Lieb-Schultz-Mattis Theorem in Higher Dimensions from Approximate Magnetic Translation Symmetry

Yasuhiro Tada *

Quantum Matter Program, Graduate School of Advanced Science and Engineering, Hiroshima University, Higashihiroshima, Hiroshima 739-8530, Japan
and Institute for Solid State Physics, University of Tokyo, Kashiwa 277-8581, Japan

(Received 27 May 2021; accepted 8 November 2021; published 3 December 2021)

We prove the Lieb-Schultz-Mattis theorem on the energy spectrum of a general two- or three-dimensional quantum many-body system with the U(1) particle number conservation and translation symmetry. Especially, it is demonstrated that the theorem holds in a system with long-range interactions. To this end, we introduce approximate magnetic translation symmetry under the total magnetic flux $\Phi = 2\pi$ instead of the exact translation symmetry, and explicitly construct low energy variational states. The energy spectrum at $\Phi = 2\pi$ is shown to agree with that at $\Phi = 0$ in the thermodynamic limit, which concludes the Lieb-Schultz-Mattis theorem.

DOI: 10.1103/PhysRevLett.127.237204

Introduction.—Understanding the low energy spectrum of a quantum many-body system is a central issue in condensed matter physics [1]. The spectrum can be either gapless in some systems or it can be gapped in other systems with spontaneously broken discrete symmetry and an intrinsic topological order [2,3], in addition to trivial uniquely gapped systems. In this context, the Lieb-Schultz-Mattis (LSM) theorem is a fundamental theorem which can put strong constraints on possible energy spectra and provide a guiding principle for searching exotic quantum states including topological states with long-range entanglement [4–18]. Especially, the original LSM theorem for one dimension holds in a system with long-range density-density interactions, and provides a lower bound of ground state degeneracy (D), $D \geq q$, for a gapped system with the filling per unit cell $\rho = p/q$ [4–6]. The wide applicability of the theorem is fundamentally important, since long-range interactions naturally exist in real systems [19–33] and they can have significant impacts on energy spectra. For example in three dimensions, the Coulomb interaction gaps out the collective charge excitations in metals and plays a crucial role in the Anderson-Higgs mechanism in superconductors [22–24]. Exotic quantum phases can be realized in various systems where long-range interactions are essential, such as in Coulomb interacting electrons [25–28] and dipolar systems [29–33]. Besides, D is closely related to the nature of ground states for both broken discrete symmetry [34,35] and a topological order [36–38], which might be affected by long-range interactions.

Unfortunately, however, the original proof cannot be applied to a higher-dimensional system with an isotropic system size, and higher-dimensional extensions were made possible more than thirty years after the original work [8–12]. Based on local twist of a short-range Hamiltonian [9–12], it was shown that $D \geq q$ for a gapped system under an assumption on matrix elements of local operators. This may be generalized to some rapidly decaying long-range interacting systems, but exact conditions are not yet known. On the other hand, the higher-dimensional LSM theorem was proved also in a different approach under an hypothesis that an excitation gap does not close when a 2π-flux quanta piercing a hole of the torus system is adiabatically inserted [8]. Although this approach is formally applicable to a system with long-range interactions, the adiabatic hypothesis is a subtle issue especially in such a system and its validity is still under debate [8–11,39,40]. Therefore, it is still not clear whether or not the LSM theorem holds in a higher-dimensional system with long-range interactions.

In this study, we discuss the LSM theorem in higher dimensions, especially focusing on long-range interacting systems. With the use of approximate magnetic translation instead of the conventional one, we can prove the theorem and extend its applicability to a wider class of systems. Technically, our proof may be regarded as a simple generalization of the original one-dimensional LSM argument and therefore long-range interactions can be treated in a straightforward way, which is an advantage of our approach. To be concrete, we consider a simple model of spinless particles (either fermions or bosons) on a two-dimensional square lattice of a linear size $L_x \simeq L_y \simeq L = \sqrt{L_x L_y}$ with the periodic boundary condition. Our proof is applicable also to a three-dimensional system with a size $L_z \simeq L$. The Hamiltonian is given by

$$H(\phi) = H_t(\phi) + H_V = -\sum_{\langle i,j \rangle} t_{ij}(\phi) c_i^\dagger c_j + \frac{1}{2}\sum_{i,j} V_{ij}\tilde{n}_i \tilde{n}_j, \qquad (1)$$

where $j = (x_j, y_j)$ is a site position and $\langle i, j\rangle$ represents a nearest neighbor pair of sites. The hopping integral includes the vector potential $t_{jk}(\phi) = te^{iA_{jk}}$ with $t \in \mathbb{R}$ corresponding to a uniform magnetic flux per plaquette $\phi = \sum_{\langle i,j\rangle \in \text{plaquette}} A_{ij}$. The second term H_V describes the density-density interaction with $\tilde{n}_j = c_j^\dagger c_j - \rho$ at the filling $\rho = p/q$ and the potential $V_{ij} = V_{|i-j|}$ can include long-range interactions in addition to short-range interactions. The Hamiltonian possesses translation symmetry when $A_{ij} = 0$. We consider a class of general interactions with stability of the Hamiltonian and extensiveness of energy eigenvalues, including stable tempered interactions and Coulomb interaction [19,20]. Then, we prove the following statement.

