Operational Characterization of Infinite-Dimensional Quantum Resources

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Recently, various nonclassical properties of quantum states and channels have been characterized through an advantage they provide in quantum information tasks over their classical counterparts. Such advantage can be typically proven to be quantitative, in that larger amounts of quantum resources lead to better performance in the corresponding tasks. So far, these characterizations have been established only in the finite-dimensional setting, hence, leaving out central resources in continuous variable systems such as entanglement and nonclassicality of states as well as entanglement breaking and broadcasting channels. In this Letter, we present a fully general framework for resource quantification in infinite-dimensional systems. The framework is applicable to a wide range of resources with the only premises being that classical randomness cannot create a resource and that the resourceless objects form a closed set in an appropriate sense. As the latter may be hard to establish for the abstract topologies of continuous variable systems, we provide a relaxation of the condition with no reference to topology. This envelopes the aforementioned resources and various others, hence, giving them an interpretation as performance enhancement in so-called input-output games.

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Introduction.—The search of a unified framework for describing the nonclassical properties of quantum mechanics has led to the development of quantum resource theories [1]. Quantum resource theories are abstract structures built on the notions of free states and free operations. The former are quantum objects that have no resource content, and the latter are maps that do not convert free states into resource states. As an example, in the resource theory of entanglement, separability is related to free states, and local operations assisted by classical communication are free operations. To quantify the resource content, the notion of a resource measure is introduced. These are functions that do not increase under the action of free operations. The well-known max-relative entropy of entanglement [2,3], the robustness of coherence [4], and the accessible information in quantum-to-classical channels [5] are examples of such measures.

When applied to quantum information theory, an important milestone of these abstract theories is their ability to point out practical communication tasks in which quantum resources provide an advantage over their classical counterparts [4–12], see also the experimental verification of such advantage [13,14]. Moreover, the provided outperformance is quantitative, in that higher resource content relates to better performance. For example, the resource measure known as the best separable approximation [15] equals the overhead that an entangled state can provide over all separable states in the task of subchannel exclusion [16,17], and the measure known as incompatibility robustness [18] equals the advantage that incompatible quantum measurements provide over all compatible ones in the task of state discrimination [19–23].

So far, the proofs of the practical advantage have been limited to the case of finite-dimensional resources, with the exception of measurement resources, for which the connection has been recently extended to include also the case of continuous variable systems [24]. In this work, we provide a method for extending the known finite-dimensional proofs to the infinite-dimensional regime in the missing cases of quantum states and channels. This encompasses various infinite-dimensional quantum resources, for which an operational advantage has not been formerly established, such as entanglement, nonclassicality, coherence, and the quantum marginal problem of quantum states, as well as broadcasting and the property of being entanglement breaking of quantum channels.

Our extension procedure relies on approximating continuous variable quantifiers by their finite-dimensional counterparts. This results in quantifiers that enjoy many basic properties required from a proper resource measure such as monotonicity, convexity, and lower semicontinuity. These technical requirements account for the fact that free operations and convex mixtures cannot increase the resource content, and to the ability of witnessing resources despite small fluctuations. However, the desirable property of faithfulness and the central result of this Letter, that is proving that infinite-dimensional resources give an operational advantage, require more careful consideration. The discussion on these properties can be divided into two cases: either one requires them precisely, a case for which we provide sufficient topological constraints, or one that requires them approximately, in which case we provide sufficient algebraic constraints.

Concerning the first case, in the finite-dimensional setting, faithfulness and the operational advantage require the set of free states to be closed. This is also expected in the infinite-dimensional case, where, however, sets might have nonequivalent closures. We show that a natural choice for the appropriate topology arises from the relevant quantum tasks. For sets of free objects that are closed in such topology, our extension procedure results in established faithful infinite-dimensional quantifiers. As each step of the procedure is finite dimensional and, hence, carries the connection between resource content and a quantum advantage, the limit procedure gives infinite-dimensional resources the sought connection. More precisely, we show that infinite-dimensional quantum resources lead to a nonclassical advantage in so-called input-output games. These are games in which a fixed set of quantum messages has to be communicated through a channel in an optimal manner. We exemplify the use of the topological criterion with central resources related to nonclassicality, coherence and asymmetry of states, as well as the entanglement breaking property of channels.

