Renormalization Group Flows on Line Defects

Gabriel Cuomo,[1,2,*] Zohar Komargodski,[1,2,†] and Avia Raviv-Moshe[1,‡]
[1]*Simons Center for Geometry and Physics, SUNY, Stony Brook, New York 11794, USA*
[2]*C. N. Yang Institute for Theoretical Physics, Stony Brook University, Stony Brook, New York 11794, USA*

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We consider line defects in d-dimensional conformal field theories (CFTs). The ambient CFT places nontrivial constraints on renormalization group (RG) flows on such line defects. We show that the flow on line defects is consequently irreversible and furthermore a canonical decreasing entropy function exists. This construction generalizes the g theorem to line defects in arbitrary dimensions. We demonstrate our results in a flow between Wilson loops in four dimensions.

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Introduction.—In lattice systems, in order to understand the physics on different length scales, we perform block-spin transformations, eliminating degrees of freedom that live at short distances. This process obviously reduces the overall number of degrees of freedom. But one can ask whether this reduces the number of degrees of freedom per lattice site, which is much less clear. In quantum field theory, the number of degrees of freedom per lattice site is roughly speaking the number of fields and this raises the question of whether the number of fields decreases as we probe physics of longer and longer distances.

To address these questions precisely one has to give a nonperturbative definition of what the "number of fields" means and provide a prescription to evaluate it even when there is no weakly coupled description in terms of fields. Starting from the work of Zamolodchikov on the c function in two dimensions [1], several such proposals and results were discussed in diverse dimensions [2–24].

The focus of this Letter is the physics of one-dimensional defects in a CFT. Such defects can undergo nontrivial renormalization group flows while affecting the bulk very little far away from the defect. A few known examples of this kind include Wilson or 't Hooft lines in 4D gauge theories [25] and holography [26,27], symmetry defects and impurities in 3D quantum critical systems [28–33], etc. In two dimensions, line defects correspond to boundaries or interfaces and appear naturally as the low-energy limit of lattice systems with impurities (see, for instance, Refs. [34–38]).

There is already some extensive work on renormalization group flows on various defects [39–56]. For our purposes, it is important to highlight the conjecture of Affleck and Ludwig [39] for the decreasing entropy function on line defects in two dimensions and its subsequent proofs [42] and [48]. Here we will discuss the properties of line defects in arbitrary dimensions. We will define an entropy function and show that it monotonically decreases. In the Supplemental Material [57] we show how our result applies to a nontrivial flow between two different conformal Wilson lines in super Yang-Mills (SYM) theory in four dimensions.

The main idea we employ is that surrounding the line defect with conformal charges leads to nontrivial identifications in theory space when the defect is nonconformal. This can be expressed in terms of constraints on the dilaton living on the line defect. We show that these constraints translate to a monotonic entropy function.

DCFTs.—We consider local, reflection-positive Euclidean conformal field theories (CFTs) in $d \geq 2$ dimensions. We will be interested in CFTs in the presence of a line defect which preserves unitarity and locality. We will be interested in infinite straight lines or circular defects. At the fixed point of the (defect) renormalization group flow, the straight line defect preserves the subgroup $SL(2,\mathbb{R}) \times SO(d-1)$ of the full conformal group. In this case the system is called a defect CFT (DCFT). In $d = 2$, conformal line defects additionally preserve one copy of the Virasoro algebra.

DCFTs share many of the standard properties of CFTs. However, in general, the line defect does not support a stress tensor [58–60]. This statement really means that there is no possibility to localize energy on the line defect and energy always ends up being smeared into the bulk. The bulk stress tensor $T_b^{\mu\nu}$ obeys the following Ward identity (It is convenient to consider normalized correlation functions, so $\langle T_b^{\mu\nu}\rangle$ really stands for $\langle T_b^{\mu\nu} D\rangle/\langle D\rangle$, where D is the defect operator.) [47,59,61,62]:

$$\nabla_\mu T_b^{\mu\nu} = -\delta_D^{d-1} n_i^\nu D^i, \tag{1}$$

where δ_D^{d-1} is a delta function localized at the defect, $\{n_i^\nu\}$ is a basis of $d-1$ unit vectors normal to the defect, and D^i is the displacement operator [47,59] (The defect D, and the displacement operator D^i, are distinguished by the superscript i.), which parametrizes the breaking of translations in the directions normal to the defect. Finally, we mention that all bulk correlation functions may be systematically decomposed into defect correlators via the bulk-to-defect OPE [36,63]. This allows us to study the DCFT data via a systematic bootstrap approach [59,64,65], similar to the one usually adopted in standard CFTs [66–68].