Theorem.—Consider the Hamiltonian $H(\phi = 0)$. When the filling per unit cell is $\rho = p/q$ with coprime $p, q \in \mathbb{N}$, either there exist gapless excitations or the ground states are at least q-fold degenerate in the thermodynamic limit.

The proof consists of two steps. (i) We first construct approximate magnetic translation operators $\mathcal{T}_{x,y}$ in the presence of $\phi_L = 2\pi/L_xL_y = 2\pi/L^2$ and show that the low energy states of $H(\phi_L)$ are nearly q-fold degenerate in a finite size system as a consequence of a nontrivial commutation relation of $\mathcal{T}_x$, $\mathcal{T}_y$ corresponding to a projective representation of $\mathbb{Z} \times \mathbb{Z}$. (ii) Next, we demonstrate that the energy difference $\delta E_n(\Phi_0) = [E_n(\Phi_0) - E_n(0)]$ vanishes in the thermodynamic limit, where $E_n(\Phi_0)$ is the nth eigenvalue of $H(\phi_L)$ with the total magnetic flux, $\Phi_0 = L_xL_y \times \phi_L = 2\pi$. By combining these two results, we can complete the proof of the main theorem [41]. The proof can be generalized to a wide class of models with hopping beyond the nearest neighbors, lattices other than the square or cubic lattice, spins, and orbitals, and some other long-range interactions. In the following, we discuss the two steps for the Hamiltonian Eq. (1) and generalizations will be presented elsewhere.

Step (i) approximate magnetic translation and low energy states.—First, we give an explicit construction of the approximate magnetic translation operators for the Hamiltonian Eq. (1) and also of low energy variational states under the small magnetic field ϕ_L. We consider the string gauge with the period L_x, L_y which realizes the smallest flux per plaquette $\phi = \phi_L = 2\pi/L^2$ and the total flux in the system $\Phi = \Phi_0 = 2\pi$ under the periodic boundary condition [42–44]. In this study, the gauge configuration is fixed as in Fig. 1 and straightforwardly generalized for arbitrary L_x, L_y [45].

One can define an approximate magnetic translation operator in the string gauge by introducing appropriate scalar functions X_j, Y_j,

$$\mathcal{T}_x = T_xU_y = T_x \exp\left(i\sum_j Y_j\tilde{n}_j\right), \tag{2}$$

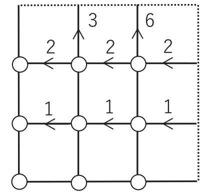

FIG. 1. The string gauge for a $L_x = L_y = 3$ system. Each number on the bonds corresponds to A_{ij} in unit of $\phi_{L=3} = 2\pi/9$ and is given in mod 9.

$$\mathcal{T}_y = T_yU_x = T_y \exp\left(i\sum_j X_j\tilde{n}_j\right), \tag{3}$$

where $T_{x,y}$ are the conventional translation operators without a magnetic field. We can determine the functions X_j, Y_j by trying to require translational symmetry of the Hamiltonian as follows. The hopping Hamiltonian is transformed as

$$\begin{aligned}\mathcal{T}_\mu c_j^\dagger e^{iA_{jk}} c_k \mathcal{T}_\mu^{-1} &= c_{j+\hat{\mu}}^\dagger e^{iZ_j^\mu} e^{iA_{jk}} e^{-iZ_k^\mu} c_{k+\hat{\mu}} \\ &\equiv c_{j+\hat{\mu}}^\dagger e^{iA_{j+\hat{\mu},k+\hat{\mu}}} c_{k+\hat{\mu}}\end{aligned} \tag{4}$$

