solution to Eq. (1) is obtained as the limit $a \to 0$ of the even solution $\phi_a(x)$ to the corresponding regular Schrödinger equation. This interpretation is physically meaningful as a pointlike potential is merely an approximation of a regular potential whose range is much smaller than the wavelength of the bosons.

The fermionic counterpart of this problem is less known and more subtle. For fermions, the only parity symmetric pointlike potential induces a discontinuity in the wave function itself (see below) [27–30]. But in contrast to the bosonic case, the interpretation of this pointlike potential in terms of (derivatives of) δ functions is problematic [31]. The heuristic analog of the Schrödinger equation (1) would be [29,32,33]

$$\psi''(x) + k^2\psi(x) = 2\beta\partial_x[\delta(x)\partial_x]\psi(x), \tag{2}$$

where β parametrizes the interaction strength. Indeed, by integrating x over the interval $-\epsilon < x < y$ and then y over $-\epsilon < y < \epsilon$ and applying the usual rules for the δ distribution yields $\psi(0^+) - \psi(0^-) = 2\beta\psi'(0)$ [34]. Through the Girardeau mapping $\phi(x) = \mathrm{sgn}(x)\psi(x)$ [1,3] the fermionic pointlike interaction (2) is dual to the bosonic one (1) with $\beta = (1/\gamma)$. However, this manipulation of δ's to obtain the jump condition is again problematic and requires a more careful analysis. In contrast to Eq. (1), this time a simple regularization of the delta function does not suffice. Indeed, replacing $\delta(x)$ by a smooth potential $\delta_a(x)$ in Eq. (2) *does not* yield the expected discontinuity in the limit $a \to 0$ and one obtains instead a continuous wave function [35]. One way of making the above manipulations well defined is to consider a generalization of distributions to discontinuous test functions [29,31–33,33]. Other generalizations of distributions were also considered [23]. Although mathematically sound, these generalizations suffer from a lack of physical meaning: one loses the interpretation of the pointlike interaction as an approximation of a very short range smooth potential. Moreover, the meaning of Eq. (2) in second quantization becomes unclear.

The physical and operational definition of this pointlike interaction in terms of a regularization is thus a nontrivial problem. It requires the construction of a smooth potential $V_a(x)$ such that the odd solution $\psi_a(x)$ to the Schrödinger equation

$$\psi_a''(x) + k^2\psi_a(x) = V_a(x)\psi_a(x), \tag{3}$$

reduces in the limit $a \to 0$ to

$$\begin{aligned} &\psi''(x) + k^2\psi(x) = 0 \quad \text{for } x \neq 0, \\ &\psi'(0^+) - \psi'(0^-) = 0, \\ &\psi(0^+) - \psi(0^-) = 2\beta\psi'(0). \end{aligned} \tag{4}$$

Solutions to this problem involving non-Hermitian potentials, nonlocal potentials or pseudopotentials were proposed [36–40]. The first solution in terms of a Hermitian, regular potential was obtained by Cheon and Shigehara in Ref. [2] and is of the form

$$V_a^{\rm CS}(x) = \left[\frac{1}{\beta} - \frac{1}{a}\right]\Big(\delta(x+a) + \delta(x-a)\Big). \tag{5}$$

Here the δ functions can be regularized on a scale that is small compared to a [41,42]. Crucially this potential describes strong interactions between two fermions for any value of β. This is in spite of the fact that for small β it results in wave functions that are close to those of free fermions. While this formulation allowed CS to establish a duality between a system of interacting fermions and the LL model, it obscured the fact that strongly interacting bosons are dual to *weakly* interacting fermions. Moreover, by its very nature it precluded any kind of perturbative calculation. The key aspect of our work is the construction of a smooth potential $V_a(x)$ that gives rise to Eq. (4) while being weak for small β.

