Symmetry-Protected Multifold Exceptional Points and Their Topological Characterization

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We investigate the occurrence of n-fold exceptional points (EPs) in non-Hermitian systems, and show that they are stable in $n-1$ dimensions in the presence of antiunitary symmetries that are local in parameter space, such as, e.g., parity-time (PT) or charge-conjugation parity (CP) symmetries. This implies in particular that threefold and fourfold symmetry-protected EPs are stable, respectively, in two and three dimensions. The stability of such *multofold* exceptional points (i.e., beyond the usual twofold EPs) is expressed in terms of the homotopy properties of a *resultant vector* that we introduce. Our framework also allows us to rephrase the previously proposed $\mathbb{Z}_2$ index of PT and CP symmetric gapped phases beyond the realm of two-band models. We apply this general formalism to a frictional shallow water model that is found to exhibit threefold exceptional points associated with topological numbers ± 1. For this model, we also show different non-Hermitian topological transitions associated with these exceptional points, such as their merging and a transition to a regime where propagation is forbidden, but can counterintuitively be recovered when friction is increased furthermore.

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Since the discovery of graphene, Dirac points have been revealed to be a ubiquitous property of various two-dimensional (2D) materials. The existence of such twofold degeneracies is enforced by symmetries [1,2], and their stability can be expressed with winding numbers of the phase of the wave functions [3–5]. Similarly, non-Hermitian Hamiltonians may exhibit degeneracy points of their complex eigenvalues, called exceptional points (EPs). The quest for EPs, their topological properties and their physical implications, stimulated tremendous efforts in the past few years [6–15]. At an EP, the number of degenerated eigenvalues (the algebraic multiplicity μ) is generically larger than the number of eigenvectors (the geometric multiplicity) leaving the Hamiltonian non-diagonalizable. Quite remarkably, twofold EPs (EP2s) do not require any symmetry to be stable in two dimensions, and this stability can be expressed in terms of a winding number [8] associated with their complex eigenvalues [16]. This is however not the case for multifold exceptional points EPμs. Those were indeed shown to be unstable in two dimensions [16], although stable in higher $D = 2\mu - 2$ dimension [6,17], so that EPμs beyond $\mu = 2$ have remained overlooked. Still, a few examples of multifold EPs were surprisingly reported recently [18–20], including in PT-symmetric systems [21]. This asks the crucial question of the conditions of existence of multifold EPs, in particular in dimensions $D \leq 3$, their robustness and the role of symmetries.

Here we answer this question by showing that μ-fold EPs have a *codimension* $\mu - 1$ provided certain antiunitary symmetries are satisfied, meaning that they appear as isolated points in $\mu - 1$ dimensions. The relevant symmetries consist in parity-time (PT) symmetry, charge-conjugation parity (CP) symmetry, and their generalizations, called pseudo-Hermiticity (psH) and pseudo-chirality (psCh), that classify non-Hermitian EPs [13,22]. It follows that threefold and fourfold EPs are stable in two and three dimensions, respectively, when such symmetries apply. Then, we propose a topological characterization for symmetry-protected EPμs through the homotopy properties of a *resultant vector* that we introduce. Finally, we illustrate our theory by discussing a frictional fluid model that exhibits several EP3s with opposite windings. These EPs constitute a threshold beyond which the propagation of all the eigenmodes vanishes in a certain range of wavelength, but can counterintuitively be recovered when increasing friction furthermore.

To investigate μ-fold EPs, we focus on $\mu \times \mu$ matrices without loss of generality, because μ-fold degeneracies are described by a $\mu \times \mu$ effective Hamiltonian H. We consider a parametrization $H(\boldsymbol{\lambda})$ with $\boldsymbol{\lambda} \in \mathbb{R}^D$ of such Hamiltonians. The degeneracies of their eigenvalues $E(\boldsymbol{\lambda})$ are generically EPs, since arbitrary non-Hermitian matrices require fine-tuning to be diagonalizable at the degeneracy point [23]. Therefore, looking for μ-fold EPs essentially amounts to searching for the conditions such that the characteristic polynomial $P_{\boldsymbol{\lambda}}(E) \equiv \det[H(\boldsymbol{\lambda}) - E] = a_n(\boldsymbol{\lambda})E^n + a_{n-1}(\boldsymbol{\lambda})E^{n-1} + \cdots + a_1(\boldsymbol{\lambda})E + a_0(\boldsymbol{\lambda})$ has μ-fold multiple roots.