in the μ direction, where $Z_j^x = Y_j$, $Z_j^y = X_j$. In the second equality, we have required the magnetic translation symmetry. This leads to the condition $A_{i+\hat{\mu},j+\hat{\mu}} = A_{ij} + dZ_{ij}^\mu$ with $dZ_{ij}^\mu = Z_i^\mu - Z_j^\mu$. This is basically a gauge transformation $A_{ij} \to A'_{ij} = A_{i+\hat{\mu},j+\hat{\mu}}$ by the unknown scalar function Z_j^μ. Unfortunately, however, there is no solution for Z_j^μ that satisfies the simple periodic boundary condition, $Z_j^{x,y} = Z_{j+L_\mu\hat{\mu}}^{x,y}$. We have to introduce a singular gauge transformation to satisfy Eq. (4) and correspondingly decompose Z_j^μ into a singular term and regular term $Z_j^\mu = Z_j^{s\mu} + Z_j^{\mu r}$. An example of X_j and Y_j for $L_x = L_y = 3$ is shown in Fig. 2, and they are obtained in a similar way

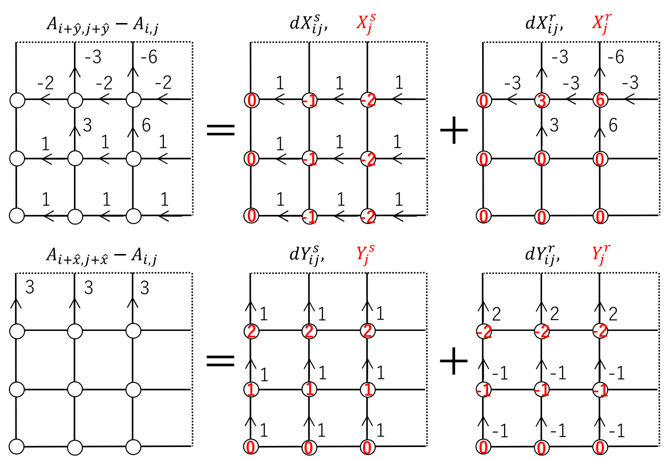

FIG. 2. The gauge transformation $A_{i+\hat{y},j+\hat{y}} - A_{i,j} = dX_{ij}^s + dX_{ij}^r$ and $A_{i+\hat{x},j+\hat{x}} - A_{i,j} = dY_{ij}^s + dY_{ij}^r$ for $L_x = L_y = 3$. The red numbers inside the circles represent $X_j^{s,r}$ and $Y_j^{s,r}$. All the numbers are defined in unit of $\phi_{L=3} = 2\pi/9$ and are in mod 9.

for other general system sizes. A singular gauge transformation is often treated with an introduction of a branch cut and it can be explicitly implemented in our system, but we will take a different approach in this study.

Here, instead of the full magnetic translation symmetry, we consider only the regular parts X_j^r, Y_j^r which approximately realize the magnetic translation, and neglect the singular parts X_j^s, Y_j^s. For simplicity, the same notation $\mathcal{T}_{x,y}$ is used for the approximated magnetic translation operator. We stress that the regular parts alone satisfy a desired commutation relation of $\mathcal{T}_{x,y}$, even when we ignore the singular parts correspond to a uniform singular vector potential $A^s_{j+\hat{\mu},j} = \phi_\mu$ with $\phi_x = -\phi_L$, $\phi_y = \phi_L$ which does not contribute to the out-of-plane flux. Indeed, one can easily derive the commutation relation of the approximate magnetic translation operator $\mathcal{T}_{x,y}$,

$$\mathcal{T}_y^{-1}\mathcal{T}_x^{-1}\mathcal{T}_y\mathcal{T}_x = e^{i\phi_L N}, \tag{5}$$

where $N = \sum_j n_j = \rho L_x L_y$ at the filling ρ. Therefore these operators give a projective representation of $\mathbb{Z}\times\mathbb{Z}$, which is a key in our discussion.

Now we consider the ground state of the Hamiltonian Eq. (1) and low energy variational states. In constructing the variational states, we use the following relations which are derived straightforwardly,

