Elliptic Double Box and Symbology Beyond Polylogarithms

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We study the elliptic double-box integral, which contributes to generic massless QFTs and is the only contribution to a particular 10-point scattering amplitude in $\mathcal{N} = 4$ SYM theory. Based on a Feynman parametrization, we express this integral in terms of elliptic polylogarithms. We then study its symbol, finding a rich structure and remarkable similarity with the nonelliptic case. In particular, the first entry of the symbol is expressible in terms of logarithms of dual-conformal cross ratios, and elliptic letters only occur in the last two entries.

Introduction.—Understanding of numbers and functions in QFT in general and in $\mathcal{N} = 4$ SYM theory in particular has lead to great progress in calculating scattering amplitudes as well as other quantities.

For one-loop quantities, multiple polylogarithms (MPLs) [1–6] suffice, and this continues to be the case in massless theories for higher loop orders for sufficiently low numbers of external particles. MPLs are characterized via their symbol [7], a tensor product—or word—in so-called letters $\log(\phi_\alpha)$, where ϕ_α are functions of the kinematic variables. These letters encode the singularity and branch-cut structure and their union is known as symbol alphabet. In cases where the ϕ_α are rational (or can be simultaneously rationalized), the symbol has made it possible to bootstrap the corresponding amplitudes to very high loop orders, i.e., to make an ansatz based on an assumed symbol alphabet [8] and to fix the coefficients via various constraints such as the Steinmann conditions [9–11] and cluster adjacency [12,13]; see, e.g., Refs. [14–19]. For slightly more legs, however, also symbol alphabets with ϕ_α occur that are not simultaneously rationalizable [20–23].

Beyond MPLs, infinite towers of more complicated functions occur [24–37]. The simplest of these classes of functions involve integrals over elliptic curves; they have recently been increasingly well understood in terms of so-called elliptic multiple polylogarithms (eMPLs) [38–60]. In particular, also a symbol for eMPLs has been defined [53,61]; the symbol letters in this case are $\Omega^{(j)}(\tilde{\phi}_\alpha)$, where $\tilde{\phi}_\alpha$ are functions of the images of the kinematics when

mapped to the torus, which is equivalent to the elliptic curve.

In $\mathcal{N} = 4$ SYM theory, the first time elliptic functions occur is the 10-point N³MHV amplitude at two-loop order. A particular component of it is given in terms of a single Feynman diagram, the elliptic double-box integral [62], depicted in Fig. 1. This integral was found to satisfy a first-order differential equation relating it to the 6D hexagon [63,64], as well as a further second-order differential equation [65]. A fourfold rational integral representation —and a onefold polylogarithmic one—were found [66], as well as a sum representation [67,68]. So far, however, it has not been possible to express the elliptic double-box integral in terms of eMPLs.

In this Letter, we express the double-box integral in terms of eMPLs and calculate its symbol, finding a rich structure. In particular, we observe that the symbol satisfies the first-entry condition occurring for MPLs [69]: the letters $\Omega^{(j)}(\tilde{\phi}_\alpha)$ in the first entry combine to $\log(u)$, where u is a dual conformal cross ratio. Similarly, the letters in the second entry combine to logs, such that elliptic letters only occur in the last two entries. Moreover, the symbol makes manifest the differential equation relating the elliptic double-box integral to the 6D one-loop hexagon integral.

The linear reducibility problem in the double box and its resolution.—Let us start with the dual conformal Feynman parameter representation of the elliptic double box [66]:
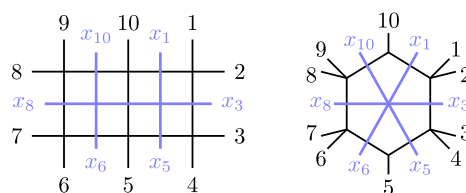

FIG. 1. The elliptic double box and the related 6D hexagon, as well as their dual graphs.