A smooth weakly coupled potential for fermions.—For a coupling strength $\beta > 0$ and a regularization parameter $a > 0$, we define the following smooth potential

$$V_{a,\beta}(x) = \frac{\beta\sigma_a''(x)}{x + \beta\sigma_a(x)}, \tag{6}$$

with $\sigma_a(x) \equiv \sigma(x/a)$, where $\sigma(x)$ is any odd regular function that satisfies

$$\begin{aligned} &\lim_{x\to\infty}\sigma(x) = 1, \qquad \lim_{x\to\infty} x^2\sigma''(x) = 0, \\ &\sigma'(0) > 0, \qquad \forall\ x, \sigma'(x) \geq 0. \end{aligned} \tag{7}$$

For example, one can choose $\sigma_a(x) = \tanh x/a$. Let $\psi_{a,\beta}(x)$ be the odd solution to the Schrödinger equation

$$\psi_{a,\beta}''(x) + k^2\psi_{a,\beta}(x) = V_{a,\beta}(x)\psi_{a,\beta}(x), \tag{8}$$

with a fixed boundary condition $\psi_{a,\beta}(1) = 1$. The key result of this Letter is that in the limit $a \to 0$ at fixed $\beta > 0$, the wave function $\psi_{a,\beta}(x)$ of the potential (6) satisfies Eq. (4).

We now briefly sketch the proof of this statement. The idea is to treat $k^2\psi_{a,\beta}(x)$ in Eq. (8) as an inhomogeneous term in the homogeneous equation obtained for $k = 0$. The even $\phi_{a,\beta}^+$ and odd $\phi_{a,\beta}^-$ independent solutions of this homogeneous equation are

$$\begin{aligned} \phi_{a,\beta}^-(x) &= x + \beta\sigma_a(x), \\ \phi_{a,\beta}^+(x) &= \frac{1}{1 + \beta\sigma_a'(x)} + [x + \beta\sigma_a(x)] \\ &\quad \times \int_0^x dy \frac{\beta\sigma_a''(y)}{[y + \beta\sigma_a(y)][1 + \beta\sigma_a'(y)]^2}. \end{aligned} \tag{9}$$

Applying the method of variation of parameters, one obtains the following self-consistency condition for the odd solution to Eq. (8) for $k \neq 0$

$$\psi_{a,\beta}(x) = k^2\sum_{\sigma=\pm}\sigma\phi_{a,\beta}^{\sigma}(x)\int_0^x dy\psi_{a,\beta}(y)\phi_{a,\beta}^{-\sigma}(y) + A\phi_{a,\beta}^-(x), \tag{10}$$

where A is an integration constant. An analysis of $\phi_{a,\beta}^+(x)$ based on the assumptions (7) shows that it can be bounded independently of a and $x \in [-1, 1]$ and that for $x \neq 0$

$$\lim_{a\to 0}\phi_{a,\beta}^+(x) = -\frac{|x|}{\beta}. \tag{11}$$

From this and Eq. (10), Grönwall's inequality [43] implies then that $\psi_{a,\beta}(x)$ itself can be bounded independently of a and $x \in [-1, 1]$. This allows us to commute the limit $a \to 0$ and the integrations in Eq. (10). The resulting self-consistency equation for $\lim_{a\to 0}\psi_{a,\beta}(x)$ establishes then that it satisfies Eq. (4).

N-particle Cheon-Shigehara gas.—Following Refs. [3,44] the above two-particle result can be readily extended to a gas of N fermions with Hamiltonian

$$H_{a,\beta}^f = -\sum_{j=1}^N \partial_{x_j}^2 + 2\sum_{j<k} V_{a,\beta}(x_j - x_k), \tag{12}$$

which from now on we will refer to as CS gas. The reasoning goes as follows. First, one observes that since the eigenstates $\psi_{a,\beta}^f(x_1, \ldots, x_N)$ are antisymmetric functions of $x_1, \ldots, x_N$ it suffices to know them on $D = \{x_1 < \ldots < x_N\}$. Second, one notes that, because the potential (6) satisfies Eq. (4) when $a \to 0$, the many-body wave function fulfills $\sum_j \partial_{x_j}^2 \psi_{0,\beta}^f = 0$ inside D, and obeys the following conditions at the boundary of D:

$$\psi_{0,\beta}^{\mathrm{f}}\Big|_{x_j=x_{j+1}^-} = -\frac{\beta}{2}[\partial_{x_{j+1}} - \partial_{x_j}]\psi_{0,\beta}^{\mathrm{f}}\Big|_{x_j=x_{j+1}^-}. \quad (13)$$