$$\mathcal{T}_x H(\phi_L;0,0)\mathcal{T}_x^{-1} = H(\phi_L;0,-\phi_y), \tag{6}$$

$$\mathcal{T}_y H(\phi_L;0,0)\mathcal{T}_y^{-1} = H(\phi_L;-\phi_x,0), \tag{7}$$

where $H(\phi_L;\phi_x,\phi_y)$ is the Hamiltonian with the magnetic field ϕ_L along the z direction and the constant vector potential $A^s_{j+\hat{\mu},j} = \phi_\mu$ along the μ direction with $\phi_x = -\phi_L$, $\phi_y = \phi_L$ [48]. These equations mean that $\mathcal{T}_{x,y}$ describe magnetic translation symmetry up to the small quantity $\phi_\mu = O(L^{-2})$, and H and $\mathcal{T}_\mu H \mathcal{T}_\mu^{-1}$ are unitary equivalent with the same spectra. In the following, we regard $\mathcal{T}_y$ as a twist operator and $\mathcal{T}_x$ as a near symmetry operator. Given the ground state which satisfies $H(\phi_L;0,0)|\Psi_0\rangle = E_0(\Phi_0)|\Psi_0\rangle$ for the total flux $\Phi_0 = 2\pi$, the variational states are defined by $|\Psi_{0k}\rangle = (\mathcal{T}_y)^k|\Psi_0\rangle$ with $k\in\mathbb{Z}$. Then, it follows from Eq. (7) that $E_{01}(\Phi_0) = \langle\Psi_{01}|H(\phi_L;0,0)|\Psi_{01}\rangle = \langle\Psi_0|H(\phi_L;\phi_x,0)|\Psi_0\rangle$ is evaluated as

$$E_{01} = E_0 + \phi_x h_1 + \phi_x^2 h_2 + \cdots, \tag{8}$$

where we have Taylor expanded $H(\phi_L;\phi_x,0)$ with respect to $\phi_x = O(L^{-2})$ and $h_l = \langle\Psi_0|\partial^l_{\phi_x}H(\phi_L;0,0)|\Psi_0\rangle/l!$. Clearly, the second correction term behaves as $\phi_x^2 h_2 = O(L^{-4})\times O(L^2) = O(L^{-2})$ in two dimensions. The first correction term $\phi_x h_1$ is odd in ϕ_x and its sign can be flipped by considering another variational state $\mathcal{T}_y^{-1}|\Psi_0\rangle$ in addition to $\mathcal{T}_y|\Psi_0\rangle$. The absolute value of $\phi_x h_1$ must be smaller than that of $\phi_x^2 h_2$ so that the variational energies of $H(\phi_L;0,0)$ for the two states $\mathcal{T}_y^{\pm 1}|\Psi_0\rangle$ are greater than or equal to E_0, which is a variant of Bloch's theorem for the persistent current [49,50]. The higher order corrections are even smaller, and we end up with $E_{01} = E_0 + O(L^{-2})$. One also obtains $E_{0k} = E_0 + O(L^{-1})$ in three dimensions.

Next, we discuss approximate orthogonality of these states based on Eq. (6) which is now regarded as a near symmetry of $H(\phi_L;0,0)$. We first consider a case where the ground state is uniquely gapped and later move on to a multiply degenerate case. Following the previous study [8], we introduce a unitary evolution operator $\mathcal{F}_y$ which adiabatically inserts a flux $\Phi_y = \sum_{y_j} A^s_{j+\hat{y},j} = L_y\phi_y$ through the noncontractible hole of the torus in y direction [51]. Since $|\Psi_n(\Phi_y)\rangle = \mathcal{F}_y(\Phi_y)|\Psi_n(0)\rangle$ [52,53], Eq. (6) leads to $H(0)\mathcal{T}_x\mathcal{F}_y|\Psi_0(0)\rangle = E_0(\Phi_y)\mathcal{T}_x\mathcal{F}_y|\Psi_0(0)\rangle$, where $E_0(\Phi_y)$ is the ground state energy with the flux, $H(\Phi_y)|\Psi_0(\Phi_y)\rangle = E_0(\Phi_y)|\Psi_0(\Phi_y)\rangle$. When the spectrum of $H(0)$ has a gap $\Delta(0) = O(L^0) = O(1)$ above the unique ground state, the gap does not close for a flux $\Phi'_y \in [0,\Phi_y]$ essentially because the inserted flux $\Phi'_y = O(L^{-1})$ is vanishingly small [54], which implies that $E_0(0\le\Phi'_y\le\Phi_y)$ stays at the lowest energy. Because the spectra of $H(0)$ and $H(\Phi_y)$ are unitary equivalent, this means $E_0(0) = E_0(\Phi_y)$ and hence $|\Psi_0(0)\rangle$ is an eigenstate of the combined unitary operator $\mathcal{T}_x\mathcal{F}_y$. Therefore, with use of the commutation relation Eq. (5), $\langle\Psi_0(0)|\Psi_{01}(0)\rangle = e^{-i\phi_L N}\langle\Psi_0(0)|(\mathcal{F}_y^{-1}\mathcal{T}_x^{-1})\mathcal{T}_y(\mathcal{T}_x\mathcal{F}_y)|\Psi_0(0)\rangle + O(L^{-1})$, we obtain in two dimensions

$$\langle\Psi_0|\Psi_{01}\rangle = e^{i2\pi\rho}\langle\Psi_0|\Psi_{01}\rangle + O(L^{-1}). \tag{9}$$

To be consistent with the preassumed unique gapped ground state, ρ must be an integer. The contraposition corresponds to a part of the LSM theorem. In three dimensions, the corresponding factor is $e^{i2\pi\rho L_z}$, which also requires an integer ρ for suitably chosen L_z similarly to the previous study [8].

