PHYSICAL REVIEW LETTERS **127,** 228301 (2021)

Editors' Suggestion **Featured in Physics**

Mutual Information Disentangles Interactions from Changing Environments

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(Received 20 July 2021; revised 14 September 2021; accepted 22 September 2021; published 22 November 2021)

Real-world systems are characterized by complex interactions of their internal degrees of freedom, while living in ever-changing environments whose net effect is to act as additional couplings. Here, we introduce a paradigmatic interacting model in a switching, but unobserved, environment. We show that the limiting properties of the mutual information of the system allow for a disentangling of these two sources of couplings. Further, our approach might stand as a general method to discriminate complex internal interactions from equally complex changing environments.

DOI: 10.1103/PhysRevLett.127.228301

Real-world systems exhibit interactions among their internal degrees of freedom. Furthermore, they are usually coupled with a noisy, ever-changing environment. Modeling together these two distinct contributions is often a problem too hard to be tackled, and a common approach would prescribe to simply ignore environmental effects altogether.

In the last twenty years, however, it was realized in many different fields that the effects of a noisy environment are often as fundamental as the internal interactions. Biological systems [1,2], biochemical [3] and gene regulatory networks [4–6], swarming, oscillatory, and ecological systems [7–9] are only a few examples of noisy interacting systems living in an equally noisy environment, and being consequently affected by it. In the last years, it has been also shown that many observed properties believed to be distinctive of neural interactions can be solely explained by an environmental-like dynamics that affects all neurons in the same way [10–13]. Alternatively, some of these properties might be stochastic by nature and not reflect any particular feature of the underlying degrees of freedom [14].

From a different perspective, crucial non-equilibrium features in chemical systems, such as thermophoresis [15,16], and pattern formation [17], have been recently shown to be sheer consequences of the interplay between environmental and internal interactions acting on different timescales [18].

To make things more interesting, an ever-growing wealth of data are populating the realm of biological, chemical, and neural systems, thus fueling the possibility of a direct extrapolation of some properties belonging to the underlying dynamics. In fact, when dealing with experimental data, it is not unusual to solve a given inverse problem, for example, using a maximum entropy principle [19–21], to reconstruct the interactions between the internal degrees of freedom that shape the observed behavior. However, one might ask whether these reconstructed effective couplings could possibly be a pure consequence of nothing but our ignorance about the unobserved environment in which the system lives. This question is often particularly hard to assess, as effective interactions arise even in noninteracting systems under the influence of a correlated noise [22].

In this work, we introduce a complete dynamical model, which includes both the internal dynamics, i.e., the one stemming from internal physical couplings, and stochastic environmental changes. While the internal dynamics is independent of the environment, and fully determined by system features, the environmental changes affect model parameters shared by all degrees of freedom. Recently, this problem has gained momentum from a theoretical perspective in different contexts [23–26], but the general question of how we can possibly disentangle the effects of internal interactions from those of a stochastic environment is very much open and elusive. Here, we consider the paradigmatic case of an environment affecting only the diffusion coefficient, hence changing the stochastic variability of the dynamics. Nonetheless, our modeling approach can be immediately generalized to diverse scenarios, from more complicated environments to spatially inhomogeneous media [27].

We will directly tackle the disentangling problem by using the mutual information to quantify the similarity between different interacting processes. In fact, the mutual information captures all cross-dependencies between two random variables [28]. We will show that in the presence of linearized interactions the mutual information of the whole system does encode both internal and environmental couplings as distinct contributions, and that they can always be fully disentangled in suitable limits. Although characterizing the specific nature of internal interactions through mutual information remains a challenge, our results suggest that fast-varying environments might reveal the presence of underlying *real* couplings in any general system.

The mutual information between two stationary processes $x_1(t)$ and $x_2(t)$ is the Kullback-Leibler divergence between their joint stationary probability distribution $p(x_1, x_2)$ and the product of their marginalized stationary distributions $p(x_1)p(x_2)$,

$$\begin{aligned} I &= \int dx_1 dx_2 p(x_1, x_2) \log \frac{p(x_1, x_2)}{p(x_1)p(x_2)} \\ &= H_1 + H_2 - H_{12}, \end{aligned} \tag{1}$$

where H_{12} is the differential entropy of the joint distribution, and, similarly, H_μ is the differential entropy of the marginalized probability distributions for $\mu = 1, 2$.