$$I_{\text{db}}^{\text{ell}} = \int_0^\infty d^4\vec{\beta}\, \frac{1}{f_1 f_2 f_3}, \tag{1}$$

where

$$f_1 = \beta_4(1+\beta_1) + \beta_1,$$
$$f_2 = 1 + u_1\beta_4 + v_1\beta_1 + u_2\beta_2 + v_2\beta_3,$$
$$f_3 = (1+u_3\beta_4)\beta_2 + (1+u_4\beta_1)\beta_3 + \beta_2\beta_3 + u_3u_4u_5 f_1. \tag{2}$$

The cross ratios are defined by

$$u_1 = x_{1,3;5,8}, \qquad u_2 = x_{3,6;8,10},$$
$$v_1 = x_{1,8;5,3}, \qquad v_2 = x_{3,10;6,8},$$
$$u_3 = x_{1,3;5,10}, \qquad u_4 = x_{1,6;5,3}, \qquad u_5 = x_{1,5;6,10}, \tag{3}$$

where $x_{a,b;c,d} = (x_{a,b}^2 x_{c,d}^2)/(x_{a,c}^2 x_{b,d}^2)$ with $x_{a,b} = x_a - x_b$ and dual momenta defined as $x_a - x_{a+1} = p_a$.

In addition to the manifest dual conformal symmetry, the double-box integral has two reflections symmetries R_1 and R_2 along the horizontal and vertical direction in Fig. 1. The action of R_1 and R_2 on the cross ratios $\{u_1, v_1, u_2, v_2, u_3, u_4, u_5\}$ gives $\{v_1, u_1, v_2, u_2, u_4, u_3, u_5\}$ and $\{u_2, v_2, u_1, v_1, u_4u_2/v_1, u_3v_2/u_1, u_5\}$, respectively.

As indicated in Ref. [66], three integrations can be performed in terms of polylogarithms, such that the double-box integral can schematically be expressed as

$$I_{\text{db}}^{\text{ell}} \sim \int_0^\infty \frac{d\beta_1}{\sqrt{Q(\beta_1)}} H(\beta_1), \tag{4}$$

where $Q(\beta_1)$ is an irreducible quartic polynomial in β_1 and $H(\beta_1)$ is a pure combination of MPLs of weight three. The obstacle in performing the last integration in terms of elliptic polylogarithms is that the letters of $H(\beta_1)$ involve not only $\sqrt{Q(\beta_1)}$ but also square roots of two quadratic polynomials in β_1. These polynomials share no roots, hence there is no way to rationalize the square roots of the two quadratics without increasing the degree of $Q(\beta_1)$.

To overcome this obstacle, one needs to trace the origin of these additional square roots which are related to the linear reducibility problem of the Feynman parameter integrals [24,57,70,71]. In our case, these square roots of quadratics are introduced in the third integration. More precisely, consider the integral after integrating out β_3 and β_4,

$$\int_0^\infty d^4\vec{\beta}\, \frac{1}{f_1 f_2 f_3} = \int_0^\infty \frac{d\beta_1 d\beta_2}{\mathcal{P}(\beta_1, \beta_2)} \mathcal{G}_2(\beta_1, \beta_2), \tag{5}$$

where the polynomial $\mathcal{P}$ has degree 3 and 2 in β_1 and β_2, respectively, and $\mathcal{G}_2(\beta_1, \beta_2)$ is a pure combination of MPLs of weight two. Three of the letters of $\mathcal{G}_2(\beta_1, \beta_2)$ are quadratic in β_1 and β_2, while the other letters are linear

in β_1 and β_2. It is these three letters that introduce additional square roots in the third integration. To perform the third integration without introducing additional square roots, one needs to make a variable substitution for β_1, β_2 such that the letters of $\mathcal{G}_2$ are linear in one of the new integration variables. A crucial observation here is the following: these three letters, which we denote by q_1, q_2, and q_3, can be expressed as