The above discussions can be extended to a gapped system with general degeneracy D, from which we can conclude $D\ge q$ for $\rho = p/q$. A fractionally filled system is either gapless or gapped with $D>1$ as shown above, and here we consider the latter case with a gap $\Delta = O(1)$ from the D-dimensional ground state sector to excited states for $H(\phi_L;0,0)$. The ground state sector consists of the states $\{|\Psi_n\rangle\}_{n=0}^{D-1}$ whose energies agree in the thermodynamic limit and we neglect possible vanishingly small energy differences for brevity. Then we construct variational states $|\Psi_{nk}\rangle = (\mathcal{T}_y)^k|\Psi_n\rangle$ for $k = 1,\ldots,K$ and evaluate their energy expectation values E_{nk}. We can just repeat the same argument as above and obtain $E_{nk} = E_0 + O(L^{d-4})$ in d dimensions. To discuss their (near) orthogonality, we introduce a vector $I = (I_0, \ldots, I_{D-1})^T$ with $I_n = \langle \Psi_n | \Psi_{nk} \rangle = \langle \Psi_n | \mathcal{T}_y^k | \Psi_n \rangle$. Then, one obtains $I = e^{i2\pi k\rho} I$ in two dimensions similarly to Eq. (9) [55] and it suggests $1 \leq \exists\, k_0 \leq K$ such that $k_0 \rho \in \mathbb{Z}$ when $K = D$ since the number of linearly independent variational states must be smaller than or equal to D. This implies $D \geq q$.

Step (ii) stability of many-body eigenvalues to magnetic fields.—Here, we discuss stability of eigenvalues $E_n(\Phi = 0)$ of $H(\phi = 0)$ to a small magnetic field in z direction, and show that $\delta E_n(\Phi_0) = [E_n(\Phi_0) - E_n(0)] \to 0$ as $L \to \infty$. One of the difficulties in discussing such stability is that the uniform magnetic field ϕ_L is not a small perturbation in the usual sense, and $|e^{iA_{jk}} - 1|$ is not vanishing for a large number of bonds, which prevents us from Taylor expanding the Hamiltonian only up to a small finite order in ϕ_L. It is nontrivial whether or not $\phi_L = 2\pi/L^2$ can be simply regarded as the $\phi \to 0$ limit, since the corresponding total flux $\Phi_0 = 2\pi$ is $O(1)$, which could potentially lead to $\delta E_n(\Phi_0) = O(1)$.

On the other hand, one may naively expect the stability of the many-body eigenvalues, $\delta E_n(\Phi_0) \to 0$, as has been assumed in numerical calculations [56]. To explicitly demonstrate it, we use the stability of single-particle eigenvalues $\varepsilon_n(\phi = 0)$ to a magnetic field, which was mathematically proved in the literature [57–59]. To use this result, we have to appropriately modify our Hamiltonian by introducing an on-site potential term $H_U = \sum_i U_i n_i$ which can lift the degeneracy of the single-particle eigenvalues. Here, we choose U_j to be a fixed random potential in $[-u, u]$ for a given system size so that the degeneracy of $\varepsilon_n(\phi = 0)$ due to spatial (rotation, inversion, and translation) symmetries is lifted. Besides, the corresponding single-particle eigenfunctions will be nonzero anywhere in the system, because of the random potential which suppresses accidental zeros. Then, one has $\delta\varepsilon_n(\phi_L) = [\varepsilon_n(\phi_L) - \varepsilon_n(0)] \sim \phi_L^2 = O(L^{-4})$ possibly with a u-dependent coefficient [57–59].

This immediately leads to eigenvalue stability of the noninteracting Hamiltonian $H_{tU}(\phi_L, u) = H_t(\phi_L) + H_U(u)$, namely, $\delta E_n(\Phi_0, u, V = 0) \sim \phi_L^2 N = O(L^{d-4})$ in d dimensions. We keep $u > 0$ to show $\delta E_n(\Phi_0, u) \to 0$ in the thermodynamic limit, and then turn off the random potential, $u \to 0$ [60], which eventually implies $\delta E_n \to 0$ in absence of the artificial potential U_j. We can also see that corresponding changes in eigenvectors of $H_{tU}(\phi_L, u)$ are vanishingly small; a direct calculation gives $\| |\delta\Psi_n(\phi_L, u)\rangle \|^2 = \| |\Psi_n(\phi_L, u)\rangle - |\Psi_n(0, u)\rangle \|^2 = O(\phi_L^2 N) = O(L^{d-4})$. Therefore the eigenvalue stability implies that the resolvent $R_{tU}(\phi_L, u; E) = [H_{tU}(\phi_L, u) - E]^{-1}$ approaches $R_{tU}(0, 0; E)$ in the above mentioned limit.

