Operator-Algebraic Renormalization and Wavelets

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(Received 13 February 2020; revised 26 May 2021; accepted 15 October 2021; published 1 December 2021)

We report on a rigorous operator-algebraic renormalization group scheme and construct the free field with a continuous action of translations as the scaling limit of Hamiltonian lattice systems using wavelet theory. A renormalization group step is determined by the scaling equation identifying lattice observables with the continuum field smeared by compactly supported wavelets. Causality follows from Lieb-Robinson bounds for harmonic lattice systems. The scheme is related with the multiscale entanglement renormalization ansatz and augments the semicontinuum limit of quantum systems.

DOI: 10.1103/PhysRevLett.127.230601

Introduction.—Lattice regularization is a standard procedure for defining continuum quantum field theories [1] which has led to extraordinary results in the *ab initio* determination of the Hadron mass spectrum [2] and may serve as a starting point for the quantum simulation of quantum field theories [3]. While interacting models have been rigorously constructed in the classical works of Glimm-Jaffe and others [4], the lattice and continuum theories are often indirectly related in terms of correlation functions.

A recent attempt to build a continuum conformal field theory (CFT) by embedding a quantum spin chain from coarser to finer lattices, coined the semicontinuum limit and inspired by block-spin renormalization, resulted in a discontinuous action of symmetries, even the translations [5–8]. Here, we explain how this deficiency can be remedied by utilizing an observable-based, i.e., operator-algebraic, approach to the Wilson-Kadanoff renormalization group (RG) [9–11] for lattice field theories [12,13]. As an important, instructive example [14,15], we construct the massive continuum free field with its continuous action of spacetime translations via the scaling limit of lattice systems in their ground states approaching the unstable, massless fix point (see [16] for details and proofs). More recently, the presented method has been extended to CFTs based on free fermions [17] invoking the Koo-Saleur formula [18].

Our RG is defined in terms of compactly supported, regular wavelets [19] allowing for simultaneous control of locality properties in real and momentum space. We take inspiration from renormalization in classical systems [20] and use a scaling function and its multiresolution analysis to define a RG step: While block-spin renormalization would correspond to a step function, we use a Daubechies's scaling function (see Fig. 1), cf. [21,22]. Thereby, we avoid the obstacles encountered in [5,7,8] to implement continuous symmetries in the scaling limit, cf. [23]. Mapping observables from coarser to finer lattices results in a real-space RG dual to coarse graining the Hamiltonian or density matrices, e.g., the density matrix renormalization group [15,24,25]. Our method applies in all dimensions as we explicitly demonstrate for scalar lattice fields. Moreover, our approach yields a rigorous proof that spacetime locality (in the sense of the Haag-Kastler axioms [26]) in the continuum follows from Lieb-Robinson bounds [27–31].

As real-space RG schemes have received rapidly growing interest in recent years, especially in the context of tensor networks [32] and the multiscale entanglement renormalization ansatz (MERA) [33–35], we show, as an important application, that our approach yields a rigorous analytic MERA in any dimension d which is not restricted to critical (massless) models [36,37]. The discrete dimension of the $d+1$-dimensional tensor network of the MERA is identified with the sequence of scales at which the given quantum system is observed.

The Letter is organized as follows. First, we outline our general renormalization scheme. Then, we apply it to lattice scalar fields by constructing explicit renormalization maps

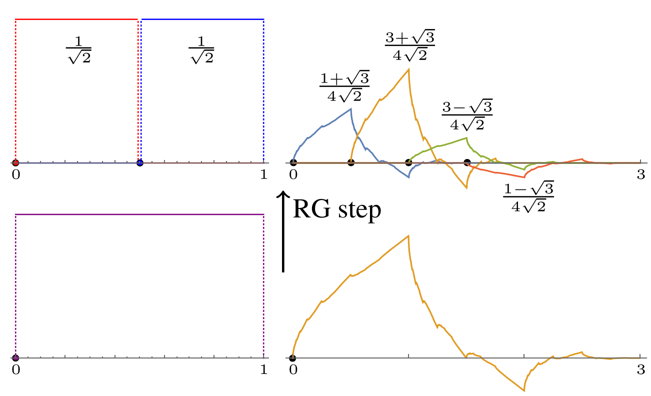

FIG. 1. Illustration of the decomposition of lattice sites for $d=1$ by an RG step determined by the scaling equation (5): On the left: The block-spin RG and its weights. On the right: The wavelet-based RG with weights determined by the low-pass filter of Daubechies's $D4$ scaling function.