Concerning the approximate case, our extension procedures are not unique, i.e., different approximation procedures may lead to different quantifiers. Whereas this is not the case when the above closure requirement is fulfilled, we introduce another condition on the free set, under which there exists a family of equivalent extension procedures. This requires the possibility of an extension procedure that uses only free operations. Under this condition, the extension procedures result in quantifiers, for which the game interpretation holds. Furthermore, these quantifiers are faithful on the closure of the free set. This is in line with the finite-dimensional setting, where quantifiers separate the free set from objects that are outside of its closure. This algebraic constraint is demonstrated with entanglement of states, the quantum marginal problem, and the broadcastability of channels.

Resource quantification in the finite-dimensional setting.—We concentrate on two types of resource quantifiers. The first consists of robustness measures $\mathcal{R}_{F,N}$, where F is the (convex and closed) free set and N is the noise set. In our applications, the noise set will be either all quantum channels (yielding the generalised robustness) or the free set (yielding the free robustness). Robustness measures quantify the amount a given resource channel Λ can resist mixing with noise channels $\tilde{\Lambda} \in N$ before the resource is lost. Formally, we have, [9,12,25]

$$\mathcal{R}_{F,N}(\Lambda) = \min\left\{t \geqslant 0 \middle| \frac{\Lambda + t\tilde{\Lambda}}{1+t} \in F, \tilde{\Lambda} \in N\right\}. \quad (1)$$

The second type of resource quantifier is the convex weight W_F, i.e., the best free approximation of a resource channel Λ. Formally, we define, [15–17]

$$\mathcal{W}_F(\Lambda) = \min\{\mu \geqslant 0 | \Lambda = \mu\Gamma + (1-\mu)\Lambda_F\}, \quad (2)$$

where the optimization is over all channels Γ and all free channels $\Lambda_F \in F$. A possible intuition behind the convex weight is the question of how frequently a free channel Λ_F can be used in the preparation procedure of a resource channel Λ.

Both quantifiers $\mathcal{R}_{F,N}$ and $\mathcal{W}_{\mathcal{F}}$ can be cast as conic programs, which allows their evaluation in the dual form; see, for example, Refs. [9,12] for the robustness and Refs. [16,17] for the weight. For the dual formulation, we need the Choi presentation of the channels, i.e., $J_\Lambda \coloneqq (1/d)\sum_{ij} |i\rangle\langle j| \otimes \Lambda(|i\rangle\langle j|)$, so that

$$\begin{aligned} 1 + \mathcal{R}_F(\Lambda) = \max_Y \quad & \mathrm{tr}[YJ_\Lambda] \\ \text{s.t.:} \quad & Y \geqslant 0, \quad \mathrm{tr}[YT] \leqslant 1 \quad \forall\ T \in J_F, \end{aligned} \quad (3)$$

and

$$\begin{aligned} 1 - \mathcal{W}_F(\Lambda) = \min_Y \quad & \mathrm{tr}[YJ_\Lambda] \\ \text{s.t.:} \quad & Y \geqslant 0, \quad \mathrm{tr}[YT] \geqslant 1 \quad \forall\ T \in J_F, \end{aligned} \quad (4)$$

where Y constitutes a witness and J_F is the image of the free set F under the Choi isomorphism.

It should be noted that the evaluation through the dual holds when the Slater conditions hold. First, this requires the problems to be finite, and, second, for the convex weight, the required (feasible interior point) condition can be verified by choosing $Y = \alpha\mathbb{1}$ for large $\alpha > 0$. For the robustness one requires the existence of a point $\Lambda_F \in F$ and a number $\alpha > 1$ such that $\alpha J_{\Lambda_F} - J_\Lambda$ is an interior point of the cone C_{J_N} defined by the noise set N. In our noise sets, the existence of some full-rank point (such as the maximally mixed state) in the free set guarantees the Slater condition to hold; cf. Refs. [12,17] for more details.