$$q_1 = \beta_1(\beta_2 u_2 + \beta_1 v_1) + \cdots,$$
$$q_2 = -u_3(\beta_2 + \beta_1 u_4 u_5)(\beta_2 u_2 + \beta_1 v_1) + \cdots,$$
$$q_3 = (\beta_2 + \beta_1 u_4 u_5)(\beta_2 u_2 + \beta_1 v_1) + \cdots, \tag{6}$$

where "$\cdots$" denote terms linear in β_1 and β_2. Then it is natural to introduce the variable substitution

$$x = \beta_1 v_1 + \beta_2 u_2, \qquad \tilde{\beta}_2 = u_2 \beta_2 / v_1, \tag{7}$$

so that all letters of $\mathcal{G}_2$ are linear in $\tilde{\beta}_2$ [72]. Now the integration over $\tilde{\beta}_2$ gives

$$I_{\text{db}}^{\text{ell}} = \int_0^\infty \frac{dx}{y} \mathcal{G}_3(x, y), \tag{8}$$

where

$$y^2 = x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0$$
$$= \left(\frac{v_1}{u_4} [(1-u_4)(x+1-v_2) - u_1 + u_3 v_2] + h_1 + h_2 \right)^2$$
$$- 4h_1 h_2, \tag{9}$$

with

$$h_1 = \frac{u_2 u_4}{v_1} [x^2 + (1 - u_1 + v_1)x + v_1],$$
$$h_2 = \left(x + \frac{v_1}{u_4} \right) \left[(1 + x - u_1) \left(\frac{u_2 u_4}{v_1} - 1 \right) + (1 - u_3) v_2 \right]. \tag{10}$$

Here, the coefficients a_i are polynomials in the cross ratios, and $\mathcal{G}_3$ is a pure combination of MPLs of weight three whose letters are rational functions of x and y.

At this stage, there is no obstacle to performing the integration over x and evaluating it in terms of E_4 functions which are recursively defined as [50]

$$E_4 \begin{pmatrix} n_1 \ldots n_k \\ c_1 \ldots c_k \end{pmatrix}; x = \int_0^x dx'\, \psi_{n_1}(c_1, x') E_4 \begin{pmatrix} n_2 \ldots n_k \\ c_2 \ldots c_k \end{pmatrix}; x', \tag{11}$$

with $E_4(; x) = 1$, where

$$\psi_0(0,x) = \frac{1}{y}, \qquad \psi_{-1}(\infty, x) = \frac{x}{y},$$

$$\psi_1(c,x) = \frac{1}{x-c}, \qquad \psi_{-1}(c,x) = \frac{y_c}{y(x-c)}, \qquad (12)$$

with $y_c = y|_{x=c}$. The definition of $\psi_n(c,x)$ for $|n| > 1$ can be found in Ref. [50]; the kernels (12) are sufficient for the computation of the double-box integral, though.

We give the final result in terms of E_4 functions in the Supplemental Material [73]. Here, we only record the arguments c_i of the E_4's, which make up the set

$$\Bigg\{ 0, -1, \infty, -u_2, -v_1, -\frac{v_1}{u_4}, -1 + \frac{u_1}{u_3}, -u_2 u_4 u_5, -u_2(1-u_4) - v_1, \frac{u_2(u_3 + u_4 - 1) - v_1}{1 - u_3}, \frac{u_2 u_3 u_4 u_5 - v_1}{1 - u_3}, $$

$$\frac{u_2 u_3 u_4 u_5 - v_1}{u_4(1 - u_3 u_5)}, \frac{u_2(u_3 u_4 u_5 v_2 - u_1)}{u_3 v_2 - u_1}, \frac{v_1(u_3 u_4 u_5 v_2 - u_1)}{u_4(u_1 - u_3 u_5 v_2)}, \frac{u_4 u_5 [u_2(u_4 - 1) - v_1] + v_1}{u_4(u_5 - 1)}, \frac{u_1 u_2(u_4 - 1) - v_1(u_1 - u_3 v_2)}{u_1 - u_3 v_2}, $$

