Quantum Energy Lines and the Optimal Output Ergotropy Problem

Salvatore Tirone,[1,*] Raffaele Salvia,[1,†] and Vittorio Giovannetti[2]
[1]*Scuola Normale Superiore, I-56127 Pisa, Italy*
[2]*NEST, Scuola Normale Superiore and Istituto Nanoscienze-CNR, I-56127 Pisa, Italy*

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We study the transferring of useful energy (work) along a transmission line that allows for partial preservation of quantum coherence. As a figure of merit we adopt the maximum values that ergotropy, total ergotropy, and nonequilibrium free energy attain at the output of the line for an assigned input energy threshold. For phase-invariant bosonic Gaussian channel (BGC) models, we show that coherent inputs are optimal. For (one-mode) not phase-invariant BGCs we solve the optimization problem under the extra restriction of Gaussian input signals.

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Introduction.—Quantum technologies, which are extremely successful in delivering groundbreaking improvements for information processing procedures [1], have a chance of being essential also in the management of energy sources. In particular the tremendous advances in experimental techniques witnessed in the last decade [2–7] suggest the possibility of realistically enhancing the performances of thermal machines by designing new types of devices that maintain some degree of quantum coherence in their functioning (quantum thermal machines) [8–36]. Furthermore recent studies [37–42] indicate that using genuine quantum systems as energy storing devices (quantum batteries) could be crucial in speeding up energy charging processes—see Ref. [43] for a first experimental proof-of-concept implementation. In view of these results, it makes sense to study the impact that quantum effects may have on energy transmission procedures. Here we present a first study of quantum energy lines (QELs) which, at variance with traditional models, are capable of preserving a certain degree of quantum coherence during the transfer of energy pulses. Previous works on the subject can be found in quantum biology, where the rather counter-intuitive feature of noise-enhanced speedup [44–46] was observed, which may actually contribute to the efficiency of the light-harvesting complexes responsible for photosynthesis [47–51]. Moreover, there have been theoretical attempts to teleport energy using ground state fluctuations of quantum fields [52,53]. At variance with those studies, where the goal of the transmission line is to send down energy from *classical* energy sources to *classical* users, in our vision QELs could be employed to improve the connectivity between energy power plants and energy storing sites that are capable of handling energy in a quantum coherent fashion, avoiding the need to pass through unnecessary quantum-to-classical and classical-to-quantum conversion stages (see, e.g., Ref. [54]).

While rather unconventional, prototypical examples of QELs already exist in the form of free space or optical fiber transmission lines which are currently under development by various international agencies [55–60]. These schemes have been extensively studied by the quantum information community [61–70] and admit a formal description in terms the formalism of bosonic Gaussian channels (BGCs) [61–65] which we shall adopt hereafter. Within this setting, the main goal of our analysis is to identify the pulses that have to be sent through the QEL to ensure the lowest level of energy waste. Such a task is not dissimilar from the optimization problem one faces with more conventional power lines: in the present case however the issue is complicated by the absence of a clear distinction between heat and work in a purely quantum mechanical setting [71–74]. Singling out the useful part (work) of the internal energy of a quantum system (in our case the output signals of the QEL) does indeed strongly depend upon the resources available to the process [75]. The maximal amount of energy one can recover by means of reversible coherent (i.e., unitary) processes [76] is the *ergotropy* $\mathcal{E}$, a nonlinear functional of the state of the quantum system [76]. This quantity is relevant in the study of cycles [23–28], and it has important connections with the theories of entanglement and coherence [29–31]. If we have instead access to many copies of the system, the relevant quantity is the *total ergotropy* $\mathcal{E}_{\rm tot}$, which is the work that we reversibly extract from an ensemble of asymptotically many copies of the system of interest [75,77]. Finally, granting access to a thermal bath, we can push the work extraction process to a further level quantified by the *nonequilibrium free energy* $\mathcal{F}^\beta$ of the state [78–80].