The dual formulation can be used to relate the resource measures to the performance a channel Λ provides in a discrimination [9,12] or an exclusion [17] input-output game. The relevant games $\mathcal{G}$ consists of input quantum states $\{\varrho_a\}_a$, i.e., positive unit-trace operators, a quantum measurement $\{M_b\}_b$, that is a positive operator valued measure (or POVM for short), i.e., $M_b \geqslant 0$ for every b and $\sum_b M_b = \mathbb{1}$, where $\mathbb{1}$ is the identity operator, and a score assignment $\{\omega_{ab}\}_{ab}$. The payoff $\mathcal{P}$ of the game $\mathcal{G}$ for a given channel Λ reads

$$\mathcal{P}(\Lambda,\mathcal{G})=\sum_{a,b}\omega_{ab}\mathrm{tr}[\Lambda(\varrho_a)M_b]. \tag{5}$$

To relate such games to the resource quantifiers, one can interpret a witness Y as a game. We write $Y=\sum_{ab}\omega_{ab}\varrho_a^T\otimes M_b$, cf. Ref. [26], which makes the object functions of Eqs. (3) and (4) special instances of the payoff function in Eq. (5). Clearly, optimal witnesses Y^r for the robustness and Y^w for the weight, with their respective instances of the game $\mathcal{G}_{Y^r}$ and $\mathcal{G}_{Y^w}$, lead to $\mathcal{P}(\Lambda,\mathcal{G}_{Y^r})\geqslant[1+\mathcal{R}_{F,N}(\Lambda)]\max_{\Lambda_F\in F}\mathcal{P}(\Lambda_F,\mathcal{G}_{Y^r})$ and $\mathcal{P}(\Lambda,\mathcal{G}_{Y^w})\leqslant[1-\mathcal{W}_F(\Lambda)]\min_{\Lambda_F\in F}\mathcal{P}(\Lambda_F,\mathcal{G}_{Y^w})$. To make these inequalities tight, one writes Λ using Eqs. (1) and (2), and notices that the games are linear in Λ. This leads to

$$\sup_{\mathcal{G}_N}\frac{\mathcal{P}(\Lambda,\mathcal{G}_N)}{\max_{\Lambda_F\in F}\mathcal{P}(\Lambda_F,\mathcal{G}_N)}=1+\mathcal{R}_{F,N}(\Lambda) \tag{6}$$

and

$$\inf_{\mathcal{G}}\frac{\mathcal{P}(\Lambda,\mathcal{G})}{\min_{\Lambda_F\in F}\mathcal{P}(\Lambda_F,\mathcal{G})}=1-\mathcal{W}_F(\Lambda), \tag{7}$$

where the optimization goes as follows: when the noise set coincides with the set of channels, or one considers the convex weight, the optimization runs over those games G_N that have a non-negative payoff for any channel, and for the case $N=F$, one optimizes over those games that have a non-negative payoff for the free set. Also, the games resulting in a zero denominator are excluded.

We note that the above connections between resource quantifiers and quantum games hold also for channel tuples. As the extension is straightforward, i.e., the channels, the respective witnesses, and the games are replaced by tuples, we have spelled the connection in the Supplemental Material, Sec. A [27]. We further note that quantum states can be seen as quantum channels with a trivial input. Hence, all the above results on quantum channels work also for quantum states. In this case, the corresponding game has a trivial input as well, a fact that renders the games into subchannel discrimination [7,9,12,23] or subchannel exclusion tasks [16,17].

Resource quantification in the infinite-dimensional setting.—From here on, we concentrate explicitly on the robustness measures, as the procedure for the convex weight can be obtained with very similar techniques. The free set of channels F is now a convex subset of the set of all Schrödinger channels (between operators on separable Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$).

