Constrained Reversible System for Navier-Stokes Turbulence

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Following a Gallavotti's conjecture, stationary states of Navier-Stokes fluids are proposed to be described equivalently by alternative equations besides the Navier-Stokes equation itself. We discuss a model system symmetric under time reversal based on the Navier-Stokes equations constrained to keep the enstrophy constant. It is demonstrated through highly resolved numerical experiments that the reversible model evolves to a stationary state which reproduces quite accurately all statistical observables relevant for the physics of turbulence extracted by direct numerical simulations (DNS) at different Reynolds numbers. The possibility of using reversible models to mimic turbulence dynamics is of practical importance for the coarse-grained version of Navier-Stokes equations, as used in large-eddy simulations. Furthermore, the reversible model appears mathematically simpler, since enstrophy is bounded to be constant for every Reynolds number. Finally, the theoretical interest in the context of statistical mechanics is briefly discussed.

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Nonequilibrium macroscopic systems are generally described in the framework of irreversible hydrodynamics [1–5]. In some cases, the hydrodynamic level is obtained from the microscopic molecular through coarse graining [6,7], and the laws that emerge through coarse graining break the fundamental time-reversal symmetry inherent to the microscopic laws [1,8–11]. The foremost physical example of irreversible processes is given by an incompressible fluid which is described by the Navier-Stokes equations [12–14]. In this framework, the molecular effects are represented by the viscosity ν that is also responsible for the dissipation of energy, and may lead to a stationary state when energy is injected. In the limit of vanishing viscosity, the fluid becomes turbulent [12,15] and displays the outstanding feature of "anomaly dissipation," which means that the mean rate of kinetic energy dissipation $\langle\epsilon\rangle$ remains finite and independent of ν. Thus, the trace of irreversibility is kept through this singular limit [16,17]. The rigorous explanation of such a feature remains an open issue, and is at the basis of the mathematical problem of the existence and smoothness of the Navier-Stokes solution in three dimensions [18–20]. Furthermore, nontrivial features of irreversibility have been found in Lagrangian statistics [21], and such extreme events have been shown to be possibly related to singularities in Navier-Stokes equations [22,23]. One problem of such an approach is the asymptotic nature of turbulence, which makes it difficult to disentangle in actual experiments Reynolds number effects from genuine features [24,25]. An alternative approach was proposed by Gallavotti through the conjecture that the same system can be described by different yet equivalent models, notably for fluids [26]. In particular, phenomenological irreversible macroscopic systems could be described by suitable reversible dissipative models, at least in some respect. This idea was rooted in several developments in statistical physics, and notably in the use of thermostats in molecular dynamics simulations [27,28].

The possibility to use a time-reversible model to obtain turbulent features was pioneered in Ref. [29], and then conjectured in a more formal way by Gallavotti [30,31]. This conjecture has been called equivalence of dynamical ensemble, to clearly point out the analogy with ensembles in equilibrium statistical mechanics [32]. In this framework, in the thermodynamic limit, $N \to \infty$ with $\rho = N/V = \text{const}$, any local observable (i.e., related to a finite region of the phase space) is equal in all canonical ensembles. Following this picture, it has been proposed to replace the constant viscosity with a fluctuating one that would make it possible to have a new global invariant for the system. The thermodynamic limit is obtained in the case $1/\nu \to \infty$. Since in this fully turbulent limit the system is highly chaotic and exhibits a random behavior, it is plausible to conjecture that it may be described by an invariant distribution, as already postulated by Kolmogorov [33–35].

The conjecture has been directly tested in small 2D systems [31,36,37], for the Lorenz model [38], in shell models [39,40]. Recently, a model obtained by imposing the constraint that turbulent kinetic energy is conserved has been analyzed in 3D turbulence with a small number of modes [41]. Parallel attempts have been made to test the consequences, namely the fluctuation relations in different systems [42–45]. Yet, a clear demonstration of the validity of Gallavotti's conjecture is still lacking.