$$z_{1,3,5,8} - 1, \bar{z}_{1,3,5,8} - 1, z_{1,3,6,8} - 1, \bar{z}_{1,3,6,8} - 1, -z_{3,5,8,10}, -\bar{z}_{3,5,8,10}, -z_{3,6,8,10}, -\bar{z}_{3,6,8,10},$$

$$\frac{u_2 u_3 u_4 u_5 - v_1 + r_+}{1 - u_3}, \frac{u_2 u_3 u_4 u_5 - v_1 + r_-}{1 - u_3} \Bigg\}, \qquad (13)$$

where $z_{a,b,c,d}\bar{z}_{a,b,c,d} = x_{a,b;c,d}$, $(1 - z_{a,b,c,d})(1 - \bar{z}_{a,b,c,d}) = x_{d,a;b,c}$, and

$$r_\pm = \frac{\mathcal{G}_{45}^{-1} \det \mathcal{G} \pm \sqrt{\det \mathcal{G}^{(45)}} \sqrt{-\det \mathcal{G}}}{2(1 - u_5) x_{1,5}^2 x_{3,10}^2 x_{1,6}^2 x_{3,8}^2 x_{5,10}^2}. \qquad (14)$$

Here, we have introduced the Gram matrix $\mathcal{G} = (x_{i,j}^2)$ with i and j running over the set $\{1, 3, 5, 6, 8, 10\}$, the elements of the inverse of the Gram matrix $\mathcal{G}_{ij}^{-1} = (\mathcal{G}^{-1})_{ij}$, as well as the matrix $\mathcal{G}^{(ij)}$ obtained from $\mathcal{G}$ by deleting the ith and jth rows and columns.

From the elliptic curve to the torus: A birational approach.—To define the pureness of the double-box integral and to evaluate its symbol, one needs to express it in terms of iterated integrals on the torus. To this end, we need to find a bijection between the elliptic curve $\mathcal{C}$ and the torus $\mathbb{C}/\Lambda$, where Λ is the lattice generated by the periods ω_1 and ω_2 of the elliptic curve. Instead of using the map provided in Ref. [50], here we adopt another strategy: first we find the standard Weierstrass form $Y^2 = 4X^3 - g_2 X - g_3$ birationally equivalent to $\mathcal{C}$ based on its rational point at infinity, then use the standard map $z \mapsto (X,Y) = (\wp(z), \wp'(z))$ in terms of the Weierstrass $\wp$ function. This gives $z \mapsto (x,y) = (\kappa(z), \kappa'(z))$, where

$$\kappa(z) = \frac{6a_1 - a_2 a_3 + 12 a_3 \wp(z) - 24 \wp'(z)}{3 a_3^2 - 8[a_2 + 6\wp(z)]}. \qquad (15)$$

There are several comments in order: (i) the infinity point $(+\infty, +\infty)$ is mapped to a lattice point, (ii) each point c in kinematic space corresponds to two points $(c, \pm y_c)$ on the elliptic curve $\mathcal{C}$ and hence to two images on the torus $\mathbb{C}/\Lambda$, which we denote by $z_c^\pm$; these two images satisfy

$$z_c^+ + z_c^- = z_\infty^- + z_\infty^+ \equiv z_\infty^- \quad \text{mod } \Lambda, \qquad (16)$$

since the corresponding points $(X_c^\pm, Y_c^\pm)$, together with (X_∞^-, Y_∞^-), are on the same line. Similarly, one can find that the torus images $z_{c_i}^\pm$ of the kinematics c_i in Eq. (13) satisfy

$$\{z_{c_6}^+ + z_{c_7}^+, z_{c_{25}}^+ + z_{c_{26}}^+, z_{c_6}^+ + z_{c_{10}}^+ + z_{c_{16}}^+,$$

$$z_{c_{10}}^+ - z_{c_{17}}^+ + z_{c_{18}}^+, z_{c_{16}}^+ + z_{c_{23}}^+ - z_{c_{24}}^+\} \equiv 0 \quad \text{mod } \Lambda, \qquad (17)$$

and similar relations for $z_{c_i}^-$ obtained by Eq. (16), where c_i is the ith element in Eq. (13).