Our technique is based on a finite-dimensional approximation of the general resource measures. There are several ways to do the approximation, but we concentrate on a particular method. We say that two sequences $(\alpha_n)_n$ and $(\beta_n)_n$ of channels form an *approximation procedure* if the following five conditions hold: (i) α_n is a channel within a finite-dimensional subspace $\mathcal{H}_n$ of $\mathcal{H}$ and β_n is within a finite-dimensional subspace $\mathcal{K}_n$ of $\mathcal{K}$ and (ii) for any states ϱ on $\mathcal{H}$ and σ on $\mathcal{K}$, $\|\varrho-\alpha_n(\varrho)\|_{\mathrm{tr}}\to 0$ and $\|\sigma-\beta_n(\sigma)\|_{\mathrm{tr}}\to 0$ as $n\to\infty$. These conditions ensure that the approximations use only finite-dimensional systems and that each state can be approximated. The remaining three conditions are not relevant for understanding our main results, but they guarantee some technical details for our proofs, such as monotonicity of the approximation procedure: (iii) whenever ϱ is supported by $\mathcal{H}_n$ then $\alpha_n(\varrho)=\varrho$ and similarly for a state σ supported by $\mathcal{K}_n$, and (iv) if σ is injective (i.e., has strictly positive eigenvalues), then $\beta_n(\sigma)$ is of full rank within $\mathcal{K}_n$. Moreover, (v) we require that, when $m\leqslant n$, $\alpha_n\circ\alpha_m=\alpha_m$ and $\beta_m\circ\beta_n=\beta_m$.

As an example, take an increasing sequence $(\mathcal{H}_n)_n$ of finite-dimensional subspaces of $\mathcal{H}$ such that the closure of their union is $\mathcal{H}$. Define the channels $\alpha_n(\varrho)=P_n\varrho P_n+\mathrm{tr}[\varrho P_n^\perp]\varrho_0$ where P_n is the orthogonal projection onto $\mathcal{H}_n$ and ϱ_0 is some fixed state within the first space $\mathcal{H}_1$. By defining the channels β_n in a similar manner, we clearly get an instance of an approximation procedure.

Suppose that $\mathbb{A}=\{\alpha_n,\beta_n\}_n$ is an approximation procedure. Using this, we denote $F_n:=\{\beta_n\circ\Lambda_F\circ\alpha_n|\Lambda_F\in F\}$ and $N_n:=\{\beta_n\circ\Lambda_N\circ\alpha_n|\Lambda_N\in N\}$. We further denote the standard closures of these finite-dimensional sets by $\bar{F}_n$ and $\bar{N}_n$. We easily see that the sequence of finite-dimensional robustnesses $[\mathcal{R}_{\bar{F}_n,\bar{N}_n}(\beta_n\circ\Lambda\circ\alpha_n)]_n$ related to this approximation procedure is increasing (see Supplemental Material, Sec. B [27]). We define the *approximate robustness* $\mathcal{R}_{F,N}^{\mathbb{A}}$ as the supremum of the finite-dimensional steps. According to the above, this is the limit of $\mathcal{R}_{\bar{F}_n,\bar{N}_n}(\beta_n\circ\Lambda\circ\alpha_n)$ as $n\to\infty$.

To see that $\mathcal{R}_{F,N}^{\mathbb{A}}$ is a proper resource measure, we note that as a pointwise supremum of a family of convex lower semicontinuous functions, $\mathcal{R}_{F,N}^{\mathbb{A}}$ is also convex and lower semicontinuous. Also, $\mathcal{R}_{F,N}^{\mathbb{A}}$ is nonincreasing under free operations, i.e., operations that do not map elements of F outside of F, see Supplemental Material, Sec. B [27]. Another property required from a resource measure is faithfulness, i.e., $\mathcal{R}_{F,N}^{\mathbb{A}}(\Lambda)=0$ if and only if $\Lambda\in F$. As this property depends on the topological properties of the set F, we will comment on this later. We note that the kind of dependency is not specific to the approximate robustness, but it also affects the original one $\mathcal{R}_{F,N}$.