in terms of compactly supported wavelets, and we discuss the connection with the MERA. Finally, in the example of the free scalar field, we show that imposing a suitable renormalization condition on lattice ground states at different scales, we fully recover the continuum massive field in the scaling limit including the action of spacetime translations. The letter closes with an outlook on possible future developments.

Operator-algebraic renormalization.—As discussed in [13], the RG approach to the lattice approximation of continuum theories can be rephrased in terms of observables, that is, operator algebras, as follows. We fix a family of lattices Λ_N in $\mathbb{R}^d$ with lattice constant $\varepsilon_N = 2^{-N}\varepsilon$, and consider a sequence of Hamiltonian quantum systems $\{\mathfrak{A}_N, \mathcal{H}_N, H_0^{(N)}\}$ indexed by the scale N. At each scale N, we have an algebra of observables $\mathfrak{A}_N$ generated by (bounded functions of) basic time-zero lattice fields $\Phi_N(x)$, their momenta $\Pi_N(x)$, and a Hamiltonian $H_0^{(N)}$ both acting on the Hilbert space $\mathcal{H}_N$. The quantum state at each scale is initially given by a density matrix $\rho_0^{(N)}$, e.g., in terms of a Hamiltonian: $\rho_0^{(N)} = (Z_0^{(N)})^{-1} e^{-H_0^{(N)}}$. The RG connects systems at different scales via (coarse graining) quantum operations, mapping density matrices on the finer system to the coarser system

$$\mathcal{E}_N^{N+M}(\rho_0^{(N+M)}) = \rho_M^{(N)}, \quad \mathcal{E}_N^{N+1} \circ \mathcal{E}_{N+1}^{N+2} = \mathcal{E}_N^{N+2}, \quad (1)$$

where $\rho_M^{(N)}$ corresponds to the (M times) renormalized Hamiltonian $H_M^{(N)}$ at scale N. Because quantum states ρ are positive, linear maps $\omega : \mathfrak{A}_N \to \mathbb{C}$, by $\omega(A) = \mathrm{tr}(\rho A)$, and the field correlation functions are given by $\langle \Phi_N(x), \ldots, \Pi_N(y) \rangle^{(N)} \coloneqq \omega^{(N)}[\Phi_N(x), \ldots, \Pi_N(y)]$., we can state (1) as

$$\mathcal{E}_N^{N+M}(\omega_0^{(N+M)}) = \omega_0^{(N+M)} \circ \alpha_{N+M}^N = \omega_M^{(N)}, \quad (2)$$

where $\alpha_{N+M}^N : \mathfrak{A}_N \to \mathfrak{A}_{N+M}$ is the dual of $\mathcal{E}_N^{N+M}$ (the ascending superoperators [34]). $\omega_0^{(N)}$ and $\omega_M^{(N)}$ characterize the initial and renormalized states on $\mathfrak{A}_N$ corresponding to $\rho_0^{(N)}$ and $\rho_M^{(N)}$. We call the collection α_{N+M}^N the scaling maps or renormalization group. The structure is neatly summarized by an adaptation of Wilson's triangle of renormalization [[10], p. 790] in Fig. 2. If the limit $\omega_\infty^{(N)} \coloneqq \lim_{M\to\infty} \omega_M^{(N)}$ exists (in a suitable sense), the sequence $\omega_\infty^{(N)}$, called the scaling limit of the initial states $\omega_0^{(N)}$, is stable under coarse graining