Ergotropy, total ergotropy, and nonequilibrium free energy all qualify as bona fide figures of merit for the work we can extract from a quantum system. Accordingly we shall study the efficiency of a QEL by determining which, among the set of signals that have the same input energy, ensure the highest values of $\mathcal{E}$, $\mathcal{E}_{\rm tot}$, and $\mathcal{F}^\beta$ at the receiving end of the line. For the special case of QELs described by phase-insensitive (PI) BGCs which describe propagation loss, thermalization, and amplification noise effects, we provide an exact solution of the optimization task showing that optical coherent states [81,82] always ensure the best performances for all the three figures of merit (a result that mimics the Gaussian optimization solution observed in the study of PIBGC as quantum communication lines [83–86]). We complete our analysis presenting the solution for the optimization problem for the special case of general (non-PI) one-mode BGCs, obtained by limiting the input signals to Gaussian states [63].

The scheme.—We shall model a QEL as a collection of Bosonic (electromagnetic) modes that lose energy en route from the transmitter to the receiver while possibly undergoing events of amplification, and thermalization effects [62,66,67]. A rigorous mathematical characterization of the noise affecting the transmitted signals in these lines can be obtained in terms of phase-insensitive BGCs [63,64,87–89] whose properties we now review in brief. An n-mode continuous variables (CV) system is described by a complex separable Hilbert space $\mathcal{H}$ equipped with self-adjoint bosonic field operators $\hat{q}_1, \hat{p}_1, \ldots, \hat{q}_n, \hat{p}_n$ that obey the canonical commutation relations (i.e., $[\hat{q}_j, \hat{q}_k] = [\hat{p}_j, \hat{p}_k] = 0$, $[\hat{q}_j, \hat{p}_k] = i\delta_{jk}\hat{I}$ with $\hat{I}$ the identity operator), and by the free electromagnetic Hamiltonian $\hat{H} := \sum_{j=1}^{n}[(\hat{q}_j^2 + \hat{p}_j^2)/2] - (n/2)$, which we express using dimensionless units ($\hbar = \omega = 1$) and removing the vacuum energy contribution. The set of quantum states $\mathcal{D}(\mathcal{H})$ of the system is composed of all positive trace-class operators on $\mathcal{H}$ with trace 1. Introducing $\hat{r} := (\hat{q}_1, \ldots, \hat{q}_n, \hat{p}_1, \ldots, \hat{p}_n)^T$ and using $\{,\}$ to denote the anticommutator, for each $\hat{\rho} \in \mathcal{D}(\mathcal{H})$ we hence define its statistical mean vector $m(\hat{\rho}) := \mathrm{Tr}[\hat{\rho}\,\hat{r}] \in \mathbb{R}^{2n}$, its covariance matrix $\sigma(\hat{\rho}) \in \mathbb{R}^{2n\times 2n}$ of elements $\sigma_{jk}(\hat{\rho}) := \mathrm{Tr}[\hat{\rho}\{\hat{r}_j - m_j, \hat{r}_k - m_k\}]$, and its characteristic function $\chi(\hat{\rho}; x) := \mathrm{Tr}[\hat{\rho}\,\hat{D}(x)]$, where $\hat{D}(x) := \exp[i\hat{r}\cdot x]$ is the Weyl or displacement operator and $x \in \mathbb{R}^{2n}$. We also say that $\hat{\rho} \in \mathcal{D}(\mathcal{H})$ is a Gaussian state when the associate characteristic function is Gaussian [88,89], i.e., when $\chi(\hat{\rho}; x) = e^{-\frac{1}{4}x^T\sigma x + im\cdot x}$ for some mean vector m and convariance matrix σ. Given now an input n_I-mode CV system with Hilbert space $\mathcal{H}_I$, and an output n_O-mode system with Hilbert space $\mathcal{H}_O$, a BGC $\Phi: \mathcal{D}(\mathcal{H}_I) \to \mathcal{D}(\mathcal{H}_O)$ is a linear, completely positive and trace preserving map [64] that preserves the Gaussian character of the transmitted signals. These transformations can be formally described by assigning a vector $v \in \mathbb{R}^{2n_O}$ and matrices $Y \in \mathbb{R}^{2n_O\times 2n_O}$, $X \in \mathbb{R}^{2n_O\times 2n_I}$ that verify the condition $Y \geq i(\gamma_{n_O} - X\gamma_{n_I}X^T)$ with $\gamma_n := \left(\begin{smallmatrix} 0 & I_n \\ -I_n & 0 \end{smallmatrix}\right)$ being a $2n \times 2n$ block matrix. Explicitly given $\hat{\rho} \in \mathcal{D}(\mathcal{H}_I)$ as the state describing the input signal of the channel, the characteristic function of the corresponding state $\Phi(\hat{\rho})$ at the end of the transmission line can be written as $\chi(\Phi(\hat{\rho}); x) = \chi(\hat{\rho}; X^T x)e^{-\frac{1}{4}x^T Y x + iv\cdot x}$, implying the identities