The purpose of the present work is precisely to show to which extent the Gallavotti conjecture is accurate, using high-resolution numerical experiments at different Reynolds numbers. Different equivalent models may in principle be proposed [31], yet considering the physics of turbulence the reversible model should be related to the dissipation anomaly, where the average rate of dissipation is defined as $\langle \epsilon \rangle \equiv \langle \nu |\Delta u|^2 \rangle = 2\nu\Omega$, where $\Omega = \langle \omega^2 \rangle$ is the enstrophy, expressed in terms of the vorticity $\omega = \nabla \times \mathbf{u}$ [12]. In analogy with statistical mechanics [46], we consider the irreversible distribution as the canonical ensemble with ν corresponding to $\beta = (k_B T)^{-1}$, and the analogous of the microcanonical ensemble taking the enstrophy Ω as fixed, and letting ν fluctuate.

Giving evidence of the equivalence of reversible and irreversible Navier-Stokes (NS) equations, this work makes a first link between turbulent fluids and the general framework for nonequilibrium problems in statistical mechanics [47–49], formally based on the chaotic hypothesis [32,50–52]. The main difficulty is that the general theory applies only to time-reversible dynamical systems, whereas NS does not. However, our results show that many nonequilibrium systems, and most notably turbulent fluids, could be considered *in practice* as reversible as far as statistical observables are considered, and therefore the Gallavotti-Cohen theory could be applied to the correct observables. Moreover, a multiscale approach is crucial to tackle complex systems with decimated models [53], like in climate and meteorological sciences. In this case, only large scales can be simulated and small scales are modeled often in an irreversible dissipative way [54,55]. The present study aims to give some insight on a new possible way to propose reversible models, since it is known that such models may better describe the cascade process [56]. Finally, the conjecture is related to the issue of a rigorous proof of existence of unique solutions of the Navier-Stokes equations [20,57,58]. Indeed, the reversible model proposed should admit a smooth solution, since the vorticity remains bounded for any value of the viscosity. While the original mathematical problem would remain open, the conjecture should provide an answer at least from the statistical point of view, since the same statistical results can be obtained with a well-posed set of equations.

We consider here an incompressible fluid, with constant density $\rho = 1$, subjected to viscosity and an external forcing term. The motion is described by the NS equation:

$$\partial_t \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = -\nabla p + \nu \nabla^2 \boldsymbol{u} + \boldsymbol{f}, \qquad \nabla \cdot \boldsymbol{u} = 0, \tag{1}$$

where ν is the cinematic viscosity, p the pressure, and $\boldsymbol{f}$ a forcing term which acts at large scales. Clearly, the dissipative term breaks up the symmetry for temporal inversion; i.e., the equation is not invariant under the transformation: $\mathcal{T}: t \to -t$; $\boldsymbol{u} \to -\boldsymbol{u}$. The corresponding reversible model is obtained replacing the viscosity coefficient ν with a time-dependent term which makes the equation invariant under the symmetry $\mathcal{T}$. Imposing the conservation of enstrophy $\Omega \equiv \int_{\mathcal{V}} |\nabla \times \boldsymbol{u}|^2 d\boldsymbol{x}$, Eq. (1) becomes the reversible Navier-Stokes (RNS) $\partial_t \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = -\nabla p + \alpha[\boldsymbol{u}]\nabla^2 \boldsymbol{u} + \boldsymbol{f}$ with the *fluctuating* viscosity defined as

$$\alpha[\boldsymbol{u}] = \frac{\int_{\mathcal{V}} [\boldsymbol{g} \cdot \omega + \omega \cdot (\omega \cdot \nabla)\boldsymbol{u}] d\boldsymbol{x}}{\int_{\mathcal{V}} (\nabla \times \omega)^2 d\boldsymbol{x}}, \tag{2}$$

where the integrals are defined over the whole volume of the fluid $\mathcal{V}$; the vorticity $\omega = \nabla \times \boldsymbol{u}$ and $\boldsymbol{g} = \nabla \times \boldsymbol{f}$ are used.

While the stationary states of NS define a *nonequilibrium ensemble* $\mathcal{E}_\nu$, RNS equation will generate stationary states that form a collection of new *reversible viscosity ensemble* $\mathcal{E}_\Omega$. Denoting $\langle\rangle_\nu$, $\langle\rangle_\Omega$ the averages over the two corresponding distributions, the content of the Gallavotti's conjecture of equivalence is the following: for small enough ν, it can be expected that the system is highly chaotic and $\alpha(\boldsymbol{x})$ fluctuates wildly leading to a multiscale or homogenization phenomenon [6,59]; that is, a large class of observables have the same statistics in the two ensembles, provided that $\langle \alpha \rangle_\Omega = \nu$ or equivalently $\langle \Omega \rangle_\nu = \Omega$. More details about the theory are given in Supplemental Material [60].

