Complete Incompatibility, Support Uncertainty, and Kirkwood-Dirac Nonclassicality

Stephan De Bièvre*
Univ. Lille, CNRS, Inria, UMR 8524 - Laboratoire Paul Painlevé, F-59000 Lille, France

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For quantum systems with a finite dimensional Hilbert space of states, we show that the complete incompatibility of two observables—a notion we introduce—is equivalent to the large support uncertainty of all states. The Kirkwood-Dirac (KD) quasiprobability distribution of a state—which depends on the choice of two observables—has emerged in quantum information theory as a tool for assessing nonclassical features of the state that can serve as a resource in quantum protocols. We apply our result to show that, when the two observables are completely incompatible, only states with minimal support uncertainty can be KD classical, all others being KD nonclassical. We illustrate our findings with examples.

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Introduction.—The nonclassical features of quantum mechanical states can be of a very diverse nature. Incompatible, noncommuting, and complementary or conjugate observables, (de)coherence, interference, uncertainty principles, negativity or nonreality of quasiprobability distributions, entanglement, contextuality, and nonlocality are concepts used to evaluate the degree to which quantum states of a variety of physical systems may exhibit manifestly nonclassical behavior in various experimental situations. Partially in order to obtain a better understanding of quantum mechanics and partially because such nonclassical behavior has proven essential for tasks in quantum information theory and metrology, the study of their properties and of the relationship between them attracts continued attention [1–18].

That strong links exist between the trio composed of incompatibility, uncertainty, and (non)classicality is familiar from standard quantum mechanics and from quantum optics. As the prototypical pair of very strongly incompatible observables one may consider Q and P, two conjugate observables. The *total noise* $(\Delta Q)^2 + (\Delta P)^2$ [19] can then be seen as a measure of the uncertainty of a state. It satisfies $(\Delta Q)^2 + (\Delta P)^2 \geq 1$ for all states, and this lower bound is reached only for the coherent states which are, in this context, considered "classical." These results are well known to be intimately linked to the canonical commutation relation $[Q, P] = i$. It expresses the very strong sense in which Q and P fail to commute, and—in this sense—their very strong incompatibility. In the first part of this Letter we will show that, provided suitably adapted notions of incompatibility, of uncertainty, and of nonclassicality are adopted, precisely the same situation occurs in quantum systems with a $d < +\infty$ dimensional Hilbert space $\mathcal{H}$, as for systems of qudits or qubits.

In that context, the starting point is not the choice of two observables such as Q and P, but of two orthonormal bases $\mathcal{A} = \{|a_i\rangle\}$ and $\mathcal{B} = \{|b_j\rangle\}$, that can be thought of as eigenbases of two observables A and B. If, for example, A and B have nondegenerate spectra, and if they are compatible in the usual sense that $[A, B] = 0$, then all their eigenprojectors commute so that each $|a_i\rangle$ is up to a phase equal to some $|b_j\rangle$. This notion, and others close to it [3,10,13,17,18], is a very strong form of compatibility, the negation of which therefore yields a very weak notion of incompatibility. For our purposes, we need a stronger such notion, the *complete incompatibility* of the bases, a concept we introduce in Definition 1. We then show that it is equivalent to the statement that the smallest possible support uncertainty [defined in Eq. (1)] of any pure state is $d + 1$ (Theorem 2). This can be paraphrased as saying that complete incompatibility of observables is equivalent to large (support) uncertainty of states. Indeed, when on the contrary the bases $\mathcal{A}$ and $\mathcal{B}$ are compatible in the above sense, the smallest possible support uncertainty of any pure state is 2: it is attained for any of the basis vectors.