The approximate robustness has an interpretation as performance in input-output games. To see this, we note that the finite-dimensional results from the previous section apply to any convex and compact free set F, when N is either F or the whole set of channels. As the set N maps surjectively to the corresponding set N_n of channels (free or whole) from $\mathcal{H}_n$ to $\mathcal{K}_n$, the robustness measure $\mathcal{R}_{\bar{F}_n,\bar{N}_n}$ has a game interpretation for each n. More specifically, Eq. (6) leads to

$$\sup_n \sup_{\mathcal{G}_n} \frac{\mathcal{P}(\beta_n \circ \Lambda \circ \alpha_n, \mathcal{G}_n)}{\sup_{\Lambda_F \in F_n} \mathcal{P}(\Lambda_F, \mathcal{G}_n)} = 1 + \mathcal{R}^{\mathbb{A}}_{F,N}(\Lambda), \qquad (8)$$

where the second supremum runs over all games $\mathcal{G}_n$ on the set of channels between $\mathcal{H}_n$ and $\mathcal{K}_n$ such that $\mathcal{P}(\Lambda_n, \mathcal{G}_n) \geqslant 0$ for all $\Lambda_n \in N_n$.

In principle, the quantity $\mathcal{R}^{\mathbb{A}}_{F,N}$ can depend on the approximation procedure. However, it is always a lower bound on the actual robustness, as defined in Eq. (1), and there is a simple sufficient condition for these robustnesses to agree. Namely, if the sets F and N are suitably closed, then the approximation procedure reaches $R_{F,N}$.

A natural choice for the topology is related to the games. One can ask for any given input, i.e., a trace-class operator ϱ_a, and any given output, i.e., a bounded operator M_b, whether a sequence (or a net) of channels $(\Lambda_n)_n$ converges to a channel Λ in the sense that $\mathrm{tr}[\Lambda_n(\varrho_a)M_b] \to \mathrm{tr}[\Lambda(\varrho_a)M_b]$ as $n \to \infty$. We denote the topology associated with this type of convergence by τ. When F is closed with respect to τ and N is the whole set of channels or N is τ closed, $\mathcal{R}^{\mathbb{A}}_{F,N} = \mathcal{R}_{F,N}$ for any approximation $\mathbb{A}$. In this case, the approximate robustness (as well as the original one) are faithful. We summarize these ideas in the following Theorem, the detailed proof of which is presented in the Supplemental Material, Sec. C [27].

Theorem 1.—Let the free set be τ closed. Whenever N is τ closed or the whole set of channels, we have $\mathcal{R}^{\mathbb{A}}_{F,N}(\Lambda) = \mathcal{R}_{F,N}(\Lambda)$, and the analogically defined approximate weight $\mathcal{W}^{\mathbb{A}}_F(\Lambda) = \mathcal{W}_F(\Lambda)$ for any approximation procedure, and all quantifiers are faithful. If, furthermore, $N = F$ or N is the whole set we have

$$\sup_{\mathcal{G}} \frac{\mathcal{P}(\Lambda, \mathcal{G})}{\sup_{\Lambda_F \in F} \mathcal{P}(\Lambda_F, \mathcal{G})} = 1 + \mathcal{R}_{F,N}(\Lambda), \qquad (9)$$

where the outer supremum runs over those games that have a positive payoff in the set N, whenever the right-hand side is finite. Moreover, one has

$$\inf_{\mathcal{G}} \frac{\mathcal{P}(\Lambda, \mathcal{G})}{\inf_{\Lambda_F \in F} \mathcal{P}(\Lambda_F, \mathcal{G})} = 1 - \mathcal{W}_F(\Lambda), \qquad (10)$$

where the infimum runs over those games that have a positive payoff for any channel, and we omit the games that result in a zero denominator.