In what follows, we will work in the region given by positive momentum-twistor kinematics [74,75], where the four roots of $y^2(x)$ come in complex conjugate pairs as shown in Fig. 2. In this case, the torus image z_c^+ for any real c is simply given by

$$z_c^+ = \int_{-\infty}^c \frac{dx}{y}. \qquad (18)$$

Hence, z_∞^+ is one period of the torus, and we choose it to be ω_2. The image z_c^- can be obtained by Eq. (16) together with $z_\infty^- = \int_{\gamma_-} dx/y$, and the other period is $\omega_1 = \int_{\gamma_1} dx/y$, where the integration contours are defined in Fig. 2.

Now one can introduce iterated integrals on the normalized torus with periods $(1, \tau = \omega_2/\omega_1)$ [50,53]:

$$\tilde{\Gamma}\begin{pmatrix} n_1 \dots n_k \\ w_1 \dots w_k \end{pmatrix}; w = \int_0^w dw' \, g^{(n_1)}(w' - w_1) \tilde{\Gamma}\begin{pmatrix} n_2 \dots n_k \\ w_2 \dots w_k \end{pmatrix}; w', \qquad (19)$$
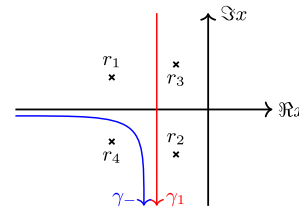

FIG. 2. Four roots of $y^2(x)$ in the positive kinematics region and two integration contours. The contour γ_2 which defines ω_2 runs along the real axis.

with $\tilde{\Gamma}(; w) = 1$. Such an iterated integral is said to have length k and weight $\sum_k n_k$. The integration kernels $g^{(n)}(z)$ are generated by the Eisenstein-Kronecker series

$$\frac{\partial_z \theta_1(0)\theta_1(z+\alpha)}{\theta_1(z)\theta_1(\alpha)} = \sum_{n \geq 0} \alpha^{n-1} g^{(n)}(z), \qquad (20)$$

where $\theta_1(z) = \theta_1(z|\tau)$ is the odd Jacobi theta function.

With these conventions, it is not hard to find

$$\psi_1(c,x)dx = [g^{(1)}(w-w_c^+) + g^{(1)}(w-w_c^-)$$
$$- g^{(1)}(w-w_\infty^+) - g^{(1)}(w-w_\infty^-)]dw, \quad (21a)$$

$$\psi_{-1}(c,x)dx = [g^{(1)}(w-w_c^+) - g^{(1)}(w-w_c^-)$$
$$+ g^{(1)}(w_c^+) - g^{(1)}(w_c^-)]dw, \qquad (21b)$$

$$\psi_{-1}(\infty,x)dx = [g^{(1)}(w-w_\infty^-) - g^{(1)}(w)$$
$$+ g^{(1)}(w_\infty^-) - \omega_1 a_3/4]dw, \qquad (21c)$$

as well as $\psi_0 dx = \omega_1 dw$, where $w_c^\pm$ are the normalized torus images $z_c^\pm/\omega_1$. It is then trivial to express the double-box integral in terms of $\tilde{\Gamma}$ functions as [76]

$$I_{\mathrm{db}}^{\mathrm{ell}} = \omega_1 T_{\mathrm{db}}^{\mathrm{ell}},$$

where $T_{\mathrm{db}}^{\mathrm{ell}}$ is a pure combination of $\tilde{\Gamma}$'s of length four and weight three.

Equivalently, the functions $\tilde{\Gamma}$ can be expressed in terms of the functions $\mathcal{E}_4$ [54], which are defined in complete analogy to Eq. (11) in terms of kernels ($n \geq 0$)

$$\Psi_{\pm n}(c,x)dx = [g^{(n)}(w-w_c^+) \pm g^{(n)}(w-w_c^-)$$
$$- \delta_{\pm n,1}(g^{(1)}(w-w_\infty^+) + g^{(1)}(w-w_\infty^-))]dw. \tag{22}$$

We provide the more compact expression for $T_{\mathrm{db}}^{\mathrm{ell}}$ in terms of $\mathcal{E}_4$'s, as well as code expanding it in terms of $\tilde{\Gamma}$'s, in the Supplemental Material [73].