Having thus closely linked the complete incompatibility of observables to the support uncertainty of states, it remains to understand how both relate to the nonclassicality of states, a task we turn to in the second part of this Letter. The notion of nonclassicality arising naturally in systems with a finite dimensional Hilbert space is Kirkwood-Dirac (KD) nonclassicality. Recall that a state is said to be KD nonclassical if its KD distribution [see Eq. (4)], a finite dimensional analog of the well-known Wigner distribution, has negative or complex values. KD nonclassicality has come to the forefront in recent years because of its use in quantum tomography [5,6,9] as well as in the theory and applications of weak measurements, contextuality, and their relation to nonclassical effects in quantum mechanics [2,7,8,12]. In addition, KD nonclassicality has been linked to out-of-time-ordered correlators [11] and proposed as a measure for scrambling [14]. It was furthermore shown to provide an operational quantum advantage in postselected metrology [15]. Since the KD distribution and hence the

KD nonclassicality of ψ depend not only on ψ, but also on $\mathcal{A}$ and $\mathcal{B}$, the question that arises naturally is what properties of $\mathcal{A}$ and $\mathcal{B}$ will ensure the prevalence of KD-nonclassical states in Hilbert space? We will show that the bases $\mathcal{A}$ and $\mathcal{B}$ need to be completely incompatible for this to occur. In that case, only states with minimal support uncertainty can be KD classical, all others being KD nonclassical (Corollary 3). The strong analogy with the situation of two conjugate variables and the corresponding coherent states that we recalled above is manifest. Complete incompatibility implies furthermore that the support uncertainty of a state provides a very efficient and convenient KD-nonclassicality witness (Theorem 2). We illustrate our findings on a number of examples, in particular mutually unbiased and spin bases.

$\mathcal{A}$ and $\mathcal{B}$ representations. Support uncertainty.—Given a basis $\mathcal{A}$, we associate to each state ψ its $\mathcal{A}$ representation, which is the vector of its components on the $\mathcal{A}$ basis: $(\langle a_1|\psi\rangle, \ldots, \langle a_d|\psi\rangle) \in \mathbb{C}^d$. The $\mathcal{A}$ support S_ψ of ψ is then defined as the set of outcomes $i \in [\![1,d]\!]$ that occur with nonzero probability when a measurement in the $\mathcal{A}$ basis is made: $S_\psi = \{i \in [\![1,d]\!] | \langle a_i|\psi\rangle \neq 0\}$. We will write $n_\mathcal{A}(\psi) = |S_\psi|$, where $|S|$ denotes the number of elements in the set $S \subset [\![1,d]\!]$. Hence $n_\mathcal{A}(\psi)$ [respectively, $n_\mathcal{B}(\psi)$] counts the number of nonvanishing overlaps $\langle a_i|\psi\rangle$ (respectively, $\langle b_j|\psi\rangle$). One should think of $n_\mathcal{A}(\psi)$ and $n_\mathcal{B}(\psi)$ as the size of the support or the "spread" of the probability distributions $|\langle a_i|\psi\rangle|^2$ and $\langle b_j|\psi\rangle|^2$ of the state ψ in the $\mathcal{A}$ and $\mathcal{B}$ representations, which is one possible measure of their uncertainty. Many other such measures exist, notably the entropic ones [20,21].

In our analysis of the incompatibility of two bases $\mathcal{A}$ and $\mathcal{B}$, the *support uncertainty* $n_{\mathcal{A},\mathcal{B}}(\psi)$ of ψ arises naturally. It is defined as

$$n_{\mathcal{A},\mathcal{B}}(\psi) \coloneqq n_\mathcal{A}(\psi) + n_\mathcal{B}(\psi) \tag{1}$$

and has proven useful in other contexts previously [22–26]. We also introduce the *minimal support uncertainty* $n^{\min}_{\mathcal{A},\mathcal{B}}$ of the bases $\mathcal{A}$, $\mathcal{B}$ as $n^{\min}_{\mathcal{A},\mathcal{B}} = \min_{\psi\neq 0} n_{\mathcal{A},\mathcal{B}}(\psi)$. Clearly, $2 \leq n^{\min}_{\mathcal{A},\mathcal{B}} \leq d+1$, as can be seen by considering for ψ the basis vectors $|a_i\rangle$ or $|b_j\rangle$. When $\mathcal{A}$ and $\mathcal{B}$ are compatible in the usual sense recalled above, $n^{\min}_{\mathcal{A},\mathcal{B}} = 2$.