$$m[(\Phi(\hat{\rho})] = Xm(\hat{\rho}) + v, \quad \sigma[(\Phi(\hat{\rho})] = X\sigma(\hat{\rho})X^T + Y. \tag{1}$$

A BGC map Φ is finally said to be PI if for all input states $\hat{\rho} \in \mathcal{D}(\mathcal{H}_I)$, and for all $t \in \mathbb{R}$ we have

$$\Phi(e^{-i\hat{H}_I t}\hat{\rho}e^{i\hat{H}_I t}) = e^{\mp i\hat{H}_O t}\Phi(\hat{\rho})e^{\pm i\hat{H}_O t}, \tag{2}$$

with $\hat{H}_I$ and $\hat{H}_O$ the free Hamiltonians of $\mathcal{H}_I$ and $\mathcal{H}_O$ respectively [specifically a Φ verifying Eq. (2) with the upper signs in the rhs is said to be *gauge-covariant*]. From a practical point of view the multimode PI BGC models described here provide an idealized yet commonly used version of broadband communication lines, because they characterize quantum states transferred through an optical medium via electromagnetic pulses whose bandwidth is small compared with their central reference frequency [61,68–70].

Quantum work extraction.—At variance with purely classical settings, discriminating which part of the internal energy of a quantum system (e.g., the output signal of a QEL) can be identified with heat or work is difficult [71–73] due to correlation-induced entropy increases that may arise when coupling the system to an external load [75]. Nonetheless, limiting the allowed operations to be local, fully reversible, and coherent (i.e., unitary), the amount of work we can extract from a single copy of a density matrix $\hat{\rho}$ of a system is given by the *ergotropy* functional $\mathcal{E}(\hat{\rho})$ [76]. Letting $\mathfrak{E}(\hat{\rho}) := \mathrm{Tr}[\hat{\rho}\,\hat{H}]$, we can write

$$\mathcal{E}(\hat{\rho}) := \mathfrak{E}(\hat{\rho}) - \mathfrak{E}(\hat{\rho}^{\downarrow}), \tag{3}$$

where $\hat{\rho}^{\downarrow}$ is the *passive counterpart* [90,91] of $\hat{\rho}$, i.e., the special element of $\mathcal{D}(\mathcal{H})$ which has the lowest energy among those with the same spectrum of $\hat{\rho}$ [92]. Since passive states are not necessarily *completely passive* [93,94], ergotropy turns out to be a nonextensive, superadditive quantity. Accordingly, when operating with reversible coherent operations on N copies of a given state $\hat{\rho}$, it is possible to increase the total amount of extractable energy by acting jointly on the whole set of subsystems. The maximum amount of energy per copy that is attainable under this new paradigm is quantified by the *total ergotropy* $\mathcal{E}_{\mathrm{tot}}(\hat{\rho})$, a functional fulfilling the inequality $\mathcal{E}_{\mathrm{tot}}(\hat{\rho}) \geq \mathcal{E}(\hat{\rho})$ which can obtained via a proper regularization of Eq. (3), i.e.,

$$\mathcal{E}_{\mathrm{tot}}(\hat{\rho}) := \lim_{n\to\infty}\frac{1}{n}\mathcal{E}(\hat{\rho}^{\otimes n}) = \mathfrak{E}(\hat{\rho}) - \mathfrak{E}(\hat{\tau}_{\beta(\hat{\rho})}), \tag{4}$$