In order to fix the ideas, let us consider the paradigmatic example of two interacting Ornstein-Uhlenbeck processes [29]. This particular choice is twofold. First, an Ornstein-Uhlenbeck process is one of the simplest multidimensional stochastic process with a non-trivial stationary distribution. Second, Ornstein-Uhlenbeck processes can often be seen as a linearization of more complex, nonlinear internal interactions. Therefore, we introduce an internal dynamics by means of an interaction matrix $\mathbf{A}$ between the internal degrees of freedom $x_1(t)$ and $x_2(t)$. Then, we consider an archetypal description of the environmental changes, which we will regard as unobserved degrees of freedom acting on both x_1 and x_2 in the same way. At all times, x_1 and x_2 share the same diffusion coefficient, and the diffusion coefficient itself is a stochastic variable. In particular, we take it to be a dichotomous process $D_{i(t)}$ between the states $i \in \{-, +\}$, so that the diffusion coefficient jumps between two states D_- and $D_+ > D_-$, with transition rates $W(- \to +) = w_+$ and $W(+ \to -) = w_-$. All in all, our model can be written as the set of Langevin equations

$$\frac{\partial x_\mu}{\partial t} = -\sum_\nu A_{\mu\nu} \frac{x_\nu}{\tau} + \sqrt{2D_{i(t)}}\xi_\mu(t), \tag{2}$$

where $i(t)$ is a realization of the jump process between $\{-, +\}$ and ξ_1 and ξ_2 are independent white noises with zero mean. Here, the environment is encoded in the two distinct diffusion coefficients D_i, whereas the internal couplings stem from the off-diagonal elements of $\mathbf{A}$. Our goal is to understand whether these two distinct contributions to the dynamics can be disentangled, and, if so, under which conditions.

With this aim in mind, let us begin with the simple case $\mathbf{A} = \mathbb{1}$, i.e., x_1 and x_2 do not interact, so that the only contribution to the mutual information has to come from the environmental changes. The system is described by a joint probability distribution function $p_i(\mathbf{x}, t)$ to have values $\mathbf{x} = (x_1, x_2)$ at time t and to be in the environmental state $i \in \{-, +\}$. This probability is governed by the Fokker-Planck equation

$$\begin{aligned} \partial_t p_i(\mathbf{x}, t) = \sum_{\mu=1}^{2} \partial_\mu \left[\frac{x_\mu}{\tau} p_i(\mathbf{x}, t)\right] + D_i \sum_{\mu=1}^{2} \partial_\mu^2 p_i(\mathbf{x}, t) \\ + \sum_{j \neq i} [w_i p_j(\mathbf{x}, t) - w_j p_i(\mathbf{x}, t)]. \end{aligned} \tag{3}$$

This model corresponds, for instance, to a switching environment in a chemical [3,16] or biological [2,6] system, or to different regimes of neural activity [10,13]. Furthermore, being related to "diffusing diffusivity" processes, it can also describe spatially disordered or inhomogeneous environments [27,30]. An extension to N different processes $(x_1, \ldots, x_N)$ and M environmental states $i_1, \ldots, i_M$ is possible once we choose a multivariate generalization of the mutual information (see the Supplemental Material [31] for details).

Let us note beforehand that the mutual information, Eq. (1), can only depend on dimensionless quantities. The relevant dimensionless parameters of this model are (i) τw_{sum}, where $w_{\text{sum}} = w_+ + w_-$, which governs the timescale separation between the internal degrees of freedom and the jump process of the environmental states; (ii) w_-/w_+, which determines the relative persistence of the environmental states; (iii) D_-/D_+, which describes the separation between the environmental states. Importantly, the joint probability does not depend on these three parameters' combinations only. Hence, to find a general solution to Eq. (3) proves to be a particularly challenging task. Therefore, we resort to a timescale separation that corresponds to the two limits in which the jumps are either much faster or much slower than the relaxation time of x_1 and x_2 (see the Supplemental Material [31]).