Let us close this section by remarking on the shuffle regularization. It will be convenient to introduce $\Omega^{(j)}$, defined via

$$\partial_z \Omega^{(j)}(z,\tau) = (2\pi i)^{1-j} g^{(j)}(z,\tau),$$
$$\partial_\tau \Omega^{(j)}(z,\tau) = j(2\pi i)^{-j} g^{(j+1)}(z,\tau). \tag{23}$$

As we shall see in the next section, these $\Omega^{(j)}$ appear as the symbol letters. In contrast to Refs. [40,53], we have included factors of $2\pi i$ such that all letters have weight 1 and we can find linear relations with rational coefficients among them. By definition, $(2\pi i)^{1-j}\tilde{\Gamma}(\genfrac{}{}{0pt}{}{j}{0}; w) = \Omega^{(j)}(w) - \Omega^{(j)}(0)$. However, $\Omega^{(1)}(w)$ is singular at $w = 0$, and the usual shuffle regularization [53] takes $\tilde{\Gamma}(\genfrac{}{}{0pt}{}{1}{0}; w) = \Omega^{(1)}(w) - 2\log \eta(\tau)$ with Dedekind eta function $\eta(\tau)$. Here, to be consistent with the shuffle regularization $G(0;x) = \log(x)$ for MPLs which we implicitly used in Eq. (8), we take the shuffle regularization of $\tilde{\Gamma}(\genfrac{}{}{0pt}{}{1}{0}; w)$ to be

$$\tilde{\Gamma}\left(\genfrac{}{}{0pt}{}{1}{0}; w\right) = \Omega^{(1)}(w) - 2\log \eta(\tau) - \log \frac{2\pi i}{\omega_1 y_0}. \tag{24}$$

Symbology.—The symbol of $\tilde{\Gamma}$ can be defined recursively via the differential of $\tilde{\Gamma}$ in a similar way as for MPLs [53]. The differential of $\tilde{\Gamma}_k^{(n)}$ of weight n and length k schematically takes the form

$$d\tilde{\Gamma}_k^{(n)} = \sum_i (2\pi i)^{j_i-1} \tilde{\Gamma}_{k-1}^{(n-j_i)} d\Omega^{(j_i)}(y_i), \tag{25}$$

where the $\Omega^{(j)}$ are given in Eq. (23) with $\Omega^{(-1)} = -2\pi i\tau$; the precise formula is given in Ref. [53]. It is easy to see that there would be an overall factor $(2\pi i)^{n-k}$ if we keep taking the differential recursively. Thus, it is natural to define the symbol for $(2\pi i)^{k-n}\tilde{\Gamma}_k^{(n)}$ rather than $\tilde{\Gamma}_k^{(n)}$ as

$$\mathcal{S}((2\pi i)^{k-n}\tilde{\Gamma}_k^{(n)}) = \sum_i \mathcal{S}((2\pi i)^{k-n+j_i-1}\tilde{\Gamma}_{k-1}^{(n-j_i)}) \otimes \Omega^{(j_i)}. \tag{26}$$

For the double box, the resulting symbol is of the form

$$\mathcal{S}(T_{\mathrm{db}}^{\mathrm{ell}}) = \frac{1}{2\pi i} \sum_j \Omega^{(j_1)}(w_{j_1}) \otimes \cdots \otimes \Omega^{(j_4)}(w_{j_4}), \quad (27)$$

where $\sum_{i=1}^4 j_i = 3$. Naively, there would be $\Omega^{(6)}$'s at most due to the existence of $\Omega^{(-1)}$, but all $\Omega^{(j>3)}$'s drop out after using $\Omega^{(j)}(-w) = (-1)^{j+1}\Omega^{(j)}(w)$. At this stage, the symbol has around 10^6 terms.