Now we consider eigenvalue stability of the interacting Hamiltonian $H(\phi_L, u) = H_{tU}(\phi_L, u) + H_V$. We can see that the eigenvalues and eigenvectors of $H(\phi_L, u)$ approach those at $\phi = 0$ in a similar manner. This follows from the resolvent equation

$$
\begin{aligned}
&[H_{tU}(\phi_L, u) + H_V - E]^{-1} \\
&\quad = [H_{tU} - E]^{-1}[1 + H_V[H_{tU} - E]^{-1}]^{-1}, \qquad (10)
\end{aligned}
$$

where $[H_{tU}(\phi_L, u) - E]^{-1} \to [H_{tU}(0, u) - E]^{-1}$ as already discussed. Therefore, we conclude $[H_{tU}(\phi_L, u) + H_V - E]^{-1} \to [H_{tU}(0, u) + H_V - E]^{-1}$, which means stability of the eigenvalues and eigenvectors of $H(\phi_L, u) = H_{tU}(\phi_L, u) + H_V$ to the small magnetic field ϕ_L at $u \neq 0$. Finally, we take the limit $u \to 0$ and conclude that the eigenvalues of the clean many-body Hamiltonian for the sufficiently large system approach $E_n(\Phi = 0)$. Since the eigenvectors of $H(\phi_L)$ also converge to those of $H(0)$, the (near) orthogonality Eq. (9) is kept down to $\phi = 0$. This completes our proof of the LSM theorem.

In summary, with use of the approximate magnetic translation symmetry, we have extended the LSM theorem to higher-dimensional long-range interacting systems and derived the lower bound, $D \geq q$, for gapped ground state degeneracy at a fractional filling $\rho = p/q$.

We are grateful to Y. Yao, M. Oshikawa, A. Ueda, T. Koma, M. G. Yamada, M. Sato, S. C. Furuya, K. Shiozaki, and S. Kamimoto for valuable discussions. This work was supported by JSPS KAKENHI Grant No. JP17K14333.

*ytada@hiroshima-u.ac.jp

[1] T. Cubitt, D. Perez-Garcia, and M. Wolf, Nature (London) **528**, 207 (2015).
[2] X. G. Wen, *Quantum Field Theory of Many-body Systems: From the Origin of Sound to an Origin of Light and Electrons* (Oxford University Press, New York, 2004).
[3] X.-G. Wen, Rev. Mod. Phys. **89**, 041004 (2017).
[4] E. Lieb, T. Schultz, and D. Mattis, Ann. Phys. (Berlin) **16**, 407 (1961).
[5] I. Affleck and E. H. Lieb, Lett. Math. Phys. **12**, 57 (1986).
[6] M. Yamanaka, M. Oshikawa, and I. Affleck, Phys. Rev. Lett. **79**, 1110 (1997).
[7] T. Koma, J. Stat. Phys. **99**, 313 (2000).
[8] M. Oshikawa, Phys. Rev. Lett. **84**, 1535 (2000).
[9] M. B. Hastings, Phys. Rev. B **69**, 104431 (2004).
[10] M. B. Hastings, Europhys. Lett. **70**, 824 (2005).
[11] M. B. Hastings, arXiv:1008.5137.
[12] B. Nachtergaele and R. Sims, Commun. Math. Phys. **276**, 437 (2007).
[13] S. A. Parameswaran, A. M. Turner, D. P. Arovas, and A. Vishwanath, Nat. Phys. **9**, 299 (2013).
[14] H. Watanabe, H. C. Po, A. Vishwanath, and M. Zaletel, Proc. Natl. Acad. Sci. U.S.A. **112**, 14551 (2015).
[15] Y.-M. Lu, Y. Ran, and M. Oshikawa, Ann. Phys. (Amsterdam) **413**, 168060 (2020).
[16] Y. Yao and M. Oshikawa, Phys. Rev. X **10**, 031008 (2020).
[17] X. Chen, Z.-C. Gu, and X.-G. Wen, Phys. Rev. B **83**, 035107 (2011).
[18] Y. Ogata, Y. Tachikawa, and H. Tasaki, Commun. Math. Phys. **385**, 79 (2021).