At an EPμ of energy E_0, $P_\lambda(E)$ and its $\mu-1$ successive derivatives $P_\lambda^{(j)}(E) \equiv \partial^j P_\lambda/\partial E^j$ must vanish at $E = E_0$. This property is encoded in the zeros of the so-called *resultant* $R_{P,P^{(j)}}(\lambda)$. The resultant of two polynomials $P_1(E)$ and $P_2(E)$ is an elementary concept in algebra [24]. It reads $R_{P_1,P_2} \equiv \det S_{P_1,P_2}$, where S_{P_1,P_2} is the Sylvester matrix of P_1 and P_2. Importantly, this quantity vanishes if and only if the two polynomials P_1 and P_2 have a common root. Actually, we demonstrate (see the Supplemental Material [23]) that a polynomial $P(E)$ of degree μ has a μ-fold multiple root if and only if the resultants of its successive derivatives $R_{P^{(j-1)},P^{(j)}}$ vanish. Put formally,

$$\begin{array}{l}\mu\times\mu \text{ non-Hermitian matrices}\\ H(\lambda) \text{ have an EP}\mu \text{ at } \lambda=\lambda_0\end{array} \Leftrightarrow \begin{array}{l} R_{P^{(j-1)},P^{(j)}}(\lambda_0)=0\\ \text{for } j=1\ldots\mu-1.\end{array} \quad (1)$$

$R_{P^{(j-1)},P^{(j)}}(\lambda)$ is in general a complex-valued function of λ. The equivalence (1) thus yields $2(\mu-1)$ constraints to be satisfied, which gives the codimension of an EPμ in the absence of symmetry, in agreement with [6,17]. Note that the usual EP2s are defined by $R_{P,P'}(\lambda_0) = 0$, where $R_{P,P'}(\lambda)$ is proportional to the discriminant Δ of the characteristic polynomial. Since Δ is in general a complex function of λ, the stability of EP2s can be expressed by the winding of $\arg[R_{P,P'}(\lambda)]$ along a close circuit around λ_0 in a 2D parameter space [16].

As we discuss now, antiunitary symmetries that are local in parameter space, have important consequences. First, they decrease the codimension of the EPs. Second, they make the discriminant Δ real. Thus $\mathrm{sgn}(\Delta)$ becomes a well-defined quantity that one can use to characterize a spontaneous symmetry breaking. The winding of $\arg(\Delta)$ becomes ill defined, but we shall see that another natural homotopy property can be assigned to multifold EPs.

The symmetries we consider are

$$\text{PT symmetry} \qquad U_{\rm PT}H(\lambda)U_{\rm PT}^{-1} = H^*(\lambda), \qquad (2a)$$

$$\text{pseudo-Hermiticity} \qquad U_{psH}H(\lambda)U_{psH}^{-1} = H^\dagger(\lambda), \qquad (2b)$$

$$\text{CP symmetry} \qquad U_{\rm CP}H(\lambda)U_{\rm CP}^{-1} = -H^*(\lambda), \qquad (2c)$$

$$\text{pseudo-chirality} \qquad U_{psCh}H(\lambda)U_{psCh}^{-1} = -H^\dagger(\lambda), \qquad (2d)$$

where $*$ stands for complex conjugation, $\dagger$ stands for Hermitian conjugation, and the U's are unitary operators. The consequences of these symmetries on the existence of EPs are readily obtained from the characteristic polynomial that must fulfill

$$P_\lambda(E) = \det[UH(\lambda)U^{-1} - E]. \qquad (3)$$

Let us first proceed with PT symmetry (2). Then, Eq. (3) implies $P_\lambda(E) = a_n^*(\lambda)E^n + a_{n-1}^*(\lambda)E^{n-1} + \cdots + a_1^*(\lambda)E + a_0(\lambda)^*$, so that $a_l^*(\lambda) = a_l(\lambda)$. All the coefficients $a_l(\lambda)$ must therefore be real. Thus, the resultants $R_{P^{(j-1)},P^{(j)}}(\lambda)$ are real too. This is due to the fact that the elements of the Sylvester matrix $S_{P^{(j-1)},P^{(j)}}$ are essentially the coefficients a_l [multiplied adequately by numbers of the form $n(n-1)\cdots(n-j+1)$ due to the j successive derivatives] [24]. The reality of the resultants implies that the number of conditions in Eq. (1) to have an EPμ reduces to $\mu-1$. By definition, the codimension of a PT-symmetry protected EPμ is therefore $\mu-1$.