Finally, performing the Girardeau mapping [1,45,46]

$$\psi_{2/\beta}^b(x_1,\ldots,x_N) = \psi_{0,\beta}^f(x_1,\ldots,x_N)\prod_{j<k}\mathrm{sgn}(x_j - x_k), \quad (14)$$

one finds that the function $\psi_{2/\beta}^b$ exactly satisfies the conditions of the LL eigenstates at the boundary of D [44]. This establishes that when $a \to 0$ at fixed β, $H_{a,\beta}^f$ is equivalent to the LL Hamiltonian

$$H_c^b = -\sum_{j=1}^N \partial_{x_j}^2 + 2c\sum_{j<k}\delta(x_j - x_k), \quad (15)$$

with $\beta = 2/c$. Having established that Eq. (12) provides a dual description to Eq. (15) we now show that our formulation allows one to carry our perturbative calculations in the large-c limit that are in agreement with exact results. When $\beta = 0$, the energy levels of the free fermion Hamiltonian (12) on a ring of size L are given by $\sum_{\lambda_i \in \boldsymbol{\lambda}} \lambda_i^2$ with $\boldsymbol{\lambda}$ any subset of $\{2\pi n/L, n \in \mathbb{Z}\}$ with N elements. In the thermodynamic limit $L \to \infty$, they are parametrized by a particle density $0 \le \rho(\lambda) \le 1/(2\pi)$. Let us now fix such a state at $\beta = 0$ through its particle density ρ and compute perturbatively in β the energy levels of (12) at fixed a. The energy per site $e_a(\beta)$ can be written as

$$e_a(\beta) = e_a^{(0)}(\beta) + e_a^{(1)}(\beta) + e_a^{(2)}(\beta) + \mathcal{O}(\beta^3), \quad (16)$$

where the successive orders in perturbation theory are

$$\begin{aligned}
e_a^{(0)}(\beta) &= \int_{-\infty}^{\infty} d\lambda \rho(\lambda)\lambda^2, \\
e_a^{(1)}(\beta) &= \int_{-\infty}^{\infty} d\lambda d\mu \rho(\lambda)\rho(\mu)[\hat{V}_{a,\beta}(0) - \hat{V}_{a,\beta}(\lambda-\mu)], \\
e_a^{(2)}(\beta) &= \pi\int_{-\infty}^{\infty} d\lambda d\mu d\nu \rho(\lambda)\rho(\mu)\rho_h(\lambda+\nu)\rho_h(\mu-\nu) \\
&\quad \times \frac{[\hat{V}_{a,\beta}(\lambda-\mu+\nu) - \hat{V}_{a,\beta}(\nu)]^2}{\nu(\mu-\lambda-\nu)}. \quad (17)
\end{aligned}$$

Here $\hat{V}_{a,\beta}(\lambda) = \int_{-\infty}^{\infty} dx V_{a,\beta}(x) e^{i\lambda x}$ and $\rho_h(\lambda) = 1/(2\pi) - \rho(\lambda)$ is the hole density. Expanding the potential (6) in β at fixed a and considering $a \to 0$ afterwards, we have up to $\mathcal{O}(a) + \mathcal{O}(\beta^3)$ corrections

$$e_a^{(1)}(\beta) = 2\beta\mathcal{D}\mathcal{E}\left[-1 + \frac{\beta}{2a}\int_{-\infty}^{\infty} dx \sigma'(x)^2\right], \quad (18)$$

$$e_a^{(2)}(\beta) = 3\beta\mathcal{D}\mathcal{E}\left[1 - \frac{\beta}{3a}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}[\hat{\sigma}'(\omega)]^2\right]. \quad (19)$$