In a fast environment, we have $\tau w_{\text{sum}} \gg 1$, and we find the stationary probability distribution

$$p^F(x_1, x_2) = \frac{1}{2\pi\tau\langle D\rangle_\pi} \exp\left[-\frac{x_1^2 + x_2^2}{2\tau\langle D\rangle_\pi}\right] \equiv p^F(x_1)p^F(x_2), \tag{4}$$

where $\langle D\rangle_\pi = (D_+ w_+ + D_- w_-)/w_{\text{sum}}$ plays the role of an effective diffusion coefficient, and the superscript F refers to the fast-jumps regime. Loosely speaking, this limit describes environmental changes affecting the internal degrees of freedom only on average, leaving the two processes independent. Hence, the joint probability factorizes and no mutual information arises (Fig. 1).

The picture is markedly different in the slow-jumps limit, when $\tau w_{\text{sum}} \ll 1$. The stationary probability distribution is now given by the Gaussian mixture

$$\begin{aligned} p^S(x_1, x_2) &= \frac{1}{2\pi\tau} \sum_i \left[\frac{\pi_i^s}{D_i} e^{-\frac{1}{2D_i\tau}(x_1^2 + x_2^2)}\right] \\ &= \pi_- p_-^S(x_1, x_2) + \pi_+ p_+^S(x_1, x_2), \end{aligned} \tag{5}$$

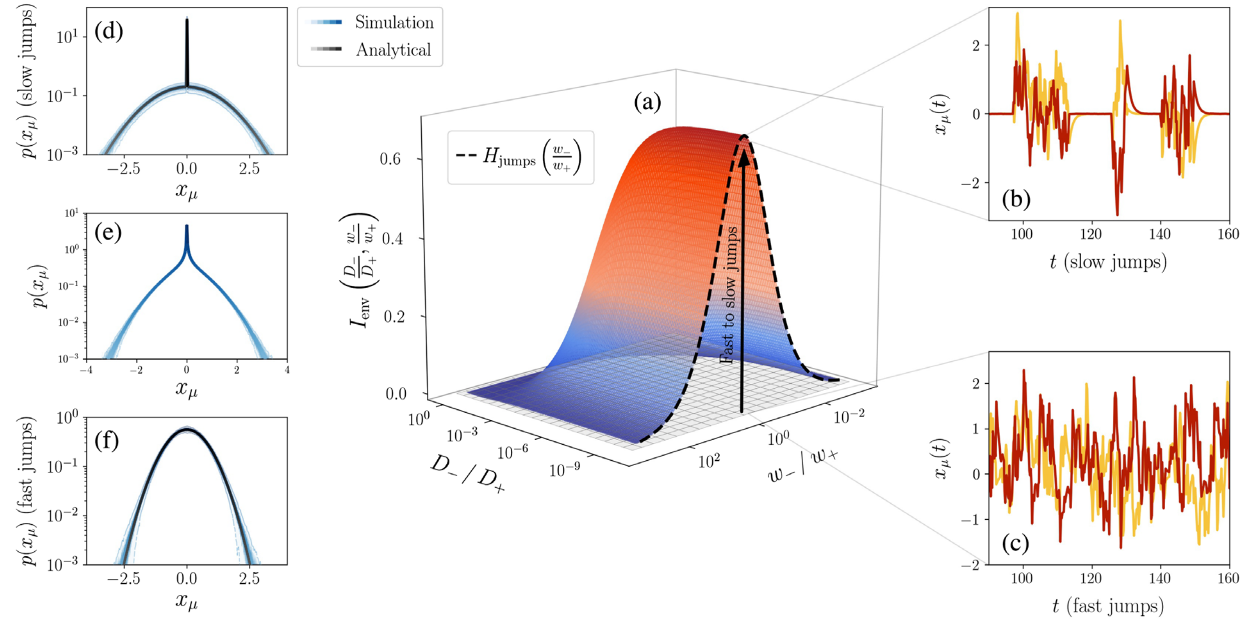

FIG. 1. The environmental contribution to the mutual information as a function of D_-/D_+ and w_-/w_+. (a) The colored surface is the result of a Monte Carlo integration with importance sampling of the mutual information in the slow-jumps limit with $\tau w_{\text{sum}} = 10^{-3}$ [see Eq. (1) and Eq. (4)]. In the $D_-/D_+ \to 0$ limit, I_{env} becomes exactly H_{jumps}, the black dashed line, which is also its maximum value. The gray plane is instead the mutual information in the fast-jumps limit, which always vanishes. (b) A realization of $x_1(t)$ and $x_2(t)$ (red and yellow curves) in the slow-jumps limit, at $w_-/w_+ = 1$ and $D_-/D_+ = 10^{-10}$. A bursty, coordinated behavior emerges due to the environmental changes. (c) Same, but in the fast-jumps limit, where both variables show a Brownian-like behavior. (d)–(f) Comparison between the marginalized probability distribution $p(x_\mu)$ from the simulated Langevin dynamics and the analytical distribution [(d) and (f) cases] for $D_- = 10^{-5}$, $D_+ = 1$, $\tau = 1$ in (d) the slow-jumps limit at $w_- = w_+ = 5 \times 10^{-4}$, (f) the fast-jumps limit at $w_- = w_+ = 50$ and (e) in between at $w_- = w_+ = 0.5$.