[19] D. Ruelle, *Statistical Mechanics: Rigorous Results* (World Scientific Pub Co Inc, Singapore, 1999).
[20] E. Lieb and R. Seiringer, *The Stability of Matter in Quantum Mechanics* (Cambridge University Press, Cambridge, England, 2009).
[21] A. Campa, T. Dauxois, and S. Ruffo, Phys. Rep. **480**, 57 (2009).
[22] J. R. Schrieffer, *Theory of Superconductivity* (Westview Press, Colorado, 1971).
[23] P. W. Anderson, Phys. Rev. **110**, 827 (1958).
[24] Y. Nambu, Phys. Rev. **117**, 648 (1960).
[25] M. Hohenadler, F. Parisen Toldin, I. F. Herbut, and F. F. Assaad, Phys. Rev. B **90**, 085146 (2014).
[26] G. Rohringer, H. Hafermann, A. Toschi, A. A. Katanin, A. E. Antipov, M. I. Katsnelson, A. I. Lichtenstein, A. N. Rubtsov, and K. Held, Rev. Mod. Phys. **90**, 025003 (2018).
[27] M. Hirata, A. Kobayashi, C. Berthier, and K. Kanoda, Rep. Prog. Phys. **84**, 036502 (2021).
[28] N. P. Armitage, E. J. Mele, and A. Vishwanath, Rev. Mod. Phys. **90**, 015001 (2018).
[29] N. Y. Yao, A. V. Gorshkov, C. R. Laumann, A. M. Läuchli, J. Ye, and M. D. Lukin, Phys. Rev. Lett. **110**, 185302 (2013).
[30] S. R. Manmana, E. M. Stoudenmire, K. R. A. Hazzard, A. M. Rey, and A. V. Gorshkov, Phys. Rev. B **87**, 081106 (R) (2013).
[31] N. Y. Yao, M. P. Zaletel, D. M. Stamper-Kurn, and A. Vishwanath, Nat. Phys. **14**, 405 (2018).
[32] O. Dutta, M. Gajda, P. Hauke, M. Lewenstein, D.-S. Lhmann, B. A. Malomed, T. Sowiński, and J. Zakrzewski, Rep. Prog. Phys. **78**, 066001 (2015).
[33] A. Browaeys and T. Lahaye, Nat. Phys. **16**, 132 (2020).
[34] M. Oshikawa, M. Yamanaka, and I. Affleck, Phys. Rev. Lett. **78**, 1984 (1997).
[35] S. C. Furuya and Y. Horinouchi, Phys. Rev. B **100**, 174435 (2019).
[36] M. Oshikawa and T. Senthil, Phys. Rev. Lett. **96**, 060601 (2006).
[37] M. Levin and X.-G. Wen, Phys. Rev. Lett. **96**, 110405 (2006).
[38] A. Kitaev and J. Preskill, Phys. Rev. Lett. **96**, 110404 (2006).
[39] G. Misguich, C. Lhuillier, M. Mambrini, and P. Sindzingre, Eur. Phys. J. B **26**, 167 (2002).
[40] H. Watanabe, Phys. Rev. B **98**, 155137 (2018).
[41] In this sense, the flux ϕ_L is regarded as a perturbation to control the system in two or three dimensions. It would be an interesting future study to consider a different perturbation controlling a system in dimensions higher than three.
[42] Y. Hatsugai, K. Ishibashi, and Y. Morita, Phys. Rev. Lett. **83**, 2246 (1999).
[43] K. Kudo, T. Kariyado, and Y. Hatsugai, J. Phys. Soc. Jpn. **86**, 103701 (2017).
[44] Y. Tada, Phys. Rev. Research **2**, 033363 (2020).
[45] Compared to the minimum flux per plaquette $\phi = O(L^{-1})$ in the Landau gauge corresponding to the magnetic length $l_B = 1/\sqrt{\phi} \sim \sqrt{L}$ [7,15,46,47], $\phi_L = O(L^{-2})$ in the string gauge corresponding to $l_B \sim L$ is even smaller and essentially important in the later discussions.
[46] J. Zak, Phys. Rev. **134**, A1602 (1964).
[47] J. Zak, Phys. Rev. **134**, A1607 (1964).
[48] The twists in the right hand side arise from the approximation of neglecting the singular terms. Indeed, with the application of $\mathcal{T}_x$ to the Hamiltonian $H(\phi_L; 0, 0)$, the hopping term becomes $c^\dagger_{j+\hat{x}} e^{iA_{jk}+idY^r_{jk}} c_{k+\hat{x}} = c^\dagger_{j+\hat{x}} e^{iA_{j+\hat{x},k+\hat{x}}-idY^s_{jk}} c_{k+\hat{x}}$, where $dY^s_{jk} = A^s_{jk} = \phi_y$ itself is well-defined although Y^s_j is not. A similar argument holds for $\mathcal{T}_y$.