Here $\mathcal{D} = \int d\lambda \rho(\lambda)$ and $\mathcal{E} = \int d\lambda \rho(\lambda)\lambda^2$ are, respectively, the particle density and the (unperturbed) energy density of the macrostate parametrized by $\rho(\lambda)$. We observe that the first and second order contributions are both divergent in $1/a$, but remarkably, using Parseval's identity we find that their sum is in fact finite:

$$e_a(\beta) = (1 - 2\beta\mathcal{D} + 3\beta^2\mathcal{D}^2)\mathcal{E} + \mathcal{O}(a) + \mathcal{O}(\beta^3). \quad (20)$$

The compensation is due to the specific form of the potential (6). The expression (20) agrees with the exact Bethe ansatz result for the LL model with $\beta = 2/c$ at order $1/c^2$. Our calculation shows that all energy levels of Eq. (12) can be computed perturbatively in β at fixed a and then the regulator a can be sent to 0 order by order in β to order β^2.

Note that the result (20) allows one to study the thermodynamics of the gas (12) up to order $1/c^2$. Indeed, one can calculate the free energy at temperature T,

$$F_{a,\beta} = \mathrm{tr}[e^{-H_{a,\beta}^f/T}], \quad (21)$$

expanding the trace in terms of the noninteracting basis. One can then use Eq. (20) and proceed as in the thermodynamic Bethe ansatz (TBA) treatment [47,48]. For example, in this way one finds that, up to order $1/c^2$, the thermal energy density is given by the expression (20) with ρ being the thermal root density satisfying the Yang-Yang equation expanded at order c^{-2}.

Cheon-Shigehara field theory.—The Hamiltonian (12) (on a ring of length L) can be expressed in second quantization as

$$\mathcal{H}_{a,\beta}^f = \sum_p (p^2 - \mu)\psi_p^\dagger \psi_p + \sum_{\boldsymbol{p}} W_{a,\beta}(\boldsymbol{p})\psi_{p_1}^\dagger \psi_{p_2}^\dagger \psi_{p_3}\psi_{p_4}, \quad (22)$$

where $\psi_p^\dagger$ and ψ_p are canonical Fermi fields in momentum space, and we have introduced the short-hand notation $\boldsymbol{p} \equiv (p_1, \ldots, p_4)$. The interaction vertex in Eq. (22) is given by

$$\begin{aligned}
W_{a,\beta}(\boldsymbol{p}) &= \frac{1}{4L}\delta_{p_1+p_2-p_3-p_4,0} \\
&\quad \times \sum_{P,Q \in S_2} \mathrm{sgn}(PQ)\hat{V}_{a,\beta}(p_{P(1)} - p_{Q(1)+2}), \quad (23)
\end{aligned}$$

where S_2 is the group of permutations of two elements.

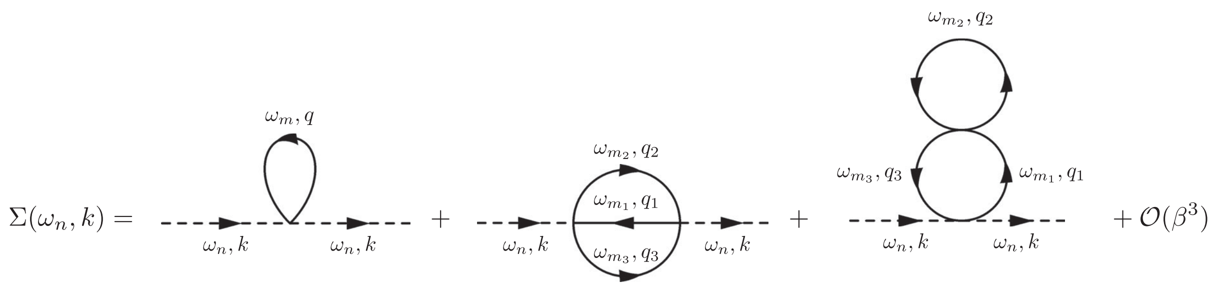
For small β the theory (22) can be analyzed using standard diagrammatic perturbation theory. Let us consider in particular the thermal propagator