where $\pi_i = w_i/w_{\text{sum}}$ are the stationary probabilities of the jump process, and the superscript S denotes the slow-jumps regime. It is clear that in this limit the two processes are not always independent. An example of a realization and its corresponding probability distribution is shown in Figs. 1(b) and 1(d), respectively. In the intermediate regime between the fast- and slow-jumps limits we cannot solve the Fokker-Planck equation explicitly, but a direct simulation of the Langevin equations [32], Eq. (2), shows that the resulting probability interpolates between Eq. (5) and Eq. (4) in a smooth fashion [Figs. 1(d)–1(f)]. Therefore, we will now focus on the slow-jumps limit, where we can tackle the problem analytically, and the mutual information takes nonzero values.

Even though no closed form exists for the entropy of a Gaussian mixture, from the bounds proposed in Ref. [33] we can build the corresponding bounds on the mutual information starting from the Chernoff-α divergence and on the Kullback-Leibler divergence between the mixture components, as reported in the Supplemental Material [31]. Notably, both the upper bound $I_{\text{env}}^{S,\text{up}}(D_-/D_+, w_-/w_+)$ and the lower bound $I_{\text{env}}^{S,\text{low}}(D_-/D_+, w_-/w_+)$ on the mixture distribution $p^S(x_1, x_2)$ only depend on the dimensionless parameters D_-/D_+ and w_-/w_+. Moreover, although in general these bounds are not tight, they do saturate in the limits $D_-/D_+ \to 0$ or $D_-/D_+ \to 1$—and these limits are particularly significant. The former corresponds to drastic environmental changes, which lead to markedly different dynamics and give rise to a bursty, seemingly coordinated behavior of the internal degrees of freedom. The latter, on the other hand, describes the trivial case in which D_- and D_+ are very similar and thus environmental changes are effectively negligible. We end up with (see Supplemental Material [31])

$$I_{\text{env}}^S\left(\frac{w_-}{w_+}\right) = \begin{cases} -\pi_+ \log \pi_+ - \pi_- \log \pi_- & \text{if } D_+ \gg D_- \\ 0 & \text{if } D_+ \approx D_-, \end{cases} \tag{6}$$

which, since the bounds saturate, are the exact limits of the mutual information in the slow-jumps regime. Clearly, when $D_-/D_+ \to 1$, the dynamics is insensitive to the environment, thus $x_1(t)$ and $x_2(t)$ are independent processes. Instead, and interestingly, the first line is nothing but the Shannon entropy of the jump distribution, $H_{\text{jumps}}(w_-/w_+)$. A Monte Carlo integration of Eq. (1) shows that H_{jumps} is also the maximum value of the mutual information that emerges due to the environment, see Fig. 1(a). This result has a quite clear intuitive interpretation. In fact, from an information-theoretic point of view, H_{jumps}

quantifies precisely the information lost once we integrate out the stochastic environment, i.e., our ignorance about the system as a whole.

So far, we have only considered the presence of an effective coupling emerging from environmental changes. Although our results have been derived for Ornstein-Uhlenbeck processes, they equivalently hold for the more general stochastic dynamics $\dot{x}_\mu(t) = f_\mu(x_\mu) + \sqrt{2D_{i(t)}}\xi_\mu(t)$, even when $f_\nu \neq f_\mu$, as we show in the Supplemental Material [31]. Now, it is time to introduce back interactions between $x_1(t)$ and $x_2(t)$ by considering the case in which the matrix $\mathbf{A}$ in Eq. (2) has nonzero off-diagonal entries. We will show that their contribution to the total mutual information, I_{tot}, can be effectively disentangled from I_{env} under suitable limits.