To make contact with the more familiar kinematic world, we can apply Eq. (21a) to $\int_a^b \psi_1(c,x)dx$ to derive the following identity:

$$\log \frac{c-a}{c-b} + \sum_{\sigma \in \pm} \Omega^{(1)}(w_c^\sigma - w_b^+) - \Omega^{(1)}(w_c^\sigma - w_a^+)$$
$$= \sum_{\sigma \in \pm} \Omega^{(1)}(w_\infty^\sigma - w_b^+) - \Omega^{(1)}(w_\infty^\sigma - w_a^+). \tag{28}$$

Further identities involving elliptic letters can be found using the PSLQ algorithm after numerically evaluating the letters via the sum representations given in the Supplemental Material [73]. For example, we found

$$\sum_{i=1}^6 (-1)^{i+1}\left[\Omega^{(1)}(w_{d_i}^- - w_\infty^+) - \Omega^{(1)}(w_{d_i}^- - w_0^+)\right]$$
$$\equiv \log \frac{d_2}{d_3 d_5} + \Omega^{(0)}(w_\infty^+ - w_0^+) \mod i\pi \tag{29}$$

with $d_i \in \{\infty, -v_1/u_4, \bar{z}_{1,3,5,8} - 1, z_{1,3,5,8} - 1, -z_{3,6,8,10}, -\bar{z}_{3,6,8,10}\}$. Moreover, we found complicated identities involving $\Omega^{(2)}$. All these identities turn out to be consequences of Abel's addition theorem and the elliptic Bloch relation [77–79].

Combining (28) and the identities found via the PSLQ algorithm, a dramatic simplification happens: all $\Omega^{(3)}$'s

drop out, only logs remain in the first two entries, and the symbol ends up with an expression of around 10^4 terms. Remarkably, the resulting simplified symbol satisfies the same physical first entry conditions found in the MPL case [69], that is the first entries can only be $\log(x_{a,b;c,d})$. Moreover, the symbol follows certain patterns for the first two entries observed in the MPL case in Refs. [23,69,80,81]: the first two entries form the symbols of $\mathrm{Li}_2(1 - x_{a,b;c,d})$, $\log(x_{a,b;c,d})\log(x_{a',b';c',d'})$, or four-mass boxes. In particular, the symbol satisfies the Steinmann conditions [9,10], i.e., discontinuities in partially overlapping channels vanish.

The complete symbol can be organized by its seven elliptic last entries of type $\Omega^{(0)}(w,\tau) = 2\pi i w$ as well as by its behavior under the two reflections R_1, R_2 [73]:

$$\mathcal{S}(T_{\mathrm{db}}^{\mathrm{ell}}) = \mathcal{S}(I_{\mathrm{hex}}) \otimes \left(w_{c_{25}}^+ - \frac{w_\infty^+}{2} \right)$$

$$+ \mathcal{S}(F_-) \otimes \left(w_\infty^- - \frac{w_\infty^+}{2} \right) + \mathcal{S}(F_+) \otimes w_\infty^+$$

$$+ \left[\mathcal{S}(F_{17}) \otimes \left(w_{c_{17}}^+ - \frac{w_\infty^+}{2} \right) + \mathrm{reflections} \right], \quad (30)$$

where I_{hex} is the 6D hexagon integral (normalized to be pure) in Fig. 1 and F_+, F_-, F_{17} are weight-3 functions whose symbols are known from $\mathcal{S}(T_{\mathrm{db}}^{\mathrm{ell}})$ and recorded in the Supplemental Material [73]. In particular, I_{hex}, F_-, and F_{17} are polylogarithmic. The symbol can be written in terms of 36 rational letters, 24 algebraic letters (in terms of momentum twistors [74]), and besides the 7 elliptic last entries, elliptic letters only appear at the third entry of $\mathcal{S}(F_+)$ and come in only 13 linear independent combinations! (For a list of symbol letters, see the Supplemental Material [73].) The first three terms in Eq. (30) are individually invariant under R_1 and R_2. The fourth term generates a 4-orbit, as indicated by the "+ reflections." The precise behavior of the torus images under the reflections is given in the Supplemental Material [73].