For completeness, we note that the statement on generalized robustness can be extended to the max-relative entropy measure, as the two bear a simple relation (see Supplemental Material, Sec. D [27]).

Examples of resources under the topological constraint.—As an application of our Theorem 1, one can take channels with a trivial input. These correspond to quantum states, and the topology τ reduces to the well-known σ-weak topology generated by bounded operators. In this topology a sequence of states $(\varrho_n)_n$ converges to ϱ if for any bounded operator B we have $\mathrm{tr}[\varrho_n B] \to \mathrm{tr}[\varrho B]$ as $n \to \infty$.

The use of this abstract topology is easiest illustrated by the resource of nonclassicality arising from the negativity of quasiprobability distributions. In this case, the free set consists of states ϱ for which the probability density $\mathrm{tr}[\varrho D(z)\Delta_\lambda D(z)^\dagger]$ is non-negative. Here $D(z)$ is the displacement operator, $\Delta_\lambda = (1-\lambda)\sum_{n=0}^{\infty} \lambda^n |n\rangle\langle n|$, and $\lambda = (s+1)/(s-1)$ with s referring to the s parametrization of quasiprobabilities. For $\lambda = 0$ we recover the Q function, for $\lambda = -1$ one has the Wigner function, and for $\lambda \to -\infty$ one gets the P function. Importantly, for $-1 \leqslant \lambda < 1$, the operator $D(z)\Delta_\lambda D(z)^\dagger$ inside the trace is bounded. This relates exactly to the topology we are concerned with, i.e., tracing states against bounded operators. Clearly, any sequence of states for which the trace is non-negative will output a non-negative number on the limit. Hence, the set of states with a non-negative quasiprobability distribution with $\lambda \in [-1, 1)$ is closed. Within this interval, only values up to $\lambda < 0$ can result in negative quasiprobabilities. Thus, Theorem 1 can be applied to the resource given by the negativity of quasiprobability distributions. For an interested reader, we have made the above statements precise in the Supplemental Material, Sec. E [27].

On top of nonclassicality, our result applies to the free sets related to coherence and asymmetry of states, as well as to the entanglement breaking property of channels, thus, generalizing the results of Refs. [4,31,37] to the infinite-dimensional setting. The only technical result needed for the generalization is the τ closedness of the related free sets, which we prove in the Supplemental Material, Sec. E [27] for the mentioned cases.

Approximation with no reference to topology.—We give another sufficient condition under which the approximate quantifiers satisfy the counterparts of Eqs. (9) and (10). Here, the approximate robustness has again a clear operational interpretation as performance in general (i.e., possibly infinite-dimensional) games, cf. Eq. (8). This is summarized in the following Observation, the proof of which is given in the Supplemental Material, Sec. F [27].

Observation 1.—Let N be the free set F or the whole set and the approximation procedure $\mathbb{A} = \{\alpha_n, \beta_n\}_n$ be such that $F_n \subseteq F$ and $N_n \subseteq N$ for all n. Now Eqs. (9) and (10) hold when one replaces the robustness $\mathcal{R}_{F,N}$ by $\mathcal{R}^{\mathbb{A}}_{F,N}$ and $\mathcal{W}_F$ by $\mathcal{W}^{\mathbb{A}}_F$. In this case we do not require τ closedness of the free set F. In particular, the approximate quantifiers are independent of the chosen approximation procedure $\mathbb{A}$, and they lower bound the robustness $\mathcal{R}_{F,N}$ and the weight $\mathcal{W}_N$, respectively.

In the setting of the above Observation, the approximate quantifiers correspond to the extensions of the original quantifiers with respect to the τ closure of the free set; for the technicalities see the Supplemental Material, Sec. F [27]. It is natural to consider such extensions, as faithfulness is difficult to establish when the free set is not closed. It follows that the approximate quantifiers are faithful with respect to the closure. Especially, they are faithful with respect to F itself, whenever the closure does not introduce any physical states or channels in addition to the original ones in F. The counterpart of these, the unphysical idealizations, are common in quantum field theory. They are states that do not respect the Born rule and channels that have no representation in the Schrödinger picture.