[49] D. Bohm, Phys. Rev. **75**, 502 (1949).
[50] Y. Tada and T. Koma, J. Stat. Phys. **165**, 455 (2016).
[51] Note that $\mathcal{F}_y = 1 + O(L^{-1})$ essentially because $\Phi_y = O(L^{-1})$. Indeed, the $\mathcal{F}_y(\Phi_y)$ can be written as $\mathcal{F}_y(\Phi_y) = \hat{T}\exp[-i\int_0^{\Phi_y} Q(\Phi'_y) d\Phi'_y]$, $\mathcal{F}_y(\Phi_y) = 1 - i\int_0^{\Phi_y} Q(\Phi'_y) d\Phi'_y + \cdots$, where $Q(\Phi_y)$ is given by $Q = i\sum_n \partial_{\Phi_y} P_n P_n$ with the projection operator $P_n(\Phi_y) = |\Psi_n\rangle\langle\Psi_n|$ for the nth eigenstate of $H(\Phi_y) = H(\phi; 0, \Phi_y/L_y)$ [52,53]. In this expression, the Hermitian operator Q is an $O(1)$ operator whose operator norm is simply $\|Q\| = O(1)$ and nearly independent of the system size, which implies $\mathcal{F}_y = 1 + O(\Phi_y) = 1 + O(L^{-1})$. This is in contrast to the 2π-flux insertion discussed previously [8] for which the corresponding unitary operator might be nontrivial.
[52] T. Kato, *Short Introduction to Perturbation Theory for Linear Operators* (Springer-Verlag, New York, 1982).
[53] T. Kato, J. Phys. Soc. Jpn. **5**, 435 (1950).
[54] More precisely, for example in two dimensions, we first Taylor expand an excited energy eigenvalue as a function of the Hamiltonian parameter φ_y by using the Hellmann-Feynman theorem, $E_n(\varphi_y + d\varphi_y) = E_n(\varphi_y) + J_n(\varphi_y) d\varphi_y + O(L^{-2})$ with $J_n(\varphi_y) = \langle\Psi_n(\varphi_y)| i\partial_{\varphi_y} H(\varphi_y) |\Psi_n(\varphi_y)\rangle$, from which we see that energy level crossing between E_n and E_0 requires a net current $|J_n|/L^2 \gtrsim (E_n - E_0)$. However, the variational excited energy for $\mathcal{T}_x^{\pm 1}|\Psi_n(0, u)\rangle$ is $E_{\rm var} - E_0 = (E_n - E_0) \pm J_n \phi_y + O(L^{-2}) \geq 0$ similarly to the Bloch's theorem [49,50]. Therefore, such energy level crossing is impossible and the gap does not close during the adiabatic process. This is consistent with the natural expectation that $E_n - E_0$ is one-sided differentiable at $\Phi_y = 0$ as a function of the total flux Φ_y in the thermodynamic limit.
[55] More precisely, after the adiabatic time evolution where the gap remains nonzero similarly to the case with $D = 1$, a state $|\Psi_n\rangle$ may change to another state $\sum_{n=0}^{D-1} P_{mn} |\Psi_n\rangle$ in the ground state sector with a unitary matrix P. So, we change the basis states such that the unitary matrix P is diagonalized, and rewrite them as $\{|\Psi_n\rangle\}_{n=0}^{D-1}$ using the same symbols for simplicity. In this basis, one obtains $I = e^{i2\pi k\rho} I$ in two dimensions.
[56] F. F. Assaad, in *Quantum Simulations of Complex Many-Body Systems: From Theory to Algorithms*, NIC Series, edited by J. Grotendorst, D. Marx, and A. Muramatsu (John von Neumann Institute for Computing, Jülich, 2002).
[57] G. Berkolaiko, Anal. PDE **6**, 1213 (2013).
[58] G. Berkolaiko and T. Weyand, Phil. Trans. R. Soc. A **372**, 20120522 (2014).
[59] Y. C. de Verdière, Anal. PDE **6**, 1235 (2013).
[60] More precisely, $\forall\, u > 0$, $\forall\, \epsilon > 0$, $\exists\, L_0 > 0$ such that $|\delta E_n(\Phi_0, u)| < \epsilon$ for $L > L_0$. Note that the order of the double limits $\lim_{u\to 0}\lim_{L\to\infty}\delta E_n$ is not interchangeable.