$$\mathcal{E}_N^{N+M}(\omega_\infty^{(N+M)}) = \omega_\infty^{(N)}. \quad (3)$$

Employing operator-algebraic techniques (see [16] for details), we obtain a Hilbert space $\mathcal{H}_\infty$ and an algebra $\mathfrak{A}_\infty$ generated by continuum fields Φ, Π, acting on it. Following [5,6,12,13,38], we call $\mathfrak{A}_\infty$ the semicontinuum limit, see, also, [39,40]. Moreover, we have isometries $V_\infty^N : \mathcal{H}_N \to \mathcal{H}_\infty$ and a state $\Omega \in \mathcal{H}_\infty$ realizing the correlations of the scaling limit

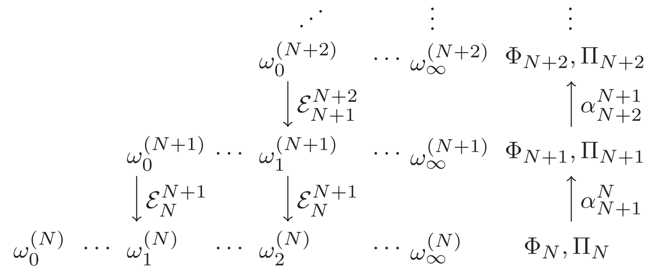

FIG. 2. Wilson's triangle of renormalization: Vertical lines represent renormalization steps, either by coarse graining states ($\mathcal{E}$'s) or by refining fields (α's). Horizontal lines represent sequences of renormalized states considered on the algebra generated by fields and momenta at a fixed scale (right column).

$\omega = \langle \Omega, .\Omega \rangle$. The finite-scale fields Φ_N, Π_N are embedded in the continuum fields Φ, Π through $\alpha_\infty^N : \mathfrak{A}_N \to \mathfrak{A}_\infty$

$$\alpha_\infty^N[\Phi_N(x)] V_\infty^N = V_\infty^N \Phi_N(x), \qquad \omega_\infty^{(N)} = \omega \circ \alpha_\infty^N. \quad (4)$$

Wavelets and the scalar field.—Now, we apply the above framework to lattice scalar fields, setting up a specific renormalization scheme involving compactly supported wavelets [19,41]. To avoid infrared divergence at finite scale, we take lattices $\Lambda_N = \varepsilon_N\{-L_N, \ldots, L_N - 1\}^d$ representing a discretization of the torus $[-L, L)^d = \mathbb{T}_L^d$ (periodic boundary conditions, $L_N \equiv -L_N$, with $\varepsilon_N L_N = L$ fixed). We denote by $\Gamma_N = (\pi/L)\{-L_N, \ldots, L_N - 1\}^d$ the dual momentum space lattices. The kinematical setup of the lattice scalar field systems is given by the Fock space $\mathcal{H}_N$, built from the action of momentum-space creation and annihilation operators $a_N(k)$, $a_N^\dagger(k)$ on the vacuum vector Ω_N subject to the canonical commutation relations (CCR), $[a_N(k), a_N^\dagger(l)] = (2L_N)^d \delta_{k,l}$, and by the algebra $\mathfrak{A}_N$ generated by the local (dimensionless) canonical lattice field for $x \in \Lambda_N$

$$\Phi_N(x) = \frac{1}{\sqrt{2}(2L_N)^d} \sum_{k\in\Gamma_N} [a_N^\dagger(k) e^{-ikx} + a_N(k) e^{ikx}],$$

and its momentum (with a similar formula) satisfying: $[\Phi_N(x), \Pi_N(y)] = i\delta_{x,y}$. The scaling maps $\alpha_{N'}^N : \mathfrak{A}_N \to \mathfrak{A}_{N'}$ are the most important input in our framework determining the existence and structure of the continuum limit. Our choice using wavelets is motivated by the block-spin case and its locality properties in real space corresponding to the smearing of continuum fields with the simplest member of the Daubechies's wavelet, the Haar wavelet $\chi_{[0,1)}$ (see Fig. 1). But, as the approximation of momenta requires higher regularity, the latter does not suffice as explained below.