It will be convenient for our purposes to consider the expectation of the $SL(2,\mathbb{R})$ charges wrapping the defect. These are obtained by integrating the stress tensor contracted with the appropriate Killing vector at a fixed distance ε from the defect:

$$Q_\xi(D) = \int_\varepsilon d^{d-1}\Sigma^\mu \langle T^b_{\mu\nu}\rangle \xi^\nu. \tag{2}$$

By conformal invariance we expect Eq. (2) to yield a vanishing result for both a straight line defect and a circular one. However, due to a subtlety with the action of conformal transformations on the point at infinity, for the straight line geometry the conformal charges vanish only when the distance between the integration surface and the defect diverges sufficiently fast as $x^d \to \pm\infty$ [25]. This issue is presumably related to the disagreement between the expectation value of circular and linear Maldacena-Wilson loops in $\mathcal{N}=4$ SYM [69–71]. We provide a detailed discussion regarding this subtlety in the Supplemental Material [57]. No issues of this sort arise for circular defects, hence we will focus on this geometry in what follows.

Let us consider for concreteness a circular defect of radius R centered around the origin on the (x^1, x^2) plane $x^3 = \ldots = x^d = 0$. The $SL(2,\mathbb{R})$ Killing vectors preserved by the circle are

$$\begin{aligned} \xi^\mu_{(a)} &= \frac{1}{2}[\delta^\mu_a(R + x^2/R) - 2x^\mu x_a/R], \\ \xi^\mu_{(\phi)} &= \delta^\mu_a \epsilon^{ab} x_b, \end{aligned} \tag{3}$$

where $a = 1, 2$ and indices are raised or lowered with the Euclidean metric. Here, $\xi^\mu_{(a)}$ are linear combinations of translations and special conformal transformations on the defect plane, while $\xi^\mu_{(\phi)}$ generates rotations in the (x^1, x^2) plane. In this geometry, there is no issue with the boundary condition at infinity and, consequently, the expectation values of the $SL(2,\mathbb{R})$ charges on a surface wrapping the defect (see Fig. 1) vanish,

$$Q_\xi(D) = 0 \qquad \text{(circular defect)}. \tag{4}$$

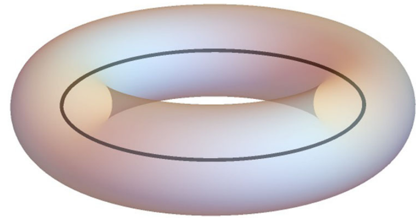

FIG. 1. An illustration of a toroidal surface wrapping a circular defect.

The statement (4) can be checked using the explicit form of the stress-tensor one-point function in a circular geometry, which depends on a unique constant [72] h_D (see, e.g., Refs. [59,79] for the explicit expressions). In particular, in contrast to the infinite line, every point of the surface can be brought arbitrarily close to the defect compatibly with the identity (4) [80]. For this reason in the following we will focus on circular defects.

Defect RG.—The main goal of this work is to study defect renormalization group (DRG) flows. A DRG may be triggered perturbing a DCFT with one or more relevant defect operators. For instance, we may consider a defect operator $\mathcal{O}$ with $\Delta_\mathcal{O} < 1$:

$$S_{\mathrm{DCFT}} \to S_{\mathrm{DCFT}} + M_0^{1-\Delta_\mathcal{O}} \int_D d\sigma \mathcal{O}(\sigma), \tag{5}$$

where $\int_D$ stand for integration along the defect and M_0 is the mass scale of the flow. Conformal invariance [i.e., $SL(2,\mathbb{R})$ transformations that preserve the defect] is now explicitly broken by the scale M_0 to just translations along the defect.

Because of the locality of the bulk CFT, the bulk stress tensor remains conserved and traceless (up to possible bulk trace anomalies in curved space) away from the line. However, now a defect stress tensor T_D is allowed. In other words, energy can now be stored on the defect. Not only T_D is allowed, such an operator must always exist away from the fixed points of the defect. The existence of the operator T_D is the reason that $SL(2,\mathbb{R})$ charges are no longer conserved. Since T_D is localized to the defect, what we mean by saying that $SL(2,\mathbb{R})$ charges are no longer conserved is that, if the charges are integrated on surfaces that intersect the defect, then they are not invariant under small deformations.