We perform numerical simulations of the 3D NS and the 3D RNS equations by using the code BASILISK [73]. The velocity field $\mathbf{u}$ is solved inside a cubic domain of side 2π, and is prescribed to be triply periodic. The NS runs are initiated from the Taylor-Green velocity field [74], then RNS runs are initiated from the final velocity field of the corresponding NS run. In both cases, we inject energy in the system by using the Taylor-Green forcing [75]. The results are independent from the choice of the initial and forcing conditions, provided forcing is at large scales, and it has been verified that numerical dissipation is negligible. Furthermore, we have verified that the RNS generates the same dynamics even if initialized with the Taylor-Green velocity and not a steady NS field. As usual in isotropic turbulence, we characterize the flow by using the dimensionless Reynolds number based on the Taylor length [12] $R_\lambda = u_{\mathrm{rms}}\lambda/\nu$; in the reversible model $R_\lambda = u_{\mathrm{rms}}\lambda/\langle \alpha \rangle$, where $\langle \alpha \rangle$ is the mean value of the fluctuating viscosity. We have performed three NS simulations at $R_\lambda = 30$, 100, 300, using the same initial conditions for the velocity field but varying the viscosity coefficient. All simulations are carried out so that the smallest scale η is very well resolved ($\Delta x/\eta \lesssim 1$ in all cases), and the corresponding number of points used are $N = 256$, 512, 1024. More numerical details are given in Supplemental Material [60].

In Fig. 1 the phenomenology of both models is illustrated by displaying the dynamics of the dissipation rate and of the enstrophy at different Reynolds numbers. It is seen from Fig. 1(a) that the reversible model at high Reynolds numbers shows wild fluctuations in $\varepsilon = 2\alpha\Omega$ because of the behavior of the fluctuating viscosity α. At more moderate Reynolds numbers the behavior is practically indistinguishable between NS and RNS. It is worth noting some sporadic negative events in dissipation at high Reynolds numbers, meaning that there is sometimes injection of energy by viscosity. The first prediction of

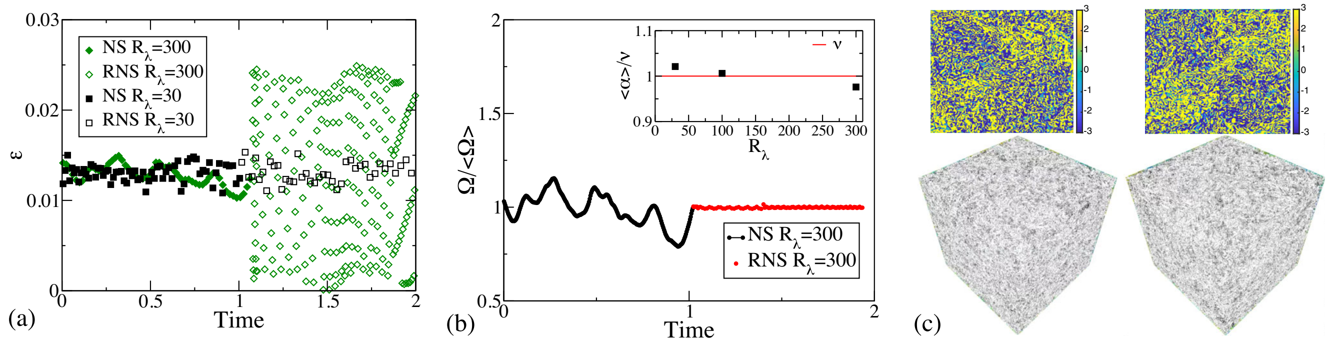

FIG. 1. Time dynamics of some observables in the irreversible NS and then switched in the reversible model. (a) Comparison between the time evolution of dissipation rate ϵ in the irreversible NS and the reversible RNS model for different Reynolds numbers. Time is normalized with the large-scale (integral) characteristic timescale. The case at $R_\lambda = 100$ is very similar to the $R_\lambda = 30$ one and is not shown for the sake of clarity. (b) Time dynamics of enstrophy Ω normalized by its average value at the highest R_λ. In the reversible model the enstrophy is kept constant. In the inset, $\langle\alpha\rangle$ normalized by the constant viscosity value is show at different Reynolds numbers. (c) Visualization of the vorticity field for the NS (left-hand panel) and RNS (right-hand panel). The 3D images are obtained with the λ_2 criterion. The snapshots are the vorticity field at a given time at the center of the cube.