Scaling maps from a scaling function: We consider an orthonormal scaling function s that satisfies the scaling equation [19,42,43]

$$s(x) = \sum_{n\in\mathbb{Z}^d} h_n 2^{\frac{d}{2}} s(2x - n), \quad (5)$$

such that its integer translates $s(\cdot - n)$ are orthonormal. Further, to build local operators, we take s compactly supported and normalized by $\hat{s}(0) = 1$. Such an s generates an orthonormal, compactly supported wavelet basis in $L^2(\mathbb{R}^d)$, and the sum (5) is necessarily finite (h_n is a finite low-pass filter [19]). We denote by $s_x^{(\varepsilon)} = \varepsilon^{-(d/2)} s[\varepsilon^{-1}(\cdot - x)]$ the scaling function localized near $x \in \varepsilon\mathbb{Z}^d$ at length scale ε, periodized on the torus $\mathbb{T}_L^d$. With the scaling relation (5) in mind, we define α_{N+1}^N using the low-pass filter h_n

$$\alpha_{N+1}^N[\Phi_N(x)] = 2^{-\frac{1}{2}} \sum_{n\in\mathbb{Z}^d} h_n \Phi_{N+1}(x + n\varepsilon_{N+1}), \quad (6)$$

and similarly for Π_N. Now, the associated semicontinuum limit algebra $\mathfrak{A}_\infty$ can be identified with the algebra generated by continuum fields smeared with the functions $s_x^{(\varepsilon_N)}$ over all scales N: The map,

$$\Phi_N(x) \mapsto \alpha_\infty^N[\Phi_N(x)] = \varepsilon_N^{-\frac{1}{2}} \int dy \Phi(y) s_x^{(\varepsilon_N)}(y), \quad (7)$$

identifies the lattice fields at scale N with the continuum fields smeared with $s_x^{(\varepsilon_N)}$ [and analogously for $\Pi_N(x)$]. The RG elements $\alpha_{N'}^N$ defined by (6) have two intriguing properties: First, the lattice field $\Phi^{(N)}(x)$ at one scale is decomposed into a linear combination of the fields at the successive scale. Second, the embedding (7) into the continuum field theory is compatible with this decomposition, $\alpha_\infty^{N+1} \circ \alpha_{N+1}^N = \alpha_\infty^N$, realizing the correct CCR

$$[\alpha_\infty^N[\Phi_N(x)], \alpha_\infty^N[\Pi_N(y)]] = [\Phi(s_x^{(\varepsilon_N)}), \Pi(s_y^{(\varepsilon_N)})] = i\delta_{x,y}.$$

Furthermore, we have $\Phi(s_x^{(\varepsilon_N)}) = \sum_{n\in\varepsilon_N} h_n \Phi(s_{x-n\varepsilon_{N+1}}^{(\varepsilon_{N+1})})$ [linearity and (5)] with an analogous formula for Π. This means that the lattice fields and their realization in terms of the continuum field have the same algebraic structure.

Concrete choice of a scaling function: The simplest scaling function, $\chi_{[0,1)}$, corresponds to the block-spin renormalization (6) (see Fig. 1). By taking a more regular scaling function, e.g., ${}_Ks$ with $K \geq 2$ of Daubechies's $D2K$ wavelet family, we achieve that the smeared continuum momentum $\Pi({}_Ks_x^{(\varepsilon_N)})$ is a well-defined operator (technically s needs to be in the Sobolev space $H^{\frac{1}{2}}$). In addition, the compact support of ${}_Ks$ leads to locality in real space, i.e., the lattice fields $\Phi_N(x)$, $\Pi_N(x)$ can be used to approximate local operators in the continuum because $\Phi(s_x^{(\varepsilon_N)})$, $\Pi(s_x^{(\varepsilon_N)})$ are spatially localized in compact regions. In comparison with the block-spin renormalization, we trade some locality (the support of the Daubechies's scaling function ${}_Ks$ is larger than the support of $\chi_{[0,1)}$) for higher regularity, improving approximations. With this price, we gain the continuum realization of $\Pi_N(x)$, and we recover the correlation functions and space-time symmetries (translations) in the scaling limit (see below).