where in the last identity $\hat{\tau}_\beta := e^{-\beta\hat{H}}/\mathrm{Tr}[e^{-\beta\hat{H}}]$ is a thermal Gibbs state of the system whose inverse temperature $\beta \in \mathbb{R}^+$ is fixed in order to ensure $S(\hat{\tau}_{\beta(\hat{\rho})}) = S(\hat{\rho}) := -\mathrm{Tr}[\hat{\rho}\log\hat{\rho}]$. $\mathcal{E}_{\mathrm{tot}}(\hat{\rho})$ represents the ultimate amount of energy that we can extract reversibly from $\hat{\rho}$ when having at our disposal an unlimited number of copies. More energy from the system can still be converted into useful work only if we are willing to admit some dissipation side effect, e.g., by coupling the system with an external thermal bath [75,78]. In this case the overall amount of extractable energy is provided by the nonequilibrium *free energy* functional $\mathcal{F}^\beta(\hat{\rho}) \coloneqq \mathfrak{E}(\hat{\rho}) - S(\hat{\rho})/\beta$, with β representing the inverse temperature of the bath.

Optimal inputs for PIBGCs.—Here we present our main result: input coherent states maximize the three functionals introduced in the previous sections at the output of any PIBGC [63]. To this aim we first observe the following fact:

Theorem 1.—Given Φ a PIBGC from n_I input to n_O output modes, for any $\hat{\rho} \in \mathcal{D}(\mathcal{H}_I)$ there exists at least a coherent input state $\hat{\varphi} \in \mathcal{D}(\mathcal{H}_I)$ having the same mean input energy of $\hat{\rho}$ such that $\mathfrak{E}[\Phi(\hat{\varphi})] \geq \mathfrak{E}[\Phi(\hat{\rho})]$.

Proof.—We recall that the mean energy of a quantum state $\hat{\rho} \in \mathcal{D}(\mathcal{H})$ of n modes can be expressed in terms of its statistical mean and covariance matrix via the compact expression $\mathfrak{E}(\hat{\rho}) = \{\mathrm{Tr}[\sigma(\hat{\rho})]/4\} + (|m(\hat{\rho})|^2/2) - n/2$. Recall also that the coherent states $\hat{\varphi}$ of a n_I-mode CV system are characterized by a covariance matrix $\sigma(\hat{\varphi}) = I_{2n_I}$; so any $\hat{\varphi} \in \mathcal{C}(\mathcal{H}_I)$ is uniquely identified by its statistical mean m: $|\varphi\rangle = \hat{D}(m)|\emptyset\rangle$ with $|\emptyset\rangle$ being the vacuum state of the model. Thanks to this identity and to Eq. (1), the mean energy at the output of a BGC Φ defined by the vector v and the matrices X, Y can be expressed as $\mathfrak{E}[\Phi(\hat{\rho})] = \{\mathrm{Tr}[X^T X \sigma(\hat{\rho})]/4\} + (|Xm(\hat{\rho}) + v|^2/2) + c$, where $c = \mathrm{Tr}[Y]/4 - n_O/2$ is a constant that is independent from the input state $\hat{\rho}$. Next we recall that any PIBGC has $v = 0$ and admits an orthogonal, symplectic transformation $V \in \mathbb{R}^{2n_I \times 2n_I}$ such that the following statement holds [63,95]:

$$(V^T X^T X V)_{jk} = \Lambda_j \delta_{jk}, \tag{5}$$

with $\Lambda_m = \Lambda_{m+n_I}$ $\forall$ $m = 1, \ldots, n_I$, and $\Lambda_m \geq \Lambda_{m+1}$ $\forall$ $m = 1, \ldots, n_I$. Observe that the above conditions are equivalent to saying that $\sqrt{\Lambda_1}$ is the highest singular eigenvalue of X, or explicitly that for every vector $w \in \mathbb{R}^{2n_I}$ it holds that $|Xw|^2 \leq \Lambda_1 w^2$. Now consider a coherent state $\hat{\varphi}$ with mean vector $m(\hat{\varphi})$ oriented in such a way to saturate the former inequality, i.e., $|Xm(\hat{\varphi})|^2 = \Lambda_1 m(\hat{\varphi})^2$, that is, with a $m(\hat{\varphi})$ which is an eigenvector of the matrix $X^T X$. Notice that such a condition can be fulfilled for any value of $|m(\hat{\varphi})|$, and hence for any possible $\mathfrak{E}(\hat{\varphi})$. For such a choice we can hence write the inequality