As a further instructive example, consider a two qubit state space $\mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^2$ with $\mathcal{A}$ the computational basis and $\mathcal{B} = \{|+,+\rangle, |+,-\rangle, |-,+\rangle, |-,-\rangle\}$ with $|\pm\rangle = (1/\sqrt{2})(|0\rangle \pm |1\rangle)$. Thinking of the qubits as spin-1/2 systems, the basis $\mathcal{A}$, respectively $\mathcal{B}$, is an eigenbasis of J_z, respectively of J_x. Considering the singlet state

$$|J^2 = 0\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) = \frac{1}{\sqrt{2}}(|+-\rangle - |-+\rangle) \in \mathcal{H},$$

one has $n_\mathcal{A}(|J^2=0\rangle) = 2 = n_\mathcal{B}(|J^2=0\rangle)$. It is also readily seen that $2 < n^{\min}_{\mathcal{A},\mathcal{B}} = 4 < d+1 = 5$.

We now show that the maximal possible value of $n^{\min}_{\mathcal{A},\mathcal{B}}$, $n^{\min}_{\mathcal{A},\mathcal{B}} = d+1$, corresponds to a strong form of incompatibility of the bases that we call *complete incompatibility.*

Complete incompatibility.—We define, for any index set $S \subset [\![1,d]\!]$, the orthogonal projector $\Pi_\mathcal{A}(S) = \sum_{i\in S} |a_i\rangle\langle a_i|$. We write $\Pi_\mathcal{A}(S)\mathcal{H}$ for the $|S|$-dimensional subspace of $\mathcal{H}$ onto which it projects: it contains all states ψ whose $\mathcal{A}$ support S_ψ lies in S.

Definition 1.—We say that two bases $\mathcal{A}$ and $\mathcal{B}$ are completely incompatible (COINC) if and only if all index sets S, T in $[\![1,d]\!]$ for which $|S| + |T| \leq d$ have the property that $\Pi_\mathcal{A}(S)\mathcal{H} \cap \Pi_\mathcal{B}(T)\mathcal{H} = \{0\}$.

While the definition is purely algebraic, its physical interpretation is readily given in terms of the quantum theory of selective projective measurements [1,27,28]. Note that the projectors $\Pi_\mathcal{A}(S)$ and $\Pi_\mathcal{B}(T)$ are observables with eigenvalues 1 and 0. Their measurement is said to be fine-grained if $|S| = 1$, $|T| = 1$, and coarse-grained otherwise. Repeated selective measurements of $\Pi_\mathcal{A}(S)$ and of $\Pi_\mathcal{B}(T)$ on a system initially prepared in ψ systematically yield the outcome 1 if and only if $\Pi_\mathcal{B}(T)\Pi_\mathcal{A}(S)\psi$ belongs to $\Pi_\mathcal{A}(S)\mathcal{H}\backslash\{0\}$. Hence, if this occurs, $\Pi_\mathcal{A}(S)\mathcal{H} \cap \Pi_\mathcal{B}(T)\mathcal{H} \neq \{0\}$. In other words, the definition of COINC bases is equivalent to the statement that such repeated *compatible selective measurements* cannot occur for any insufficiently coarse-grained measurements, i.e., for any S, T for which $|S| + |T| \leq d$. This is a very stringent requirement, whence the term "complete" incompatibility.

The link between complete incompatibility and support uncertainty is given by

Theorem 2.—$\mathcal{A}$ and $\mathcal{B}$ are COINC iff $n^{\min}_{\mathcal{A},\mathcal{B}} = d+1$.