Importantly, this codimension remains $\mu-1$ for any of the symmetries introduced in Eq. (2). This is obvious for pseudo-Hermiticity (2b), since taking the transpose of H leaves invariant the determinant in Eq. (3). The case of CP symmetry can be mapped onto the PT-symmetric one by the transformation $H \to -\mathrm{i}H$, that does not change the codimension of the degeneracies. Finally the pseudo-chiral case is deduced from the CP-symmetric one by invariance of Eq. (3) by taking the transpose of H. One can finally rephrase this result as μ-fold complex degeneracies of a $\mu\times\mu$ Hamiltonian satisfying one of the local antiunitary symmetries (2) appear as "defects" of dimension d in a D-dimensional λ-parameter space such that

$$\mathrm{codim}(\mathrm{EP}\mu) \equiv D - d = \mu - 1. \qquad (4)$$

Moreover, for each symmetry (2), the resultants $R_{P^{(j-1)},P^{(j)}}$ are real (see Ref. [23]), consistently with the $\mu-1$ constraints (1) to be satisfied to get a symmetry-protected EPμ instead of $2(\mu-1)$. The reality of the resultants naturally generates a *resultant vector* $\tilde{\mathcal{R}}$ of components $R_{P^{(j-1)},P^{(j)}}$, that maps the λ space of parameters to $\mathbb{R}^{\mu-1}$. When the dimension of the λ space is $\mu-1$ (for instance, after fixing some of the parameters), the homotopy properties of the map $\lambda \to \tilde{\mathcal{R}}/|\tilde{\mathcal{R}}|$ can be used to characterize the EPμ.

In fact, a similar construction can be done from different resultants, so that another resultant vector, $\mathcal{R}$, can be introduced. Indeed, the existence of an EPμ imposes resultants between other derivatives of $P(E)$ to vanish as well, and, in particular,

$$R_{P,P^{(j)}}(\lambda_0) = 0 \quad \text{with} \quad j = 1\ldots\mu-1. \qquad (5)$$

Actually, such a constraint, to be compared with Eq. (1), is also a necessary and sufficient condition for an EPμ to exist, at least for $\mu = 2$, 3, and 4 [23]. It turns out that those resultants are either real or purely imaginary. It is thus natural to define the resultant vector $\mathcal{R}$ from its components

$$\mathcal{R}_j \equiv \begin{cases} R_{P,P^{(j)}} & \text{–PT and psH} \\ (-\mathrm{i})^{n(n-j)} R_{P,P^{(j)}} & \text{–CP and psCh} \end{cases} \tag{6}$$

that depend on the symmetry, and where the coefficient $(-\mathrm{i})^{n(n-j)}$ guarantees the reality of $\mathcal{R}_j$.

The two resultant vectors we have introduced have the same first component $\mathcal{R}_1$, which is proportional to the discriminant Δ of the characteristic polynomial $P_{\boldsymbol{\lambda}}(E)$ of degree n as $R_{P,P'}(\boldsymbol{\lambda}) = (-1)^{n(n-1)/2} a_n(\boldsymbol{\lambda})\Delta(\boldsymbol{\lambda})$. The vanishing of the discriminant at a given $\boldsymbol{\lambda}_0$ indicates the existence of *at least* two roots of $P_{\boldsymbol{\lambda}_0}(E)$. In the presence of a symmetry (2), the discriminant is also a real quantity, since both $R_{P,P'}$ and a_n are always real. Its sign (or equivalently that of $\mathcal{R}_1$) is therefore well defined and turns out to encode crucial properties about the complex eigenenergies E_j. For instance, it is well known that for a polynomial of degree $n = 2$, $\Delta > 0$ comes along with distinct real roots, while $\Delta < 0$ indicates a pair of complex conjugated roots. These two behaviors are separated by a critical point $\Delta = 0$ where the gap separating complex energy bands closes. More generally, a change in the number of complex conjugated roots is a property of the sign of the discriminant that generalizes beyond $n = 2$ [23]. One can thus use $\mathrm{sign}(\Delta)$ as a $\mathbb{Z}_2$ index to distinguish two different regimes for non-Hermitian Hamiltonians of arbitrary size that satisfy one of the symmetries (2). The cases $n = 2$ and $n = 3$ are presented in Table I. Such an index encompasses the previously proposed topological invariants given by $\mathrm{sgn}(\det H)$ [$\mathrm{sgn}(\det \mathrm{i}H)$] for 2×2 PT (CP) symmetric Hamiltonians [25–27].