Let us consider the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & -g_1 \\ -g_2 & 1 \end{pmatrix}$$

and assume that its eigenvalues have positive real parts, so that a stationary state exists [34]. Let us also assume, for the time being, that we are in the slow-jumps limit, so that we can solve the Langevin equations separately for D_+ and D_- and then average them over $\pi_\pm$ as in Eq. (5). The two solutions are multivariate Gaussian distributions, each one with a given covariance matrix $\mathbf{\Sigma}_\pm$.

In order to try and disentangle the environmental contribution, which is due to D_i, and the one stemming from internal interactions, due to the off-diagonal elements of $\mathbf{A}$, we write the covariance matrices as $\mathbf{\Sigma}_i = D_i\tilde{\mathbf{\Sigma}}$. The matrix $\tilde{\mathbf{\Sigma}}$ then solves the Lyapunov equation (see the Supplemental Material [31])

$$\frac{1}{2}[\mathbf{A}\tilde{\mathbf{\Sigma}} + \tilde{\mathbf{\Sigma}}\mathbf{A}^T] = \mathbb{1}, \tag{7}$$

which only depends on interactions and not on the jump dynamics, nor on D_i. We are now able to bound the mutual information as

$$I_{\mathrm{tot}}^{S,\mathrm{up/low}} = I_{\mathrm{int}}(\{g_\mu\}) + I_{\mathrm{env}}^{S,\mathrm{up/low}}\left(\frac{D_-}{D_+}, \frac{w_-}{w_+}\right), \tag{8}$$

where

$$I_{\mathrm{int}}(\{g_\mu\}) = \frac{1}{2}\log\left[1 - \frac{4}{4 + (g_1 - g_2)^2} + \frac{1}{1 - g_1 g_2}\right] \tag{9}$$

is the contribution to the mutual information due to the internal interactions only, as we show in the Supplemental Material [31]. Notably, I_{int} is also the sole contribution in the fast-jumps limit, since no environmental contribution is present to begin with. On the other hand, in the slow-jumps limit we can write the two limits, as in Eq. (6),

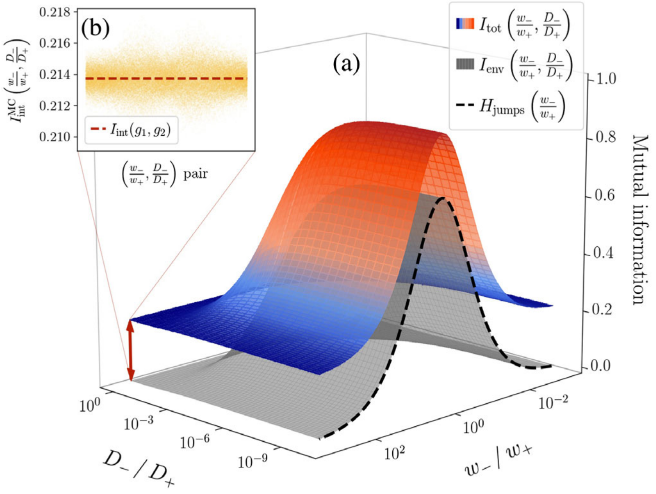

FIG. 2. The total mutual information as a function of D_-/D_+ and w_-/w_+ at fixed $\tau w_{\mathrm{sum}} = 10^{-3}$, i.e., in the slow-jumps limit. (a) The colored surface is the result of a Monte Carlo integration with importance sampling of I_{tot}, in the slow-jumps limit, for the interacting model with $g_1 = 5\tau$ and $g_2 = -0.1\tau$. The gray surface is instead the non-interacting case, I_{env}. The two contributions to the mutual information disentangle and the interactions simply result in a constant shift. (b) A comparison between the predicted shift $I_{\mathrm{int}}(g_1, g_2)$, Eq. (9), and the difference of the Monte Carlo estimates of the two surfaces for every sampling point $(w_-/w_+, D_-/D_+)$, namely, $I_{\mathrm{int}}^{MC}(w_-/w_+, D_-/D_+)$.

$$I_{\mathrm{tot}}^{S}\left(\{g_\mu\}, \frac{w_-}{w_+}\right) = \begin{cases} H_{\mathrm{jumps}} + I_{\mathrm{int}}(\{g_\mu\}) & \text{if } D_+ \gg D_- \\ I_{\mathrm{int}}(\{g_\mu\}) & \text{if } D_+ \approx D_-, \end{cases} \tag{10}$$

where the environmental bounds saturate. Finally, in the intermediate regime between the fast- and the slow-jumps limits, a Monte Carlo integration of Eq. (1) shows that the presence of linear internal interactions does simply shift the mutual information with respect to the noninteracting case (Fig. 2).