Invariance under translations along the defect implies that Eq. (1) in the presence of T_D is modified to

$$\nabla_\mu T_b^{\mu\nu} = -\delta_D^{d-1}\dot{X}^\nu \dot{T}_D - \delta_D^{d-1} n_i^\nu D^i, \tag{6}$$

where $X^\mu(\sigma)$ is the embedding function describing the defect location and the dot stands for derivatives with respect to the line coordinate σ, so that $\dot{X}^\nu$ is a tangent vector to the defect. (Here we are assuming that the defect has a trivial induced submanifold metric $g_D = \dot{X}^\mu \dot{X}^\nu g_{\mu\nu} = 1$ to simplify the notation.) Equation (6) merely expresses the energy balance between the bulk and the defect.

Spurion analysis and the dilaton.—As it often happens in the study of RG flows, it is useful to promote the renormalization group scale to a function of position $M(\sigma) = M_0 e^{\Phi(\sigma)}$ [81,82], where $\Phi(\sigma)$ is a dimensionless background dilaton field. To linear order, the partition function of the theory depends on the dilaton through to the defect energy-momentum tensor [83]:

$$\begin{aligned} \log Z|_{\Phi+\delta\Phi} = \log Z|_{\Phi} + \int_D d\sigma \delta\Phi(\sigma)\langle T_D(\sigma)\rangle_\Phi \\ + \frac{1}{2}\int_D d\sigma_1 \int_D d\sigma_2 \delta\Phi(\sigma_1)\delta\Phi(\sigma_2) \\ \times \langle T_D(\sigma_1) T_D(\sigma_2)\rangle_\Phi + \dots . \end{aligned} \tag{7}$$

The background dilaton field acts a source for the theory. This in turn modifies the conservation equation (6) as follows [61,62]:

$$\nabla_\mu T_b^{\mu\nu} = -\delta_D^{d-1} \dot{X}^\nu (\dot{T}_D - \dot{\Phi} T_D) - \delta_D^{d-1} n_i^\nu D^i. \tag{8}$$

If one views the coordinate along the defect as time, then a nontrivial $\Phi(\sigma)$ renders the theory time dependent and (8) relates the nonconservation rate of the charge associated with translations along the defect with the derivative of the dilaton source.

A position-dependent mass scale breaks the $SL(2,\mathbb{R})$ symmetry completely. What we gain by introducing the general background field $\Phi(\sigma)$ is that $SL(2,\mathbb{R})$ allows us to relate different theories instead of directly placing constraints on a given theory. Indeed, we will use Eq. (8) in what follows to derive some nontrivial identities relating theories with different values for the source $\Phi(\sigma)$.

RG flows induced by the broken charges.—It is crucial to realize that the identity (4) holds irrespectively of the breaking of scale invariance on the defect [i.e., it holds for any $\Phi(\sigma)$]. This is because the charges wrapping the defect do not intersect it and hence such charges are oblivious to what happens on the defect and they remain invariant under small deformations. They can be moved off to infinity where they annihilate the vacuum. (To see that, one can realize the wrapping surface as the difference between two S^{d-1} surfaces outside and inside the loop.)

As we explained, on general grounds one expects that $SL(2,\mathbb{R})$ transformation can be reabsorbed into a transformation of the dilaton $\Phi(\sigma)$, leading to relations between different theories. This can be made precise by using Eq. (4). To that end, consider shrinking the radius of the topological surface enclosing the defect (see Fig. 1). It is clear from Gauss's law that the only contribution in the integration of the stress tensor arises from the right-hand side in Eq. (8). We therefore conclude that Eq. (4) implies the following relation [84]:

$$\begin{aligned} 0 = Q_\xi(D) = \int d^{d-1}\Sigma^\mu \langle T^b_{\mu\nu}\rangle \xi^\nu \\ = \int_D d\sigma (\dot{\xi}_D + \xi_D \dot{\Phi})\langle T_D\rangle, \end{aligned} \tag{9}$$

where in the second line we integrated by parts and we denoted by ξ_D the projection of the Killing vectors (3) on the defect. We crucially used the fact that the normal components of the $SL(2,\mathbb{R})$ Killing vectors vanish on the defect. In fact, for more general conformal Killing vectors which do not leave the loop invariant, an analogous identity picks an additional contribution from the displacement operator in Eq. (8) but we do not study these identities here.