Connection with multiscale entanglement renormalization: Considering the embedding $I_{N+1}^N[\Phi_N(x)] = 2^{-\frac{1}{2}}\Phi_{N+1}(x)$ resulting from identifying Λ_N as a sublattice of Λ_{N+1}, and the Bogoliubov unitary,

$$U_{N+1}\Phi_{N+1}(x) = \sum_{n\in\mathbb{Z}^d} h_n \Phi_{N+1}(x + n\varepsilon_{N+1}) U_{N+1}, \quad (8)$$

implementing the redistribution of field values according to the low-pass filter h_n, the scaling map α_{N+1}^N decomposes into MERA form [13,33–35,38]

$$\alpha_{N+1}^N(\cdot) = U_{N+1}(\cdot \otimes \mathbb{1}_{N+1\setminus N}) U_{N+1}^*. \quad (9)$$

Here, $\cdot \otimes \mathbb{1}_{N+1\setminus N}$ is the tensor product with the identity on the ancillary Fock space, $\mathcal{H}_{N+1} = \mathcal{H}_N \otimes \mathcal{H}_{N+1}^{(a)}$, and the dual quantum channel $\mathcal{E}_N^{N+1} = \mathrm{Tr}_{\mathcal{H}_{N+1}^{(a)}}[U_{N+1}^*(\cdot)U_{N+1}]$ is given by a twisted partial trace on the ancillary. From (9), we find that U_{N+1} serves as MERA disentangler recovered from the isometries, $V_{N+1}^N : \mathcal{H}_N \to \mathcal{H}_{N+1}$, between Fock spaces resulting from coarse-graining stability (3)

$$\Omega_\infty^{(N+1)} = V_{N+1}^N \Omega_\infty^{(N)}, \quad (10)$$

where $\Omega_\infty^{(N)}$ is the vector implementing the scaling limit $\omega_\infty^{(N)}$ at scale N. The embedding into the continuum Hilbert space $\mathcal{H}_\infty$ can be explicitly computed from (4). Summarizing, we observe that one layer of MERA isometries and disentanglers is recovered from α_{N+1}^N and the scaling limit $\omega_\infty^{(N)}$. This structure is further elucidated by the action of the isometries V_{N+1}^N on coherent or Glauber states, $c_N(f, g) = e^{i[\Phi_N(f)+\Pi_N(g)]}\Omega_\infty^{(N)}$, using the identification (7) (see Fig. 3). In this sense, our operator-algebraic RG scheme produces an analytic MERA. Specifically, the scaling limits of free lattice ground states, which we construct below, exhibit a structure similar to an analytic MERA in arbitrary dimensions and off criticality [36,44–46].

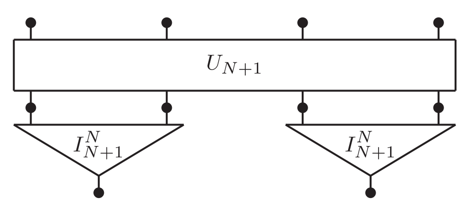

FIG. 3. Illustration of the analytic MERA in $d = 1$ induced by the wavelet scaling maps. From bottom to top: The first layer represents the isometric embedding I_{N+1}^N, and the second layer represents the action of the (dis)entangler U_{N+1} at scale $N + 1$.