Therefore, our results suggest that I_{tot} receives two distinct contributions—one from the environment, I_{env}, and one from the internal linearized interactions, I_{int}—disentangled in form:

$$I_{\mathrm{tot}}\left(\{g_\mu\}, \frac{D_-}{D_+}, \frac{w_-}{w_+}\right) = I_{\mathrm{int}}(\{g_\mu\}) + I_{\mathrm{env}}\left(\frac{D_-}{D_+}, \frac{w_-}{w_+}\right). \tag{11}$$

Although the equation above holds analytically in the fast-jumps regime, and where the slow-jumps bounds Eqs. (8)–(10) saturate, its validity has been numerically shown in the entire range of parameters. Furthermore, even if the interactions are nonlinear, we show in the Supplemental Material [31] that in the fast-jumps limit the environmental contribution vanishes exactly. Hence, and independently of the underlying interactions, any nonzero mutual information in the fast-jumps limit acts as a fingerprint of the presence of internal couplings.

This result is extremely interesting. In fact, although the environmental states, identified by D_- and D_+ in our model, are usually not experimentally accessible, it might be possible to characterize the frequency of the environmental changes. Neural activity originated by external stimuli [35–37], stirring in chemical conglomerates [38], temperature-activated chemical reactions in solutions [3,39], and population growth [40–42], are only a few examples in which our framework might apply. Even if fast-varying environments have been shown to be informative, our approach might provide hints about the presence of interactions even away from the fast-jumps limit, by bounding the environmental contribution to the mutual information. This intriguing perspective will be investigated in future works.

Although we focused on a paradigmatic, but rather comprehensive, physical model, let us note that these ideas have a much larger scope, and that disentangling the different dependencies of a system is a far-reaching question. Techniques such as Bayesian networks and other probabilistic graphical models have been successfully used in biological data, for instance to disentangle different sources of interactions and dependencies in general [43–45]. Connections may be also drawn to machine learning and artificial neural networks, particularly in the context of learning disentangled representation of the data, i.e., representations in which the informative latent factors are described by a factorized distribution [46–49], or in generative models with latent variables, such as switching state space models [50,51]. The environment in our model, in fact, can be seen as a latent variable, i.e., unobserved and independent on the observed degrees of freedom, while affecting the observed dynamics. Unlike the one presented here, these approaches are often harder to interpret and are less prone to the derivation of exact results, even though they remain extremely powerful in dealing with experimental data. Hence, a possible future perspective is to combine the physical interpretability and the analytical procedures behind our work together with tools from machine learning and data-driven approaches. This could lead to promising results in the quest of meaningfully disentangle the different sources of dependencies that emerge in complex systems.

Furthermore, there are several more possible extensions to this study. One might ask whether a stochastic environment can be mapped into a set of effective couplings with defined properties, and if such couplings can be distinguished from the internal ones. Additionally, an important and immediate generalization of our framework is to allow the environment to be a continuous variable. These problems, in principle, can be treated from a field-theoretical perspective, where the marginalization over the environment gives rise to new interaction vertexes that are not present in the original theory, i.e., before the marginalization. Ideally, this could allow for a much more general framework amenable to analytical treatments.

On the other side, the ability to analytically deal with a class of stochastic processes with tools of information theory, as shown here, opens up many fascinating possibilities. A particularly appealing question is what happens when, instead of considering a stochastic environment, the system undergoes an external perturbation—notably, how the latter changes the information content and how such information evolves over time. A first step towards this direction might be to consider two diffusion processes in a finite domain that undergo a single stochastic jump, and to study the persistence of the mutual information as a function of time, domain size, and boundary conditions of the system.

Ultimately, we believe that this work draws a path towards a deeper understanding of the different sources of couplings in real-world systems. Indeed, it is a starting point to elucidate the relations between their internal complexity and possibly equally complex, but unobserved, ever-changing environments.

We acknowledge A. Maritan, S. Azaele, and S. Suweis for insightful discussions, valuable suggestions, and a careful reading of the manuscript.

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