Because of the linear coupling between the defect stress tensor and the dilaton, we may interpret Eq. (9) as an equivalence between defects with different DRG scales $M(\sigma)$:

$$\Phi \sim \Phi + \alpha(\dot{\xi}_D + \xi_D \dot{\Phi}) \qquad |\alpha| \ll 1, \tag{10}$$

for any $SL(2,\mathbb{R})$ Killing vector ξ and any infinitesimal α. This observation is most useful when considering the expansion of the partition function (7) around $\Phi = 0$. Demanding the equivalence (10) at each order in the field expansion we then find an infinite number of identities for the correlation functions of the defect stress tensor. At second order in the field expansion we obtain the following one (omitting the subscript $\Phi = 0$ from now on):

$$\begin{aligned} \int_D d\sigma \xi_D(\sigma)\dot{\Phi}(\sigma)\langle T_D(\sigma)\rangle \\ = -\int_D d\sigma_1 \int_D d\sigma_2 \dot{\xi}_D(\sigma_1)\Phi(\sigma_2)\langle T_D(\sigma_1)T_D(\sigma_2)\rangle. \end{aligned} \tag{11}$$

Crucially, this identity holds for any $\Phi(\sigma)$. Notice that the right-hand side of Eq. (11) for generic choices of the dilaton profile is naively divergent. Our arguments however ensure that these identities must hold in any regularization scheme which preserves the invariance of the partition function under diffeomorphisms and defect reparametrizations.

At this point it is useful to specify a cylindrical system of coordinates on the defect: $x^1 = R\cos\phi$, $x^2 = R\sin\phi$ and set $\sigma = R\phi$. The projection of the three Killing vectors in Eq. (3) reads, respectively,

$$\xi_D = -\sin\phi, \qquad \xi_D = \cos\phi, \qquad \xi_D = -1. \tag{12}$$

Equation (11) is trivial for $\xi_D = -1$, but provides nontrivial constraints for the other two choices, which lead to identical constraints. A particularly useful relation is obtained choosing $\xi_D = -\sin\phi$ and $\Phi \propto \cos\phi$ in Eq. (11). This leads to

$$\begin{aligned} R\int_D d\phi \langle T_D(\phi)\rangle \\ = R^2 \int_D d\phi_1 \int_D d\phi_2 \langle T_D(\phi_1)T_D(\phi_2)\rangle \cos(\phi_1 - \phi_2), \end{aligned} \tag{13}$$

where we used trigonometric identities and invariance under translations along the defect to simplify both sides. Equation (13) will be very useful in providing a gradient formula for the DRG flow of a suitably defined defect entropy.

The defect entropy.—Our discussion thus far focused on defects in flat space, but all our considerations apply on all conformally equivalent manifolds. These include the d-dimensional sphere of radius R, with the defect spanning a maximal circle, and the cylinder $\mathbb{R}\times S^{d-1}$, with the defect on the equator of S^{d-1} at a fixed value of the Euclidean time $\tau = \log x^2/R = 0$.

We can use any of these geometries to define a defect g function, $g(M_0R)$, in terms of the partition function in the presence of the defect, normalized by the partition function without it:

$$\log g(M_0R) = \log Z_{\mathcal{M}} - \log Z_{\mathcal{M}}^{(CFT)}, \tag{14}$$

where $\log Z_{\mathcal{M}}^{(CFT)}$ is the partition function of the theory without the defect [85]. The defect contribution g depends only on the dimensionless product M_0R and it reduces to a constant at the fixed points (in a sense that we will explain below).

We must now ask to what extent is g well defined at the fixed points and away from them. $\log g$ can be shifted by the addition of a cosmological constant counterterm $\int d\sigma M_0 \sim M_0R$ with an arbitrary coefficient. All other nontrivial geometric invariants which are analytic around the flat metric have dimension larger than 1 and cannot appear as counterterms. Therefore no additional ambiguities exist in $d>2$ (we will discuss $d=2$ more in detail below). Therefore, one can obtain a scheme-independent quantity which we will refer to as the *defect entropy*, defined as [86]

$$s(M_0R) = \left(1 - R\frac{\partial}{\partial R}\right)\log g(M_0R). \tag{15}$$

At the fixed points, $s(M_0R)$ is a pure number which is scheme independent. It is equal to the perimeter-independent contribution to $\log g(M_0R)$ at the fixed point. We will refer to these fixed point values of g as $g_{\mathrm{UV}}, g_{\mathrm{IR}}$, respectively. We will show that $s(M_0R)$ decreases monotonically under DRG, implying $g_{\mathrm{UV}} > g_{\mathrm{IR}}$.

In $d=2$ Eq. (15) coincides with the interface contribution to the thermal entropy of the theory. To make the connection with $d=2$ precise, one needs to remember that in $d=2$ we can also allow the counterterm $\int d\sigma K$, where K is the extrinsic curvature [88]. Such a term vanishes for a maximal circle in S^2 and on $\mathbb{R}\times S^1$ and therefore all our conclusions hold unaltered on those manifolds. Furthermore CPT invariance implies that the coefficient of this counterterm should be purely imaginary. Therefore the definition in Eq. (15) is meaningful also in flat space provided we focus on the real part of the defect entropy.