Scaling limits of harmonic lattice systems.—Now, we are in a position to apply the RG $\alpha^N_{N'}$ defined by (6) to find the ground-state scaling limits of the free lattice Hamiltonian on $\mathcal{H}_N$

$$H_0^{(N)} = \varepsilon_N^{-1}\left(\frac{1}{2}\sum_{x\in\Lambda_N}(\Pi^2_{N|x} + \mu_N^2\Phi^2_{N|x}) - \sum_{\langle x,y\rangle\subset\Lambda_N}\Phi_{N|x}\Phi_{N|y}\right), \tag{11}$$

where $\mu_N \geq 2d$ is a "mass" parameter. The ground state $\Omega_0^{(N)}$ of $H_0^{(N)}$ can be encoded into the expectation $\omega_0^{(N)}$ on $\mathfrak{A}_N$ determined by the two-point functions

$$\omega_0^{(N)}[\Phi_N(x)\Phi_N(y)] = \frac{1}{(2L_N)^d}\sum_{k\in\Gamma_N}\frac{1}{2\varepsilon_N\gamma_{\mu_N}(k)}e^{ik(x-y)}, \tag{12}$$

with the dispersion relation $\gamma^2_{\mu_N}(k) = \varepsilon_N^{-2}(\mu_N^2 - 2d) + 2\epsilon_N^{-2}\sum_{j=1}^d[1-\cos(\varepsilon_N k_j)]$, and analogous formulas for $\omega_0^{(N)}[\Phi_N(x)\Pi_N(y)]$ and $\omega_0^{(N)}[\Pi_N(x)\Pi_N(y)]$, the latter being most singular.

Scaling limit of the ground states: We choose (12) as our initial states to generate a sequence of renormalized states $\omega_M^{(N)}$ at each scale N (Fig. 2). To avoid the RG-fixed points $\mu_N^2 = 2d$ (massless, unstable) and $\mu_N^2 = \infty$ (ultralocal, stable) and hit the unstable manifold of the relevant Φ^2 operator, we impose the renormalization condition

$$\lim_{N\to\infty}\varepsilon_N^{-2}(\mu_N^2 - 2d) = m^2, \tag{13}$$

for some $m > 0$. This leads to the massive continuum dispersion, $\lim_{M\to\infty}\gamma_{\mu_{N+M}}(k)^2 = m^2 + k^2 = \gamma_m(k)^2$, and the scaling limit [using (6) and (12), and similar for Π_N]

$$\omega^{(N)}_{m,\infty}[\Phi_N(x)\Phi_N(y)] = \frac{1}{(2L)^d}\sum_{k\in\Gamma_\infty}\frac{|\hat{s}^{(\varepsilon_N)}(k)|^2}{2\varepsilon_N\gamma_m(k)}e^{ik(x-y)}, \tag{14}$$

where $\Gamma_\infty = (\pi/L)\mathbb{Z}^d$ is the momentum space of the torus $\mathbb{T}^d_L$. Since the two-point function of the momentum Π_N is the most singular, the limit states are well defined for scaling functions with sufficient momentum-space decay, which holds for scaling functions ${}_K s$, $K \geq 2$, built from Daubechies's $D2K$ wavelet family [19]. Formulas (14), multiplied by $\varepsilon_N, \varepsilon_N^{-1}$, respectively, agree with the two point functions of the usual continuum mass-m ground state in finite volume L of the continuum smeared field operators $\Phi(s_x^{(\varepsilon_N)})$, $\Pi(s_x^{(\varepsilon_N)})$. Therefore, the semicontinuum limit algebra $\mathfrak{A}_\infty$ can be identified with a subalgebra of the algebra $\mathfrak{A}_{m,L}$ generated by the massive continuum free field $(m > 0)$ on $\mathbb{T}^d_L$, acting on the usual continuum Fock space. Because of localization and completeness of the wavelet basis associated with the scaling function s [19,41], all field operators $\Phi(f)$, $\Pi(g)$ smeared with smooth compactly supported functions can be approximated, in an appropriate sense, by operators from $\mathfrak{A}_\infty$.