In other words, two bases are COINC iff the minimal support uncertainty $n^{\min}_{\mathcal{A},\mathcal{B}}$ of all states takes on its maximal possible value, namely, $d+1$. In particular, when the bases are COINC, there are no states for which the supports S_ψ, T_ψ are small, in the sense of $n_{\mathcal{A},\mathcal{B}}(\psi) \leq d$. This property of COINC bases is reminiscent of an analogous property of conjugate operators Q and P. Indeed, it is well known that there do not exist states ψ for which both the Q representation $\langle x|\psi\rangle$ vanishes outside some bounded set $S \subset \mathbb{R}$ and the P representation $\langle p|\psi\rangle$ vanishes outside some bounded set $T \subset \mathbb{R}$ [29].

The definition of complete incompatibility transcribes this crucial property of conjugate operators to the finite-dimensional setting; in that case the restriction $|S| + |T| \leq d$ is unavoidable since, for dimensional reasons, whenever $|S| + |T| > d$, the intersection $\Pi_\mathcal{A}(S)\mathcal{H} \cap \Pi_\mathcal{B}(T)\mathcal{H}$ must be nontrivial.

The theorem implies that the two bases $\mathcal{A}$, $\mathcal{B}$ for the two-qubit system introduced above are not COINC since $4 = n^{\min}_{\mathcal{A},\mathcal{B}} < d+1 = 5$ and despite the fact that they are eigenbases of the noncommuting J_z and J_x. This illustrates that complete incompatibility is a stronger property than noncommutativity. Also, when the system is in the singlet state $|J^2 = 0\rangle$, for which $n_{\mathcal{A},\mathcal{B}}(|J^2 = 0\rangle) = 4$, successive measurements of J_x and J_z consistently give the result 0, so that the measurement of one does not, for this state, perturb the measurement of the other.

Proof.—For all states ψ, $\psi \in \Pi_{\mathcal{A}}(S_\psi)\mathcal{H} \cap \Pi_{\mathcal{B}}(T_\psi)\mathcal{H} \neq \{0\}$. If the bases are COINC, this implies $|S_\psi| + |T_\psi| > d$. So $n^{\min}_{\mathcal{A},\mathcal{B}} > d$. Since for the basis vectors we know $n_{\mathcal{A},\mathcal{B}}(|a_i\rangle) = d + 1$, we conclude $n^{\min}_{\mathcal{A},\mathcal{B}} = d + 1$. We prove the converse by proving its contraposition. Suppose $\mathcal{A}$ and $\mathcal{B}$ are not COINC. Then there exist S, T, with $|S| + |T| \leq d$ and $\Pi_{\mathcal{A}}(S)\mathcal{H} \cap \Pi_{\mathcal{B}}(T)\mathcal{H} \neq \{0\}$. Let $0 \neq \psi \in \Pi_{\mathcal{A}}(S)\mathcal{H} \cap \Pi_{\mathcal{B}}(T)\mathcal{H}$. For this state $n_{\mathcal{A}}(\psi) \leq |S|$, $n_{\mathcal{B}}(\psi) \leq |T|$. Hence $n_{\mathcal{A},\mathcal{B}}(\psi) \leq d$ and $n^{\min}_{\mathcal{A},\mathcal{B}} \leq d$.

Another property of COINC bases in which their incompatibility manifests itself, is that $\langle a_i|b_j\rangle \neq 0$ for all i, j. Indeed, if, for example, $\langle a_1|b_1\rangle = 0$, then $|a_1\rangle$ belongs to $\Pi_{\mathcal{A}}(S)\mathcal{H} \cap \Pi_{\mathcal{B}}(T)\mathcal{H}$ for $S = \{1\}$ and $T = \{2, \ldots, d\}$. Since $|T| + |S| = d$, this contradicts the definition. Hence each basis vector $|a_i\rangle$ has full $\mathcal{B}$ support and a measurement in the basis $\mathcal{B}$ on a system prepared in the state $|a_i\rangle$ can yield any postmeasurement state $|b_j\rangle$ with nonvanishing probability, a clear hallmark of incompatibility. In addition, that $\langle a_i|b_j\rangle \neq 0$ for all $i, j \in [\![1, d]\!]$ is equivalent to the property that *none* of the projectors $|a_i\rangle\langle a_i|$ commutes with *any* of the $|b_j\rangle\langle b_j|$. This is clearly stronger than the usual notion of incompatibility, which only requires that at least one such pair does not commute.