$$|Xm(\hat{\varphi})|^2 - |Xm(\hat{\rho})|^2 \geq \Lambda_1[m(\hat{\varphi})^2 - m(\hat{\rho})^2], \tag{6}$$

which holds true for all $\hat{\rho} \in \mathcal{D}(\mathcal{H}_I)$. Notice also that thanks to Eq. (5) and remembering that $\sigma(\hat{\varphi}) = I_{2n_I}$, we have

$$\begin{aligned}\mathrm{Tr}\{X^T X[\sigma(\hat{\rho}) - \sigma(\hat{\varphi})]\} &= \sum_{j=1}^{2n_I} \Lambda_j \{[V\sigma(\hat{\rho})V^T]_{jj} - 1\} \\ &\leq \Lambda_1 \mathrm{Tr}[V\sigma(\hat{\rho})V^T - \sigma(\hat{\varphi})] = \Lambda_1 \mathrm{Tr}[\sigma(\hat{\rho}) - \sigma(\hat{\varphi})].\end{aligned} \tag{7}$$

In deriving Eq. (7) we exploited the following two facts: (i) since V is a symplectic matrix, then $V\sigma(\hat{\rho})V^T$ is a covariant matrix $\sigma(\hat{\rho}')$ of a proper density matrix $\hat{\rho}'$ of the system and (ii) for all covariant matrices σ it is always true that $\sigma_{m,m} + \sigma_{m+n_I,m+n_I} \geq 2$ for all $m = 1, \cdots n_I$, which can be easily proven by noticing that the left-hand side is the trace of the covariance matrix of the reduced density matrix of the mth mode of the input system and from the fact that for any n-mode quantum state $\mathrm{Tr}[\sigma(\hat{\rho})] \geq 2n$, since $\mathfrak{E}(\hat{\rho}) \geq 0$. Finally, using Eqs. (6) and (7) we can conclude that

$$\begin{aligned}\mathfrak{E}[\Phi(\hat{\varphi})] - \mathfrak{E}[\Phi(\hat{\rho})] &= \frac{|Xm(\hat{\varphi})|^2 - |Xm(\hat{\rho})|^2}{2} - \frac{\mathrm{Tr}\{X^T X[\sigma(\hat{\rho}) - \sigma(\hat{\varphi})]\}}{4} \\ &\geq \Lambda_1\left[\frac{m(\hat{\varphi})^2 - m(\hat{\rho})^2}{2} - \frac{\mathrm{Tr}[\sigma(\hat{\rho}) - \sigma(\hat{\varphi})]}{4}\right] = \Lambda_1[\mathfrak{E}(\hat{\varphi}) - \mathfrak{E}(\hat{\rho})] \geq 0,\end{aligned} \tag{8}$$

which evaluated in the case where $\hat{\varphi}$ and $\hat{\rho}$ shares the same input energy [i.e., $\mathfrak{E}(\hat{\varphi}) = \mathfrak{E}(\hat{\rho})$] implies $\mathfrak{E}[\Phi(\hat{\varphi})] \geq \mathfrak{E}[\Phi(\hat{\rho})]$, hence the thesis.

Exploiting the above result we are now ready to present our main finding:

Theorem 2.—Given Φ, a PIBGC from n_I input to n_O output modes, for any $E \in \mathbb{R}^+$ and $\hat{\rho} \in \mathcal{D}(\mathcal{H}_I)$ with $\mathfrak{E}(\hat{\rho}) \leq E$, there exists a coherent state $\hat{\varphi} \in \mathcal{D}(\mathcal{H}_I)$ with $\mathfrak{E}(\hat{\varphi}) = E$ that achieves higher values of the output ergotropy, max ergotropy, and nonequilibrium free energy functionals, i.e.,

$$\begin{aligned}&\mathcal{E}[\Phi(\hat{\varphi})] \geq \mathcal{E}[\Phi(\hat{\rho})], \qquad \mathcal{E}_{\mathrm{tot}}[\Phi(\hat{\varphi})] \geq \mathcal{E}_{\mathrm{tot}}[\Phi(\hat{\rho})], \\ &\mathcal{F}^\beta[\Phi(\hat{\varphi})] \geq \mathcal{F}^\beta[\Phi(\hat{\rho})], \qquad \forall\ \beta > 0.\end{aligned} \tag{9}$$

Proof.—The ergotropy, the total ergotropy, and the non-equilibrium free energy are all Schur-convex functionals of the states (see the Supplemental Material [92] for details, which includes Refs. [96–100]). Now in Refs. [83,84] it has been shown that coherent states optimize the output of BGCs with respect to every Schur-convex functional. Therefore, given $\mathfrak{E}(\hat{\rho}) \leq E$ and every coherent state $\hat{\varphi}$ we can write

$$\mathfrak{E}[\Phi(\hat{\varphi})^{\downarrow}] \leq \mathfrak{E}[\Phi(\hat{\rho})^{\downarrow}], \qquad \mathfrak{E}(\hat{\tau}_{\beta[\Phi(\hat{\varphi})]}) \leq \mathfrak{E}(\hat{\tau}_{\beta[\Phi(\hat{\rho})]}), \\ S[\Phi(\hat{\varphi})] \leq S[\Phi(\hat{\rho})]. \tag{10}$$

The thesis now immediately follows from the above expressions and from Theorem 1, which guarantees that there is at least a coherent state $\hat{\varphi}$ with mean energy greater or equal to E that fulfills the inequality $\mathfrak{E}[\Phi(\hat{\varphi})] \geq \mathfrak{E}[\Phi(\hat{\rho})]$.

One-mode PIBGCs.—In the special case of one-mode (i.e., $n_I = n_O = 1$) PIBGCs, some simplification occurs that allows us to extend the result of the previous section a little further. First we remark that in this context passivity and complete passivity are equivalent notions [63]:

$$\mathcal{E}(\hat{\tau}) = \mathcal{E}_{\mathrm{tot}}(\hat{\tau}) \quad \forall \ \hat{\tau} \text{ one-mode Gaussian states.} \tag{11}$$

In this particular setting the energy at the output of PIBGCs $\mathfrak{E}[\Phi(\rho)]$ depends only on the input energy $\mathfrak{E}(\rho)$, using both this fact and the main result of Ref. [101], one can prove the following statement:

Theorem 3.—Given that Φ is a one-mode PIBGC and for any $E, s \in \mathbb{R}^+$ and one-mode bosonic state $\hat{\rho}$ with $\mathfrak{E}(\hat{\rho}) \leq E$ and $S(\hat{\rho}) \geq s$, there exists a Gaussian state $\hat{\tau}$ with mean energy E and entropy s, that achieves higher values of the output ergotropy, max ergotropy, and nonequilibrium free energy functionals, i.e.,

$$\mathcal{E}[\Phi(\hat{\tau})] = \mathcal{E}_{\mathrm{tot}}[\Phi(\hat{\tau})] \geq \mathcal{E}_{\mathrm{tot}}[\Phi(\hat{\rho})] \geq \mathcal{E}[\Phi(\hat{\rho})], \\ \mathcal{F}^{\beta}[\Phi(\hat{\tau})] \geq \mathcal{F}^{\beta}[\Phi(\hat{\rho})], \quad \forall \ \beta > 0. \tag{12}$$

It is easy to notice that in this scenario Theorem 2 is a special instance ($s = 0$) of the result above.