It is worth pointing out that a change of $\mathrm{sign}(\Delta)$ reveals that the symmetry (2) under consideration is *spontaneously* broken. Let us recall this notion in PT-symmetric systems [28]. Consider $H|\psi\rangle = E|\psi\rangle$, then $U_{PT}^{-1}\kappa|\psi\rangle$ is also an eigenstate of H with eigenvalue E^*, where κ is the complex conjugation operator. If $U_{PT}^{-1}\kappa|\psi\rangle$ is proportional to $|\psi\rangle$, then $|\psi\rangle$ is an eigenstate of the PT operator, and the eigenenergies are real. In the other case, $|\psi\rangle$ is not an eigenstate of $U_{PT}^{-1}\kappa$ and the eigenenergies come by pairs (E, E^*). PT symmetry is then said to be *spontaneously* broken: it still holds at the level of the Hamiltonian, but not at the level of the eigenstates. The same reasoning applies to CP-symmetric and pseudochiral Hamiltonians, where eigenenergies come by either by pairs $(E, -E^*)$ or are purely imaginary. In any case, the spontaneous symmetry breaking is accompanied with a change of sign of the discriminant that governs the complex nature and the pairing of the eigenvalues.

TABLE I. Properties of the eigenvalues E_i of 2×2 and 3×3 non-Hermitian matrix as a function of the $\mathrm{sgn}\Delta$ of its characteristic polynomial $\det(H - E)$ of degree n, in the presence of symmetries (2).

Degree n	sgn Δ	PT/psH	CP/psCh
	$+1$	$E_i \in \mathbb{R}$, $E_1 \neq E_2$	$E_1 = -E_2^*$
2	0	$E_1 = E_2$	$E_1 = E_2$
	-1	$E_1 = E_2^*$	$E_i \in \mathrm{i}\mathbb{R}$, $E_1 \neq E_2$
	$+1$	$E_i \in \mathbb{R}$, $E_1 \neq E_2 \neq E_3$	$E_1 = -E_2^*$ and $E_3 \in \mathrm{i}\mathbb{R}$
3	0	$E_1 = E_2$ and $E_3 \in \mathbb{R}$	$E_1 = E_2$ and $E_3 \in \mathrm{i}\mathbb{R}$
	-1	$E_1 = E_2^*$ and $E_3 \in \mathbb{R}$	$E_i \in \mathrm{i}\mathbb{R}$, $E_1 \neq E_2 \neq E_3$

Symmetry-protected multifold EPs are special points where the symmetry is spontaneously broken, since they also demand the vanishing of the other components $\mathcal{R}_j = 0$. From a geometrical point of view [13,27,29,30], the solutions $\boldsymbol{\lambda}$ of each equation $\mathcal{R}_j(\boldsymbol{\lambda}) = 0$ define a "manifold" $\mathcal{M}_j$ of dimension $D - 1$ in parameter space of dimension D (i.e., codimension 1). Thus, the coordinates $\boldsymbol{\lambda}_0$ of EPμs define a space that consists in their mutual $\mu - 1$ intersections. For instance, in $D = 2$ dimensions, $\mathcal{M}_1$ and $\mathcal{M}_2$ consist in curves in a plane, and their mutual intersections are points that correspond to EP3s. Importantly, such intersections are generically robust to perturbations. Let us illustrate this point on a concrete model, the rotating shallow water model with friction.