Translations, dynamics, locality, and Lieb-Robinson bounds: Our construction provides an explicit method for circumventing the no-go results of [5,7] concerning the implementation of continuous symmetries. In particular, the continuous extension of spatial translations by discrete vectors $a \in \cup_N \Lambda_N$ (dyadic translations as enforced by the dyadic lattice refinements) acting on $\mathfrak{A}_\infty$ to translations by arbitrary vectors $a \in \mathbb{T}^d_L$ is a consequence of the manifest continuous translations invariance of the two-point function (14), and the generators of translations are the usual momentum operators. The thermodynamical limit of (14), $L \to \infty$, exists by a Riemann-sum argument and yields the two-point functions of the free, massive vacuum in infinite volume (see [16]), which is fully Poincaré invariant. Let us, also, explicitly address the convergence of the lattice dynamics generated by the Hamiltonian $H_0^{(N)}$ of (11) to their continuum limit: From $\gamma_{\mu_N} \to \gamma_m$, we deduce

$$V_\infty^{N'}e^{itH_0^{(N')}}\alpha^N_{N'}[\Phi_N(x)]\Omega_\infty^{(N)}\overset{N'\to\infty}{\longrightarrow}e^{itH}V_\infty^N\Phi_N(x)\Omega_\infty^{(N)},$$

and similarly for Π_N, uniformly on bounded intervals of $t \in \mathbb{R}$, with the free continuum Hamiltonian H on the torus $\mathbb{T}^d_L$. Since γ_m is the free, massive relativistic dispersion relation, we know that the dynamics generated by H has propagation speed $c = 1$ and, thus, the scaling limit theory satisfies Einstein causality, i.e., $e^{itH}\alpha^N_\infty[\Phi_N(x)]e^{-itH}$ and $e^{isH}\alpha^N_\infty[\Phi_N(x)]e^{-isH}$ commute if the support of $s_x^{(\varepsilon_N)}$ at time t and the support of $s_y^{(\varepsilon_N)}$ at time s are spacelike separated on the cylinder. A more lattice-intrinsic and model-independent way to conclude recovery of causality in the scaling limit is via Lieb-Robinson bounds [28,29]. Considering the extension of the finite-scale time translations $\sigma_t^{(N)} = e^{itH_0^{(N)}}(\cdot)e^{-itH_0^{(N)}}$ to $\mathfrak{A}_\infty$ by (9), said bounds for harmonic lattice systems [30] imply

$$\lim_{N\to\infty}[\sigma_t^{(N)}(A), B] = 0, \tag{15}$$

exponentially fast and uniformly for $|t| \leq T$ with (bounded) $A, B \in \mathfrak{A}_\infty$ localized in sets $\mathcal{S}_A, \mathcal{S}_B \subset \mathbb{T}^d_L$ such that $\mathrm{dist}(x, \mathcal{S}_A) \geq c'T$ for all $x \in \mathcal{S}_B$, for some $c' > 1$. Because $c' > 1$, the causality implied by (15) is not strict, likely due to a nonoptimal bound on the Lieb-Robinson velocity [28]. Another important feature of our approximation of dynamics (or symmetries in general) is the possibility for uniform error bounds in time and within a fixed range of field and momentum amplitudes at a given scale N: For the free continuum time evolution $\sigma_t = e^{itH}(\cdot)e^{-itH}$, we have [16]

$$\|(\sigma_t^{(N')}-\sigma_t)(A)\psi\| \leq C \sup_{k\in\Gamma_\infty}\left(\frac{\gamma_m(k)^{\frac{1}{2}}|\gamma_{\mu_{N'}}(k)-\gamma_m(k)|}{(1+\varepsilon_N|k|)^\delta}\right), \tag{16}$$

for exponentials $A=\alpha_\infty^N(e^{i[\Phi_N(x)+\Pi_N(y)]})$ of fields and momenta on coherent states $\psi=c(\varepsilon_N^{-\frac{1}{2}}s_u^{(\varepsilon_N)},\varepsilon_N^{\frac{1}{2}}s_v^{(\varepsilon_N)})$ at scale N. C only depends on N, ε_N, m, T for $|t|\leq T$, and s. While the specific form of these bounds reflects the free-field situation, our general method for obtaining such uniform bounds at fixed approximation scale N is not restricted to this situation (cf. conclusion).