Finally, let us remark on the differential equation relating the double box to the 6D hexagon [63,64]. At the level of the symbol it becomes an immediate consequence of Eq. (30) since only $w_{c_{25}}^+$ in the 7 last entries depends on u_5 and $\omega_1 \partial_{u_5} w_{c_{25}}^+ = x_{1,6}^2 x_{3,8}^2 x_{5,10}^2 / \sqrt{-\det \mathcal{G}}$.

Conclusion and outlook.—In this Letter, we have calculated the 10-point two-loop massless double-box integral in terms of eMPLs and calculated its symbol. This integral is the sole contribution to a particular component of the 10-point N^3MHV superamplitude in planar $\mathcal{N} = 4$ SYM theory, thus allowing us to draw direct conclusions from our findings for scattering amplitudes.

We find that the symbol of the double-box integral shows a very rich structure. In particular, the first entry of the symbol is drawn from the letters $\log(x_{a,b;c,d})$, where $x_{a,b;c,d}$ is a dual-conformal cross ratio. This means that the double-box integral, despite being elliptic, satisfies exactly the same first-entry conditions that were argued to occur for amplitudes

built from nonelliptic polylogarithms. Moreover, the second entry of the symbol contains only letters of log type and satisfies patterns previously observed in the nonelliptic case. The last entry of the symbol is also very restricted, containing only seven possible letters, of elliptic type $\Omega^{(0)}$.

Taking the symbol of our result for the double-box integral, we observed massive cancellations and simplifications, partially due to identities which we first observed numerically via the PSLQ algorithm. As we will elaborate in upcoming work [82], these identities are consequences of Abel's addition theorem and the elliptic Bloch relation [77–79]. It would be interesting to use similar identities to better understand the 13 linearly independent combinations in which the elliptic letters occur in the third entry. Moreover, it would be very interesting to lift this simplified symbol to a simplified function.

The symbol of the double-box integral manifests the differential equation relating it to the 6D one-loop hexagon integral. This suggests that one can bootstrap the symbol via this differential equation, i.e., taking the known symbol of the hexagon, appending the elliptic final letter corresponding to the differential equation, and constructing the remainder of the symbol by imposing integrability [82]. Schematically,

$$\mathcal{S}\left(\boxplus\!\!\boxplus \right) = \mathcal{S}\left(\bigodot \right) \otimes w_{c_{25}}^+ + \mathrm{integrability}. \quad (31)$$

Traintrack integrals [27], which involve integrations over a higher-dimensional Calabi-Yau manifold, similarly satisfy differential equations relating them to n-gons [63,64]. While the functional space and corresponding symbol is not yet understood in these cases, it seems likely that a bootstrap based on the differential equation will also be possible in these cases.

In the case of MPLs with rational arguments, the symbol alphabets occurring for amplitudes as well as their adjacency conditions can be understood in terms of cluster algebras [12,13,16], and a similar understanding is currently being developed in the case of the Feynman integrals [83–85] and amplitudes including algebraic letters [86–91]. It would be very interesting to use the data we provide in this work to extend the cluster program to the elliptic case. Similarly, it would be interesting to extend the amplitude bootstrap program to the elliptic case.

The double-box integral we considered in this Letter is arguably the simplest elliptic integral contributing to planar $\mathcal{N} = 4$ SYM theory, and to massless QFTs in general. We expect that the techniques developed in this Letter can also be applied to more general elliptic integrals, such as the general 12-point double-box integral, corresponding penta-box integrals, and double-pentagon integrals. Combining these integrals with the understanding of prescriptive unitarity and the corresponding leading singularities [92,93] would directly allow us to calculate many further elliptic amplitudes in the massless case. Moreover, also the double-box integral with generic masses is elliptic [94], and should be amenable to the techniques presented here.

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