All one-mode PIBGCs can be expressed as compositions of three maps [89]: the lossy thermal channel $\mathcal{L}_{\eta,N}$, describing the interaction with a thermal environment of mean photon number $N \geq 0$ through a beam splitter of transmissivity $\eta \in [0,1]$ ($X = \sqrt{\eta} I_2$; $Y = (1-\eta)(2N+1)I_2$); the amplification thermal channel $\mathcal{A}_{\mu,N}$, describing the interaction with a thermal environment through a linear optical amplifier of gain $\mu \geq 1$ [$X = \sqrt{\mu} I_2$; $Y = (\mu - 1)(2N+1)I_2$]; and the additive classical noise channel $\mathcal{N}_N$ describing a random displacement of the signal in the phase space ($X = I_2$; $Y = 2NI_2$). It follows that, constraining the input energy to be $\mathfrak{E}(\hat{\rho}) \leq E$, the maximum ergotropy (and total ergotropy) achievable at the output are respectively $\mathcal{E}_E^{(\max)}(\mathcal{L}_{\eta,N}) = \eta E$, $\mathcal{E}_E^{(\max)}(\mathcal{A}_{\mu,N}) = \mu E$, and $\mathcal{E}_E^{(\max)}(\mathcal{N}_N) = E$ (see the Supplemental Material [92] for details). Notice that the reported values do not depend upon N and exhibit the multiplicative behavior found in Ref. [102], where an optimization of the output ergotropy for $\mathcal{L}_{\eta,N}$ and $\mathcal{A}_{\mu,N}$ was performed on the restricted setting of Gaussian inputs.

BGCs that are not PI.—If we drop the phase-invariance assumption [Eq. (2)], coherent states $\hat{\varphi}$ no longer represent the optimal choices for the output work-extraction functionals: in this case the problem is made more complex by the fact that now the channel does not admit a single input state $\hat{\rho}$ that maximizes the positive contribution of $\mathcal{E}[\Phi(\hat{\rho})]$, $\mathcal{E}_{\mathrm{tot}}[\Phi(\hat{\rho})]$, and $\mathcal{F}^{\beta}[\Phi(\hat{\rho})]$ (i.e., the term $\mathfrak{E}[\Phi(\hat{\rho})]$), and at the same time minimizes the negative one (e.g., $\mathfrak{E}[\Phi(\hat{\rho})^{\downarrow}]$ for the ergotropy). One-mode not phase-invariant BGC channels are able to describe energy exchanges of the transmitted signals with a squeezed vacuum environment [103,104]. In the Supplemental Material [92] we consider in full generality one-mode BGCs. To elucidate the difficulty of the problem, here we show the example of the map $\Gamma_{\eta,\zeta} \coloneqq \mathcal{L}_{\eta,0} \circ \Sigma_{\zeta}$, results from a concatenation of a squeezing unitary evolution ($X = \begin{pmatrix} \sqrt{\zeta} & 0 \\ 0 & \sqrt{\zeta} \end{pmatrix}$; $Y = 0$) that precedes the action of a quantum-limited attenuator (here $\zeta \geq 1$, with $\zeta = 1$ corresponding to the zero-squeezing case). On one hand, for this channel the pure displaced-squeezed states $\hat{\rho}_1$ having covariance matrix $\sigma(\hat{\rho}_1) = \begin{pmatrix} \zeta^{-1} & 0 \\ 0 & \zeta \end{pmatrix}$ can be easily shown to provide the output configurations that majorize all the others [105], minimizing the negative contributions of $\mathcal{E}[\Gamma_{\eta,\zeta}(\hat{\rho})]$, $\mathcal{E}_{\mathrm{tot}}[\Gamma_{\eta,\zeta}(\hat{\rho})]$, and $\mathcal{F}^{\beta}[\Gamma_{\eta,\zeta}(\hat{\rho})]$ by the same Schur-convex argument we used before—indeed, with this choice $\Sigma_{\zeta}(\hat{\rho}_1)$ becomes a coherent state, and the