the conjecture is the reciprocity property which states that if enstrophy is taken fixed $\Omega_{\mathrm{RNS}} = \langle\Omega\rangle_{\mathrm{NS}}$, then $\nu = \langle\alpha\rangle$. This is a prerequisite for the conjecture of equivalence. In Fig. 1(b) it is shown that this is true within the numerical errors (about 1%) at all Reynolds numbers. From a more qualitative point of view, Fig. 1(c) shows that also the geometrical features of the turbulent flow are practically indistinguishable in the reversible and irreversible dynamics. The stringent test of the conjecture is about the equivalence of statistical properties of local observables (where locality is intended in momentum space). Since dissipation takes place at small scales, the observables are local if they reside at large scale only. We compare in Fig. 2(a) the second and fourth statistical moment of the velocity field. We have computed them both from the whole field, that is containing all the wave modes, and from the large scales only. While the instantaneous value wildly oscillates, the mean values converge rapidly to the irreversible value. Key for the dynamic of turbulence are the two-point statistical observables [12,15,76]. We show both velocity time correlation and one-dimensional energy spectrum in Fig. 2(b). An excellent agreement between irreversible and reversible models is found at all scales. The analysis of the one-point probability density function (PDF) is consistent with these results (see Supplemental Material [60]).

Even more important is the scale-by-scale flux of energy, which describes the cascade of energy [77]. We compute the scale-by-scale flux from the coarse graining of the Navier-Stokes equation (1) as [16,78]

$$\Pi_\ell(\mathbf{x}) \equiv -\left(\frac{\partial \bar{u}_i}{\partial x_j}\right)\tau_{ij}, \quad \text{with} \quad (\tau_\ell)_{i,j} = \overline{(u_i u_j)}_\ell - (\bar{u}_\ell)_i(\bar{u}_\ell)_j, \tag{3}$$

where the dynamic velocity field $\mathbf{u}$ is spatially (low-pass) filtered over a scale ℓ to obtained a filtered value: $\tilde{\mathbf{u}}_\ell(\mathbf{x}) = \int d^3r G_\ell(\mathbf{r})\mathbf{u}(\mathbf{x}+\mathbf{r})$, where G_ℓ is a smooth filtering function, spatially localized, and such that $G_\ell(\vec{r}) = \ell^{-3}G(\vec{r}/\ell)$ and G satisfies $\int d\vec{r}G(\vec{r}) = 1$, and $\int d\vec{r}|\vec{r}|^2G(\vec{r}) = \mathcal{O}(1)$. The results of the flux for the different numerical experiments are displayed in Fig. 3(a) up to scale $\ell = 2\pi/256$. The global behavior is the same as obtained in analogous pseudo-spectral simulations [79,80], but what is important is that the fluxes of the reversible and irreversible model are the same at all scales, and at all R_λ. A small discrepancy is present at $R_\lambda = 300$ in the inertial range, which is probably due to different statistical convergence. These results show unambiguously that the mechanics of turbulence is the same with both the irreversible and reversible model. To complete the analysis, we have considered the higher-order structure functions $S_p(r) = \langle|\mathbf{u}(\mathbf{x}+\mathbf{r}) - \mathbf{u}(\mathbf{x})|^p\rangle$ and their scaling exponents $S_p(r) \sim r^{\zeta_p}$, which are the relevant observables for intermittency [12,81–83]. Although these kinds of observables are not included in the conjecture, the agreement displayed in Fig. 3(b) is striking. Interestingly, our direct numerical simulations results are in remarkable agreement with those obtained with shell models [84].