Examples of resources under the algebraic constraint.—As an illustrative example of state resources, we take bipartite entanglement. Here the free set is the trace-norm closure of the convex hull of product states. One possible approximation procedure is given by $\beta_n(\varrho) = P_n \otimes P_n \varrho P_n \otimes P_n + \mathrm{tr}[\varrho(P_n \otimes P_n)^\perp]\varrho_0$. Note that as states are channels with a trivial input, the approximation procedure only requires the channels $\{\beta_n\}$ on the output. Clearly, $\beta_n(\varrho)$ is separable for any separable state. Hence, this approximation procedure fulfills the conditions of Observation 1, for example, in the cases of the generalized robustness and the convex weight.

It is worth noting that in the related work [38] it was shown that the cone of separable states is closed in a coarser topology than ours. This implies the closedness in our topology as well. This means that Theorem 1 is applicable to entanglement. Importantly, closedness in our topology shows further that there are no unphysical separable states.

Similarly to entanglement, one can find simple approximation procedures for the quantum marginal problem and the broadcasting problem of channels. For the related free sets, the τ closedness remains an open problem. For details of these examples, we refer to Supplemental Material, Sec. G [27].

Conclusions.—We have presented a method for extending finite-dimensional quantum resource quantifiers into the infinite-dimensional regime. We have applied our technique to the well-established quantifiers of generalized robustness, the free robustness, and the convex weight. In the case of quantum states (quantum channels) these quantifiers were formerly known to relate to a performance boost that quantum resources give in discrimination tasks (quantum games) in the finite-dimensional setting. We have identified sufficient topological as well as algebraic conditions, under which such performance interpretation can be extended to the infinite-dimensional setting. We have presented various examples of quantum state and channel resources that fall under these conditions. These include entanglement, coherence and nonclassicality of quantum states, as well as broadcasting and a nonentanglement breaking property of quantum channels.

For future research, it will be interesting to find exact conditions under which infinite-dimensional resource theories allow faithful quantifiers with the performance interpretation. Moreover, an open question is to identify resource theories where the extended quantifiers fail to coincide with the established ones, i.e., where there is a finite gap between the two. Furthermore, our work paves the way to the resource theory of more general dynamical objects, such as quantum instruments, and more specialized quantifiers therein, such as tolerance against specific noise sets in continuous variable systems.

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Note added.—Recently, two articles by Lami, Regula, Takagi, and Ferrari [38,39] presented related results on quantum state resources. In their work, they presented a sufficient condition for the closing of the gap between their counterpart of our approximate robustness and the original robustness measure. In our setting that condition is as follows: We define the topology τ_0 on the set of non-normalized channels, i.e., the space $\mathcal{V}$ of completely bounded linear maps as the coarsest topology with respect to which the maps $\mathcal{V} \ni \Lambda \mapsto \mathrm{tr}[\Lambda(\varrho)K] \in \mathbb{C}$ are continuous for all input states ϱ and compact operators on the output space $\mathcal{K}$. The condition is that the cone corresponding to F (i.e., the cone whose intersection with the set of channels coincides with F) is τ_0 closed. (Use of cones here is motivated by noting that F itself is typically not τ_0 closed as the trace-preservation condition for channels is problematic in this topology.) As τ_0 is coarser than our topology τ, this cone is also τ closed and so is F as the intersection of this cone with the τ-closed set of completely positive unital linear maps in the Heisenberg picture (possibly without a Schrödinger description). Thus this condition implies our condition in Theorem 1. As an example, the authors proved that the cone of separable states is closed in the reduction of τ_0 topology to the case of quantum states, i.e., in the σ-weak topology generated by the compact operators. This shows, that the set of separable states is τ closed and, consequently, falls into the realm of our Theorem 1.

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