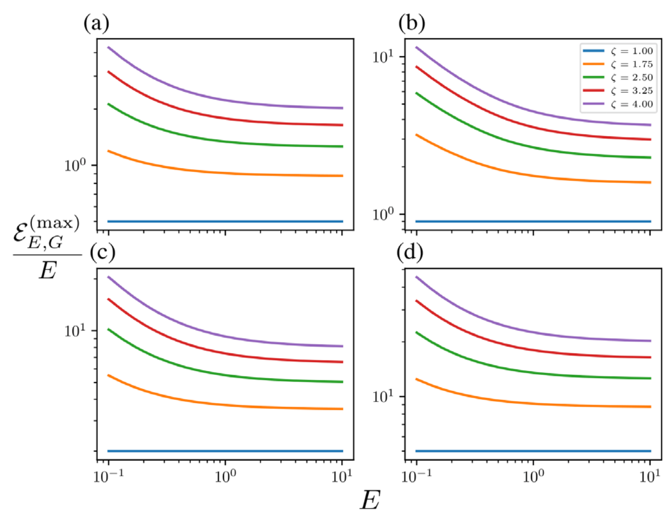

FIG. 1. Rescaled maximum output ergotropy values $\mathcal{E}_{E,G}^{(\max)}/E$ attainable with Gaussian inputs with input energy E for one-mode, not PI BGCs. (a),(b) Attenuator-squeezing channels $\Gamma_{\eta,\zeta} = \mathcal{L}_{\eta,0} \circ \Sigma_{\zeta}$ with $\eta = 0.5$ and $\eta = 0.9$, respectively. (c),(d) Amplifier-sqeezing channels $\Theta_{\mu,\zeta} = \mathcal{A}_{\mu,0} \circ \Sigma_{\zeta}$ with $\mu = 2$ and $\mu = 5$, respectively. In the no-squeezing $\zeta = 1$ regime (blue lines) the maps are PI, and the reported values coincide with the absolute maxima (η and μ) dictated by Theorem 2.

result follows directly from Refs. [83,84] by observing that $\mathcal{L}_{\eta,0}$ is phase insensitive. On the other hand, let $\hat{\rho}_2$ be the Gaussian state with moments $m(\hat{\rho}_2) = m(\hat{\rho}_1)$ and $\sigma(\hat{\rho}_2) = \left(\begin{smallmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{smallmatrix}\right)$. It is not difficult to see that $\hat{\rho}_2$ has the same energy as $\hat{\rho}_1$, but that $\mathfrak{E}[\Gamma_{\eta,\zeta}(\hat{\rho}_2)] > \mathfrak{E}[\Gamma_{\eta,\zeta}(\hat{\rho}_1)]$ for all $\zeta \geq 1$, preventing $\hat{\rho}_1$ from being the optimal choice for the positive contribution of the output work-extraction functionals.

A partial solution of the optimal work preservation problem at the output of non-PI BGCs is presented in the Supplemental Material [92] where, focusing on one-mode (not PI) BGCs, we provide an analytical characterization of the maximal output ergotropy $\mathcal{E}^{(\max)}_{E,G}$ achievable using energy constrained Gaussian inputs [incidentally, thanks to Eq. (11) these values also coincide with the Gaussian maximal values of the output total ergotropy]. The results we obtained are summarized in Fig. 1 where we plot the ratio $\mathcal{E}^{(\max)}_{E,G}/E$ for different types of channels $\Gamma_{\eta,\zeta} = \mathcal{L}_{\eta,0}\circ\Sigma_\zeta$ and $\Theta_{\mu,\zeta} = \mathcal{A}_{\mu,0}\circ\Sigma_\zeta$ obtained respectively by composing attenuator and amplifying channels with squeezing operations. Notice that the presence of squeezing tends to boost the ergotropy throughput by yielding values of the ratio which can exceed 1 even in the presence of attenuation, and that in the high energy limit the solutions approach the asymptotic limits $\lim_{E\to\infty}\mathcal{E}^{(\max)}_{E,G}(\Gamma_{\eta,\zeta})/E = \eta\zeta$ and $\lim_{E\to\infty}\mathcal{E}^{(\max)}_{E,G}(\Theta_{\mu,\zeta})/E = \mu\zeta$, respectively.

Conclusions.—The study of QELs paves the way to design improvements for quantum batteries or quantum thermal engines by facilitating the interconnections between clusters of such devices, and contributing to the stabilizing protocols for preserving the energy stored within [106–114]. Generalization of the present approach to a finite-dimensional setting may represent an interesting theoretical research line.

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*salvatore.tirone@sns.it
†raffaele.salvia@sns.it

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