Complete incompatibility: A criterion and examples.—Before turning to the link between complete incompatibility and KD nonclassicality, we provide further examples of bases that are or are not COINC. Let U be the unitary transition operator between $\mathcal{A}$ and $\mathcal{B}$, defined as $U|a_j\rangle = |b_j\rangle$, with matrix elements $U_{ij} = \langle a_i|b_j\rangle$ in $\mathcal{A}$. A useful criterion of complete incompatibility is

Lemma 3.—$\mathcal{A}$ and $\mathcal{B}$ are COINC if and only if none of the minors of the matrix U vanishes.

Recall that a k minor of U is the determinant of a k by k submatrix of U obtained by removing $d - k$ rows and $d - k$ columns from U. The statement and proof are implicit in Ref. [24]; we give a straightforward argument in the Supplemental Material [30]. As an immediate application, one sees that, in dimensions $d = 2$ and $d = 3$, two bases $\mathcal{A}$ and $\mathcal{B}$ are COINC iff for all $1 \leq i,\ j \leq d$, $\langle a_i|b_j\rangle \neq 0$, a condition that is readily checked. In dimension 2 this is obvious. In dimension 3, note that each column of U is a multiple of the complex conjugate of the vector product of the two other columns. Since its components are 2 minors, their nonvanishing follows from the nonvanishing of all matrix elements of U. Hence the bases are COINC. When $d \geq 4$, the above is no longer true, as we will see.

To synthetically represent the support properties of all states with respect to two given bases one may use an *uncertainty diagram*. This is the collection of all points $(n_{\mathcal{A}}, n_{\mathcal{B}})$ in the $n_{\mathcal{A}}$-$n_{\mathcal{B}}$ plane for which there is a ψ so that $n_{\mathcal{A}}(\psi) = n_{\mathcal{A}}$ and $n_{\mathcal{B}}(\psi) = n_{\mathcal{B}}$. Theorem 2 asserts that the uncertainty diagram of $\mathcal{A}$, $\mathcal{B}$ lies above the line segment $n_{\mathcal{A}} + n_{\mathcal{B}} = d + 1$ iff the bases are COINC. This is illustrated in Fig. 1 where, as further discussed below, panels (a) and (d) concern COINC bases, and panels (b), (c), and (e) bases that are not COINC. The uncertainty diagram is delimited from below through an uncertainty principle originally shown for the Fourier transform on finite groups [22], but which has much larger validity [25,26]. It reads

$$n_{\mathcal{A}}(\psi)n_{\mathcal{B}}(\psi) \geq M^{-2}_{\mathcal{A},\mathcal{B}}, \quad \text{where } M_{\mathcal{A},\mathcal{B}} = \max_{i,j}|\langle a_i|b_j\rangle|. \qquad (2)$$

A simple proof is provided below. It follows that the uncertainty diagram of any two bases lies above or on the hyperbola $n_{\mathcal{A}}n_{\mathcal{B}} = M^{-2}_{\mathcal{A},\mathcal{B}}$.

It is proven in Ref. [24] that none of the minors of the discrete Fourier transform (DFT) transition matrix $U_{\mathrm{DFT},i,j} \coloneqq \langle a_i|b_j\rangle = (1/\sqrt{d})\exp[i(2\pi/d)ij]$ vanish if and only if d is a prime number. Lemma 3 then implies the DFT is COINC iff d is prime. The uncertainty diagrams for the DFT in dimension 5 and 6 are displayed in Figs. 1(a) and 1(b). One clearly observes it lies above the segment $n_{\mathcal{A}} + n_{\mathcal{B}} = d + 1$ when $d = 5$, but not when $d = 6$, as expected from the previous arguments. The DFT is an example of a transition matrix for a larger family of bases, called mutually unbiased bases (MUBs). They have found