$$G(\tau, k) = -\frac{\mathrm{tr}[T_\tau[e^{\tau\mathcal{H}_{a,\beta}^f}\psi_k e^{-\tau\mathcal{H}_{a,\beta}^f}\psi_k^\dagger]e^{-\mathcal{H}_{a,\beta}^f/T}]}{\mathrm{tr}[e^{-\mathcal{H}_{a,\beta}^f/T}]}. \quad (24)$$

The usual procedure (see, e.g., Ref. [49]) is to exploit the antiperiodicity of Eq. (24) for $\tau \mapsto \tau + 1/T$, and consider its Fourier coefficients, denoted by $G(\omega_n, k)$, where $\omega_n = 2\pi T(n + 1/2)$ are called Matsubara frequencies. These coefficients can be written in the following Dyson form

$$G(\omega_n, k) = \frac{1}{i\omega_n - k^2 + \mu - \Sigma(\omega_n, k)}, \tag{25}$$

where the proper self energy, $\Sigma(\omega_n, k)$, is defined as the sum of all irreducible Feynman diagrams with two amputated legs [49]. The self energy encodes all information about the thermodynamics of the system as well as very relevant information about its dynamics. Indeed, it can be used to determine both the free energy and the spectral function [50]. Specifically, the latter is expressed as $A(\omega, k) = -2\mathrm{Im}[G^R(\omega, k)]$, where the Fourier transform of the retarded Green's function $G^R(\omega, k)$ is obtained by performing the analytic continuation $i\omega_n \mapsto \omega + i0^+$ in Eq. (25). For the theory (22), considering contributions up to order β^2, we find



(26)

where the incoming and outgoing legs (dashed lines) are amputated [30]. Remarkably, evaluating these diagrams we find that, in analogy to what happens for Eq. (20), the $1/a$ divergences in the second order contributions compensate and the final result does not require further regularization [30]. Specifically, in the thermodynamic limit we have

$$\begin{aligned}\Sigma(\omega_n, k) = {} & -2\beta(A_2 + A_0 k^2) - \frac{2\beta^2}{T}\int \frac{dq}{2\pi}(A_2 + A_0 q^2)(k-q)^2 n(q)[1 - n(q)] \\ & + 2\beta^2 \int \frac{dq_2}{2\pi}\frac{dq_3}{2\pi}\frac{[(k-q_3)^2 - (q_2 - q_3)^2]^2}{i\omega_n + q_2^2 - q_3^2 - \bar{q}_4^2 + \mu}[n(q_3)n(\bar{q}_4) - n(q_2)n(\bar{q}_4) - n(q_2)n(q_3)] \\ & - i\beta^2 \int \frac{dq_2}{2\pi}\sqrt{2(i\omega_n - k^2 + \mu) + (k - q_2)^2}(k - q_2)^2 n(q_2),\end{aligned} \tag{27}$$

where we chose the branch cut of the square root to lie along the positive real axis, $\bar{q}_4 = k + q_2 - q_3$, and

$$n(p) = \frac{1}{1 + e^{(p^2-\mu)/T}}, \qquad A_m = \int \frac{dp}{2\pi} p^m n(p). \tag{28}$$