Finally, we analyze the statistics of the time-fluctuating viscosity α, shown in Fig. 4. With respect to the equivalence conjecture, the sole crucial feature is that $\langle\alpha\rangle = \nu$, as shown in Fig. 1. The statistics of α are interesting *per se* in connection with the symmetry of fluctuations given by the fluctuation relations for time-reversible dynamical systems [30,49]. Indeed, α is related to the entropy production in the time-reversible model [31]. We plot the PDF of α computed using Eq. (2) during the reversible dynamics as well as that computed in the irreversible one at different Reynolds numbers. In the reversible dynamics, α fluctuates around

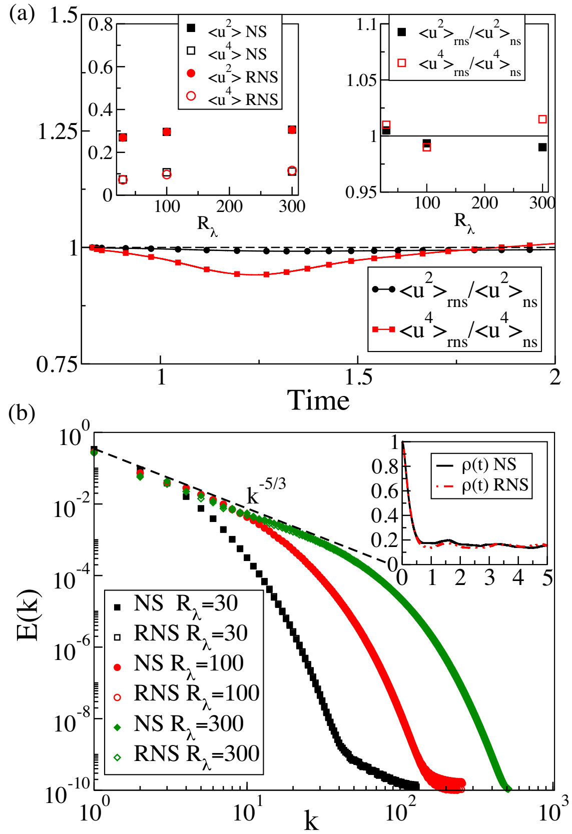

FIG. 2. Test of equivalence. (a) Running average of the ratio between the second statistical moment and fourth moment of the reversible model with respect to the irreversible one at $R_\lambda = 300$. In the left-hand inset, comparison of the same moments for NS (closed symbols) and RNS (open symbols) as a function of the Reynolds number. Right-hand inset: same moments of a velocity field component pertaining to the large scales, only the $k = 3$ mode of the Fourier transform of the field is taken. (b) Time average of the energy spectrum $E(k, t) \equiv 1/2 \sum_{\underline{\mathbf{k}}} |\mathbf{u}(\underline{\mathbf{k}}, t)|^2$, at different Reynolds number for the irreversible and reversible models. Inset: the normalized autocorrelation in time of the velocity for both NS and RNS $\rho(t) = \langle u(t_0)u(t_0 + t)\rangle/\sigma_u$, at $R_\lambda = 300$.

the "canonical" value ν, and the variance increases with the Reynolds number. At low and moderate Reynolds numbers, no negative event is recorded. Instead some are found at $R_\lambda = 300$, when the distribution turns out to be much flatter. As discussed in recent works [39,41], the limit $R_\lambda \to \infty$ and $N \to \infty$ is singular and the different behavior of the PDF reflects that. Furthermore, our results show that in the cascade regime analyzed here, it is difficult to observe extreme events on a *reasonable* observation time, notably at small R_λ. As expected for the 3D case [26], the statistics of α of the reversible and irreversible dynamics are qualitatively different. The entropy production should be the same in both dynamical ensembles, but in fact α is related to entropy only in the reversible model, whereas it bears no connection with it in the irreversible one. Our results confirm this picture with α fluctuating little in the irreversible model and not around ν, as found for the reversible model.