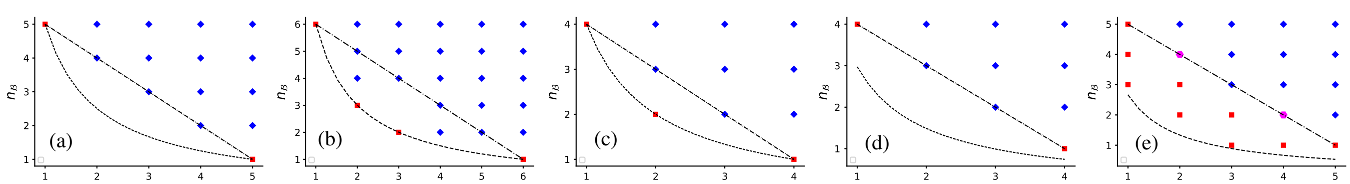

FIG. 1. Uncertainty diagrams. Dashed curve: $n_{\mathcal{A}}(\psi)n_{\mathcal{B}}(\psi) = M^{-2}_{\mathcal{A},\mathcal{B}}$. Dot-dashed line: $n_{\mathcal{A}}(\psi) + n_{\mathcal{B}}(\psi) = d + 1$. Diamonds (blue): KD-nonclassical states. Squares (red): KD-classical states. (a) DFT: $d = 5$, $M^{-2}_{\mathcal{A},\mathcal{B}} = 5$, $n^{\min}_{\mathcal{A},\mathcal{B}} = 6$. (b) DFT: $d = 6$, $M^{-2}_{\mathcal{A},\mathcal{B}} = 6$, $n^{\min}_{\mathcal{A},\mathcal{B}} = 5 < 7$. (c) Complex MUBs: $d = 4$, $M^{-2}_{\mathcal{A},\mathcal{B}} = 4$, $n^{\min}_{\mathcal{A},\mathcal{B}} = 4 < 5$. (d) Perturbed MUB as in Eq. (3); $d = 4$, $t = 0.1$, $M^{-2}_{\mathcal{A},\mathcal{B}} = 2.97 < 4$, $n^{\min}_{\mathcal{A},\mathcal{B}} = 5$. (e) Spin 2 transition matrix; $d = 5$, $M^{-2}_{\mathcal{A},\mathcal{B}} = 8/3 < 5$, $n^{\min}_{\mathcal{A},\mathcal{B}} = 4 < 6$. Hexagons (magenta): KD-classical and KD-nonclassical states.

numerous applications in various quantum information protocols [5,32–36]. We refer to Refs. [37,38] for reviews on MUBs. MUBs are characterized by the property that $|\langle a_i|b_j\rangle|^2 = d^{-1}$ for all i, j so that all measurement outcomes when measuring $\mathcal{B}$ on a system prepared in a basis vector of $\mathcal{A}$ (or vice versa) are equally probable. In view of this such bases are sometimes considered maximally incompatible. The above observation however implies MUBs are not necessarily COINC. In fact, when $d = 4$, no MUBs are COINC, as easily seen from their explicit expression [30], and Lemma 3. This is also apparent from Fig. 1(c), which shows the presence of states at $n_{\mathcal{A}} = 2 = n_{\mathcal{B}}$, so that $n_{\mathcal{A},\mathcal{B}} = 4 < 5 = d + 1$.

In the study of out-of-time-ordered correlators, one encounters pairs of bases $\mathcal{A}$ and $\mathcal{B}(t)$, where $|b_i(t)\rangle = \exp(-iHt)|b_i\rangle$, so that

$$U(t) = \exp(-itH)U, \tag{3}$$

where H is self-adjoint. Figure 1(d) shows the uncertainty diagram for a perturbed MUB matrix U of the above form, with $H_{jk} = -H_{kj} = i$, $1 \le j < k \le d$. One observes by inspecting the figure and using Theorem 2 that $U(t)$ is COINC when $t = 0.1$ while it is not when $t = 0$, as seen above. The emergence of COINC bases from the perturbation of non-COINC bases is a more general phenomenon the theoretical origin of which we shall explain elsewhere [39].