The gradient formula.—We now have all the ingredients to derive a gradient formula for the DRG flow of the defect entropy. Since g depends on M_0R only, for constant dilaton Φ, it follows that g depends on the combination RM_0e^{Φ}. We may therefore write the variation of the defect entropy s under a change in the mass scale as follows:

$$M_0\frac{\partial}{\partial M_0}s(M_0R) = \left[\left(\frac{d}{d\Phi} - \frac{d^2}{d\Phi^2}\right)\log g(RM_0e^{\Phi})\right]_{\Phi=0}. \tag{16}$$

Using the expansion (7) for constant Φ we then can write Eq. (16) in terms of correlation functions of the defect stress tensor

$$M_0\frac{\partial}{\partial M_0}s(M_0R) = R\int_D d\phi\langle T_D(\phi)\rangle - R^2\int_D d\phi_1\int_D d\phi_2\langle T_D(\phi_1)T_D(\phi_2)\rangle. \tag{17}$$

Equation (17) may not seem very useful at first sight. It is not manifestly sign definite, nor is it manifestly finite. To clarify these issues, we can rewrite the first term using Eq. (13). We obtain

$$M_0\frac{\partial s}{\partial M_0} = -R^2\int_D d\phi_1\int_D d\phi_2\langle T_D(\phi_1)T_D(\phi_2)\rangle[1-\cos(\phi_1-\phi_2)]. \tag{18}$$

The right-hand side of Eq. (18) is free of divergences and ambiguities due to the double zero of $1-\cos(\phi_1-\phi_2)$. Furthermore, (18) is manifestly negative in a reflection positive theory (note that this also applies to a connected 2-point function, as on the right-hand side of Eq. (18)). Therefore, we deduce that s monotonically decreases along defect RG flows, implying that the UV and IR DCFT satisfy

$$g_{\mathrm{UV}} > g_{\mathrm{IR}}. \tag{19}$$

Equation (18) additionally implies that s does not depend on the marginal parameters on the defect [91].

In $d=2$, Eq. (19) was originally conjectured to hold for boundaries (and therefore, using the folding trick, for interfaces) by Affleck and Ludwig [37,39]. In $d=2$, in the regime where the DRG flow can be described in terms of finitely many couplings and beta functions, a gradient formula equivalent to Eq. (18) was proposed in the context of string field theory [98–102]. It was then established by Friedan and Konechny [42]. An alternative proof of Eq. (19) in $d=2$ was also given [48] using quantum information methods. (See, for instance, also Refs. [41,44,103] for a holographic setup.) Our work provides an extension of those results to line defects in an arbitrary number of dimensions. We also remark that the inequality (19) was recently conjectured in Ref. [50] for arbitrary d. Another remark is that the trivial line has $g = 1$. However, it may *a priori* be that $g < 1$ for some nontrivial lines, as sometimes happens in two dimensions [104,105].

Equation (19) was extensively checked in $d = 2$, see, e.g., Refs. [37–39,106]. We additionally verified our results (18) and (19) in several concrete examples, including a flow between Wilson lines in $\mathcal{N} = 4$ SYM previously studied in Refs. [27,107]. Details can be found in the Supplemental Material [57].

Finally, we remark that the partition function of higher-dimensional defects is subject to further ambiguities besides a cosmological constant, rendering a generalization of our arguments not straightforward. For two- and four-dimensional defects irreversibility of the DRG flow was proven via different means, using Weyl anomaly matching [47,55].

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[*]gcuomo@scgp.stonybrook.edu
[†]zkomargodski@scgp.stonybrook.edu
[‡]araviv-moshe@scgp.stonybrook.edu

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[84] We use conventions such that the bulk stress tensor $T_b^{\mu\nu}$ does not contain any contribution proportional to δ_D^{d-1} and therefore it is traceless everywhere [47,62].
[85] $\log Z_{\mathcal{M}}^{(CFT)}$ is divergent on the infinite cylinder or in flat space, moreover, in general, it is ambiguous due to various counterterms. But these bulk issues cancel from the definition of $g(M_0 R)$.
[86] This terminology differs from that of Ref. [50], where the term *defect entropy* referred to the defect contribution to the entanglement entropy. Note that in $d = 2$ the defect entanglement entropy and ordinary entropy coincide at fixed points because $h_D = 0$. Here we see that the correct generalization to higher dimensions involves the defect entropy and not the defect entanglement entropy which is also sensitive to h_D [87].
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