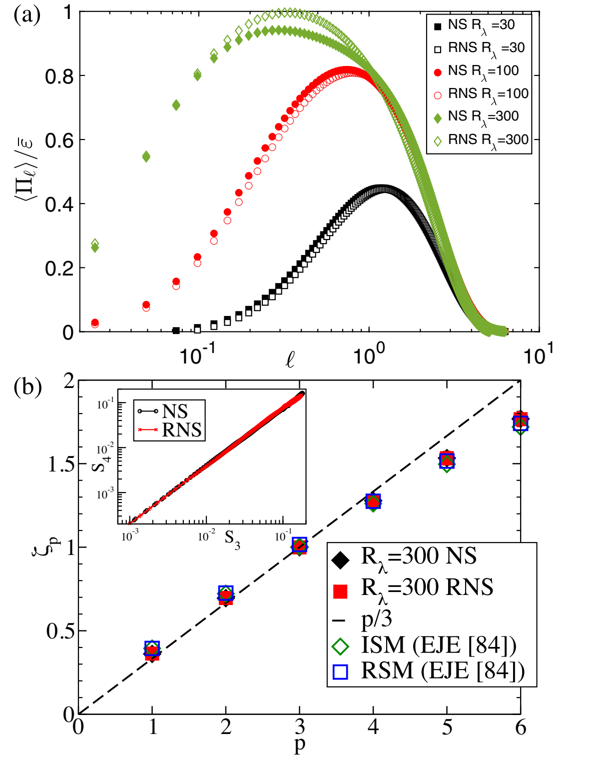

FIG. 3. (a) Scale-by-scale flux of energy normalized by the mean dissipation rate. (b) Scaling exponents extracted through extended self-similarity (ESS) procedure [85] of the structure functions up to order 6 (details in Supplemental Material [60]). Data obtained for shell models from Ref. [84] are also shown. ISM corresponds to NS, and RSM to RSN. Inset: example of comparison for the fourth order.

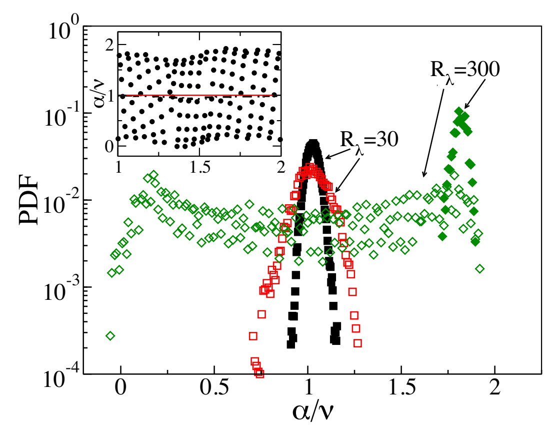

FIG. 4. Probability density function (PDF) of α. As in the previous figures, filled symbols are for NS and open symbols for RNS. The insets on the left show the corresponding typical time evolutions of α at $R_\lambda = 300$.

In conclusion, we have shown through high-resolved numerical simulations that the Gallavotti's conjecture of dynamical ensemble equivalence is correct. We observe that no matter the Reynolds number, provided sufficient resolution is kept, not only the basic requirements of the conjecture are fulfilled, but all the relevant statistical observables are found indistinguishable in the irreversible and reversible dynamical system. Furthermore, the scale-by-scale analysis of the kinetic energy flux shows negligible difference between the two models up to the dissipation range, far beyond the original formal conjecture proposition. Wild fluctuations of the reversible viscosity are encountered and at high Reynolds numbers, even negative values are recorded, which point to local antidissipative phenomena. However, these negative events remain extremely rare. Our results confirm preliminary results obtained in simplified dynamical models of turbulence [39].

Our results give empirical evidence that the *chaotic hypothesis* from which the conjecture is originally derived can be considered *morally* applicable to turbulent fluids. That means in turn that nonequilibrium statistical mechanics [52,86,87], and notably fluctuation relations, should apply in some sense also to turbulent fluids. Furthermore, it is shown that turbulence is unaffected by the precise mechanism of dissipation. This corroborates the idea that scales larger than the forcing are governed by Euler, as recently proposed [88–90]. On the other hand, it paves the way to the use of whatever phenomenological model in coarse-grained approaches, provided the correct amount of average rate of dissipation is enforced.

Some issues remain to be answered. While the reversible system appears mathematically simpler because of the constraint on the enstrophy, the presence of negative events in viscosity makes it not well posed, shifting but not solving the question of global existence of the solution. Rigorous analysis lacks. The possibility to compute nonequilibrium entropy and its behavior is appealing but the needed statistics to make predictions seems overwhelming in 3D. More notably, to exploit the new framework to get new insights on the turbulence problem remains an unexplored route.

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