Figure 1(e) shows the uncertainty diagram for the eigenbases of J_z and J_x for a spin 2. It clearly does not lie above $n_{\mathcal{A}} + n_{\mathcal{B}} = d + 1 = 6$ in that case, indicating the bases are not COINC, despite the fact that J_x and J_z do not commute and are therefore incompatible in the usual sense of the word. The same situation occurs for all integer s, as is readily shown from an examination of the Wigner rotation matrices [40] that contain zeroes [30] so that they are not COINC. In this example, there exist both KD-nonclassical and KD-classical states with $n_{\mathcal{A}} = 4$, $n_{\mathcal{B}} = 2$ and with $n_{\mathcal{A}} = 2$, $n_{\mathcal{B}} = 4$.

Further properties of completely incompatible bases and their uncertainty diagrams, as well as the link between complete incompatibility and noncommutativity will be explored elsewhere [39].

Characterizing KD nonclassicality.—Having related the complete incompatibility of two bases $\mathcal{A}$ and $\mathcal{B}$ to the support uncertainty of the states $\psi \in \mathcal{H}$, we now turn to its link with the KD nonclassicality of those states. The Kirkwood-Dirac (KD) distribution of a state ψ [41,42] is the quasiprobability distribution

$$Q(\psi)_{ij} = \langle a_i|\psi\rangle\langle\psi|b_j\rangle\langle b_j|a_i\rangle, \qquad 1 \le i, \qquad j \le d, \tag{4}$$

similar in spirit to the Wigner distribution [43,44] in continuous variable quantum mechanics. It is complex valued and satisfies $\sum_{ij} Q(\psi)_{ij} = 1$, with marginals $\sum_j Q(\psi)_{ij} = |\langle a_i|\psi\rangle|^2$, $\sum_i Q(\psi)_{ij} = |\langle b_j|\psi\rangle|^2$. A state $\psi \in \mathcal{H}$ is *KD classical* if its KD distribution is real nonnegative everywhere so that its KD distribution is a probability distribution. If not, it is *KD nonclassical*. A measure for KD nonclassicality is provided by [14] $\mathcal{N}_{\rm NC} = \sum_{ij} |Q_{ij}|$; a state ψ is KD nonclassical if and only if $\mathcal{N}_{\rm NC}(\psi) > 1$. It follows that

$$\begin{aligned} 1 = \left|\sum_{ij} Q_{ij}\right| &\le \mathcal{N}_{\rm NC} \le M_{\mathcal{A},\mathcal{B}} \sum_{ij} |\langle a_i|\psi\rangle||\langle b_j|\psi\rangle| \\ &\le M_{\mathcal{A},\mathcal{B}} \left(\sum_{i\in S_\psi} |\langle a_i|\psi\rangle|\right)\left(\sum_{j\in T_\psi} |\langle b_j|\psi\rangle|\right) \\ &\le M_{\mathcal{A},\mathcal{B}} \sqrt{n_{\mathcal{A}}(\psi) n_{\mathcal{B}}(\psi)}. \end{aligned} \tag{5}$$

This proves Eq. (2) and in addition it shows that any state for which $n_{\mathcal{A}}(\psi) n_{\mathcal{B}}(\psi) = M_{\mathcal{A},\mathcal{B}}^{-2}$ is KD classical. This is illustrated in Fig. 1.

One finally observes in all panels of Fig. 1 that there are no KD-classical states above the *nonclassicality edge*, by which we mean the line segment in the first quadrant defined by $n_{\mathcal{A}} + n_{\mathcal{B}} = d + 1$. This is explained by the following theorem.