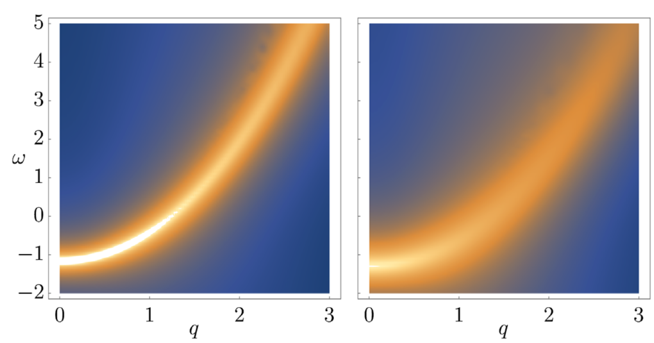


FIG. 1. Spectral function of the CS gas $A(\omega, q)$ for $\beta = 2/c = 0.5$ (left) and $\beta = 1$ (right) in an equilibrium state at temperature $T = 1$ and chemical potential $\mu = 1$. The color scale is the same for both plots. The free fermion $\beta = 0$ spectral function is $2\pi\delta(\omega - q^2 + \mu)$.

To the best of our knowledge (27) is the first calculation of the self energy in the CS model at second order in β—where interactions generate a non-vanishing imaginary part—and represents our second main result. We verified that (i) the spectral function $A(\omega, k)$ obtained from Eq. (27) fulfils the exact sum rule $\int d\omega/(2\pi)A(\omega, k) = 1$; (ii) the internal energy per volume computed using (27) agrees up to order $1/c^2$ with the exact result for the LL model [47,48]. The nontrivial effects of the interactions are best appreciated by considering the spectral function, cf. Fig. 1. We see that as a result of the interactions the fermions created by $\psi_k^\dagger$ acquire a finite lifetime and the dispersion gets renormalized. We note that the appearance of a finite lifetime is not in contradiction with the integrability of the theory (22) because the integrability-protected stable quasiparticles differ from the fermions created by $\psi_k^\dagger$ for finite c [51].

Discussion.—In this Letter we presented a one-dimensional quantum mechanical potential that induces a discontinuity in the wave function proportional to its derivative and, at the same time, can be expanded perturbatively for small but finite interaction strengths. This addresses the long-standing problem of how to best regularize pointlike interactions in quantum mechanics. We used this potential to obtain a reformulation of the Cheon-Shigehara gas, the fermionic theory dual to the Lieb-Liniger model, that makes it manifest that strongly interacting bosons correspond to weakly coupled fermions. Our results open the door to the systematic analysis of strongly interacting bosons away from the hard-core limit by means of perturbation theory, going considerably beyond the current state of the art. As a first application we have obtained the spectral function of the CS model at order $1/c^2$, displaying the previously inaccessible broadening shown in Fig. 1.

Our work can be generalized in a number of ways. It can, for example, be directly extended to treat Bose-Fermi dualities in multispecies systems [52], allowing one to access cases of high experimental relevance [5–13,53] that up to now have been treated only in the limit of infinite repulsion [54–57], via low-energy approximations [58–62], or in the hydrodynamic regime [13,14,63]. Crucially, our method does not rely on integrability and allows one to study *any* such strongly coupled theory. The case of *attractive* interactions corresponding to $\beta < 0$ is not covered by our potential (6) and deserves attention. On physical grounds, we expect a drastic change of potential from small positive to small negative β as it corresponds to going from a Tonks-Girardeau to a Super-Tonks-Girardeau gas. Applying our results to a strong coupling expansion in the LL model is particularly appealing due to the recent accounts of uniform convergence—in space and time—of the perturbative series for correlation functions both in [64] and out of equilibrium [65]. For instance, this opens the door to a systematic investigation of quantum quenches, explicitly accessing the late time regime where homogeneous systems are expected to locally relax [66] and inhomogeneous ones to follow the predictions of generalised hydrodynamics [67–69]. This could potentially lead to *ab initio* derivations of these expectations in the presence of interactions and a full characterization of the relaxation mechanisms, a task that has currently been accomplished only for a special quantum cellular automaton [70,71]. Finally, our work paves the way for establishing the Bose-Fermi mapping at an operatorial level directly in the (regularized) respective field theories. Such a mapping is highly desirable in order to be able to calculate quantities like the boson propagator in the fermionic setting.

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