Conclusions and outlook.—Our results show that the existence and properties of continuum limits depend decisively on the choice of a renormalization scheme. Correctly choosing the initial states allows us to reconstruct the continuum field theory from the lattice approximation through the semicontinuum limit. For the free massive scalar field, our renormalization scheme, given by compactly supported wavelets, yields continuous spacetime translations, avoiding the apparent no-go results stated in [5,7]. Obtaining a similar convergence statement for Lorentz transformations or even conformal transformations requires further work [17]. Apart from the question of approximation of symmetries, our method proves [(14) and (16)] that time-dependent and spatially translated correlation functions of the continuum field theory for any insertions of fields and momenta, $A_N=\Phi_N(x_1)\ldots\Pi_N(x_n)$ and $B_N=\Phi_N(x_{n+1})\ldots\Pi_N(x_{n+m})$, at any scale N are approximated by the correlation functions of the lattice models (suppressing scaling maps $\alpha_{N'}^N$, α_∞^N)

$$|\omega_0^{(N')}[A_N\sigma_{(t,x)}^{(N')}(B_N)]-\omega[A_N\sigma_{(t,x)}(B_N)]|\overset{N'\to\infty}{\longrightarrow}0, \tag{17}$$

where $\sigma_{(t,x)}$ and $\sigma_{(t,x)}^{(N')}$ are the continuum, respectively, discrete spacetime translations for $(t,x)\in\mathbb{R}\times\Lambda_N$. We point out that the convergence in (17) only mildly depends on the choice of scaling function s (requiring sufficient regularity). This presents a significant conceptual and presumably computational difference in comparison with a related construction using wavelet theory [46] focusing on locality in one-particle space and relying on a continuous adaptation of the choice of scaling function to achieve a given accuracy goal for the approximation of equal-time correlation function similar to (17). An application of the wavelet method to (free) lattice fermions has lead to similar results as those presented here [17,47]. Our general framework can also include interacting lattice systems, e.g., Φ^4 models, although we will need approximations by analytical and numerical expansion or perturbative methods [25,48,49]. Moreover, Lieb-Robinson bounds for anharmonic lattice systems [31] offer a possibility for obtaining spacetime locality directly from the lattice [28,29]. In view of the classical results by Glimm-Jaffe and others [4] on $P(\Phi)$ models in $d=1$, our method is directly applicable to those using a low-pass filter implementing momentum-space cutoffs [16], thereby providing the same regularized continuum fields as in [50], and we expect a possible extension to the wavelet setting. Therefore, it would be interesting whether the convergence to the scaling limit can be shown exploiting the results in [51] supplemented by explicit error bounds similar to (16).

Helpful discussions with T. Osborne, A. Abdesselam, and M. Fröb are acknowledged by A. S.. We would also like to thank the unknown referees for their careful consideration of our manuscript thereby improving the clarity of the presentation. V. M. and G. M. are partially supported by the European Research Council Advanced Grant No. 669240 QUEST. A. S. was supported by the Humboldt Foundation through a Feodor Lynen Return Fellowship. V. M. was supported by an Assegno di Ricerca dell'Istituto Nazionale di Alta Matematica (INdAM fellowship) and by the Alexander-von-Humboldt Foundation through a Humboldt Research Fellowship for Experienced Researchers hosted by the Mathematics Department, FAU Erlangen-Nürnberg. Y. T. is supported by the Programma per giovani ricercatori, anno 2014 "Rita Levi Montalcini" of the Italian Ministry of Education, University and Research. V. M., G. M. and Y. T. also acknowledge the MIUR Excellence Department Project awarded to the Department of Mathematics, University of Rome "Tor Vergata", Grant No. CUP E83C18000100006 and the University of Rome "Tor Vergata" funding scheme "Beyond Borders," Grant No. CUP E84I19002200005.

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