Theorem 4.—Let $\mathcal{A}$, $\mathcal{B}$ be orthonormal bases in a d-dimensional Hilbert space $\mathcal{H}$ and suppose that $\langle a_i|b_j\rangle \neq 0$ for all $i, j \in [\![1, d]\!]$. Then, if $\psi \in \mathcal{H}$ satisfies

$$n_{\mathcal{A},\mathcal{B}}(\psi) > d + 1, \tag{6}$$

then ψ is KD nonclassical. Equivalently, if ψ is KD classical then $n_{\mathcal{A},\mathcal{B}}(\psi) \le d + 1$.

The proof of Theorem 4, which sharpens an argument in Ref. [16], is given in the Supplemental Material [30]. Theorems 4 and 2 together imply

Corollary 5.—When two bases are COINC, all KD-classical states ψ have minimal support uncertainty: $n_{\mathcal{A},\mathcal{B}}(\psi) = n_{\mathcal{A},\mathcal{B}}^{\rm min} = d + 1$.

This can be observed in Figs. 1(a) and 1(d). It is nevertheless not true that all states with minimal support uncertainty are KD classical, contrary to what happens with conjugate continuous variables Q and P. Indeed, one observes in Figs. 1(a) and 1(d) both KD-classical and KD-nonclassical states on the nonclassicality edge. In other words, when the bases are COINC, all states except possibly some of those with minimal support uncertainty are KD nonclassical so that complete incompatibility guarantees the strong prevalence of nonclassicality.

Theorem 4 further explains why in Figs. 1(b) and 1(c), which correspond to MUBs that are not COINC, there are also no classical states above the nonclassicality edge. Indeed, for MUBs the condition of the theorem is clearly satisfied. In that case however, there may be both KD-nonclassical and KD-classical states with support uncertainty below $d + 1$ as observed in Fig. 1(c). One observes the same phenomenon in Fig. 1(e) in spite of the fact the theorem does not apply there. An extension of the result that covers this and similar cases where zeroes appear in the transition matrix U will be proven elsewhere [39]. The importance of being able to deal with such zeroes was pointed out in Ref. [11].

In Ref. [16], it was shown that, if none of the $|a_i\rangle$ are equal (up to a phase) to any of the $|b_j\rangle$, then the condition

$$n_{\mathcal{A}}(\psi) + n_{\mathcal{B}}(\psi) > \lfloor 3d/2 \rfloor \quad (7)$$

implies ψ is KD nonclassical; here $\lfloor x \rfloor$ is the integer part of x. While this estimate holds under weaker conditions on the overlaps $\langle a_i|b_j\rangle$ than their nonvanishing, it is not optimal when this condition is indeed satisfied and as soon as $d \geq 4$, since then $\lfloor 3d/2 \rfloor > d + 1$; the difference between the lower bounds is increasingly pronounced for larger d. For example, for $d = 6$ [see Fig. 1(b)], our optimal bound shows that all states with $n_{\mathcal{A},\mathcal{B}} \geq 8$ are KD nonclassical, while Eq. (7) only implies this if $n_{\mathcal{A},\mathcal{B}} \geq 10$.

Conclusion.—It has been observed recently that KD-nonclassical states can furnish a quantum advantage. This raises the question under what conditions on the observables used to define the KD distribution such KD nonclassicality prevails in Hilbert space? It was argued in Ref. [16] that incompatiblity does not suffice for this. We have established that complete incompatibility—a notion we introduce—furnishes the right condition: it implies only states with minimal support uncertainty can be KD classical, all others being KD nonclassical. More generally, our findings provide an improved understanding of the fundamental notions of incompatibility of observables and the related uncertainty in states as they arise in quantum mechanics on finite dimensional Hilbert spaces. Further work is needed to establish, beyond the prevalence of nonclassical states, the strength of the nonclassicality they provide and the quantitative impact this nonclassicality has on the physical phenomena in which it plays a role.

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*stephan.de-bievre@univ-lille.fr

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