The linearized rotating shallow water model describes waves propagating in a thin layer of fluid in rotation. It is currently used to describe atmospheric and oceanic waves over large distances where the Coriolis force, encoded into a parameter f, is relevant [31]. We consider a non-Hermitian version of this model that reads $H\Psi = E\Psi$ with

$$H = \begin{pmatrix} \mathrm{i}\gamma_x & \mathrm{i}f & k_x \\ -\mathrm{i}f & \mathrm{i}\gamma_y & k_y \\ k_x & k_y & \mathrm{i}\gamma_N \end{pmatrix}, \qquad \Psi = \begin{pmatrix} \delta u_x \\ \delta u_y \\ \delta h \end{pmatrix} \tag{7}$$

with δu_x and δu_y the small variations of the horizontal fluid velocity, δh a small variation of the fluid's thickness, k_x and k_y the in-plane wave numbers, γ_x and γ_y the Rayleigh friction terms, and γ_N the Newtonian friction. The competition between the two friction terms plays, for instance, a crucial role in the phenomenon of superrotation [32]. Actually, this minimal model equivalently describes active fluids where the thickness field h is formally replaced by a pressure field. [33] In that context, it was shown that the γ terms can also be negative, thus allowing for gain [34]. We now show that the friction terms generate symmetry-protected EP3s.

Assuming isotropic friction $\gamma_x = \gamma_y \equiv \gamma_R$, the model remains rotational symmetric and can thus be simplified by fixing a direction (say $k_y = 0$). Moreover, any choice of γ_R can be reabsorbed in γ_N up to a global shift of the spectrum, that does not affect the existence of degeneracies, as $H = \gamma_R \mathbb{I} + \tilde{H}$ with

$$\tilde{H} = \begin{pmatrix} 0 & \mathrm{i}f & k_x \\ -\mathrm{i}f & 0 & 0 \\ k_x & 0 & \mathrm{i}\gamma \end{pmatrix}, \tag{8}$$

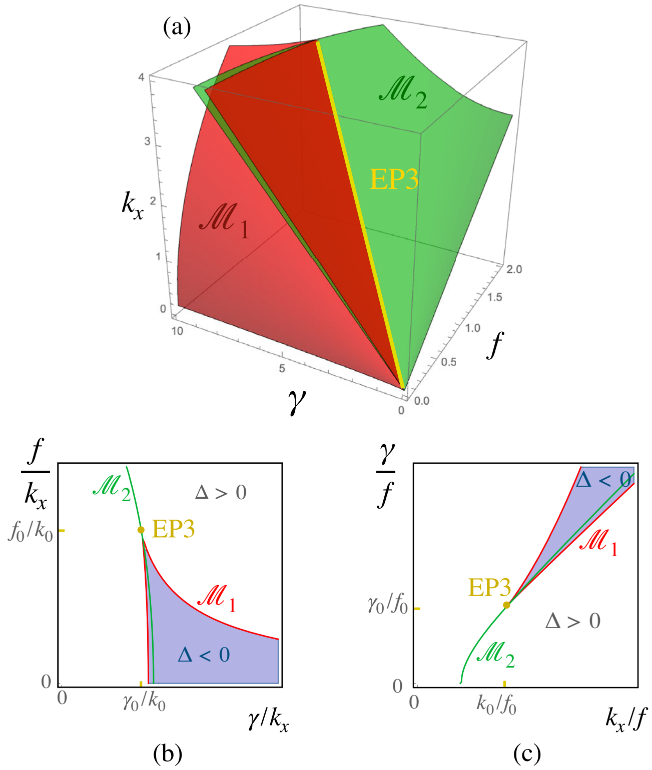

FIG. 1. EP3s (yellow) of the frictional shallow water model in 3D (a) and 2D (b),(c) parameter spaces. Those points result from the intersection of the surfaces $\mathcal{M}_1$ and $\mathcal{M}_2$ each of codimension 1.

where $\gamma \equiv \gamma_N - \gamma_R$. This matrix satisfies CP symmetry (2c) with $U_{\mathrm{CP}} = \mathrm{diag}(1, 1, -1)$, and one thus expects EP3s of codimension 2, according to the relation (4). Since the parameter space has a dimension $D = 3$, with $\boldsymbol{\lambda} = \{k_x, f, \gamma\}$, the set of EP3s must consist in a manifold of codimension 2, that is a line. For this specific model, the equation of this line can be derived explicitly from $\mathcal{R}_1(\boldsymbol{\lambda}_0) = 0$ and $\mathcal{R}_2(\boldsymbol{\lambda}_0) = 0$, which yields $\gamma_0 = \pm 3\sqrt{3} f_0$ and $k_0 = \pm 2\sqrt{2} f_0$. Geometrically, this line of EP3s corresponds to the intersection of the two spaces $\mathcal{M}_1$ and $\mathcal{M}_2$, each of codimension 1. $\mathcal{M}_1$ and $\mathcal{M}_2$ are thus surfaces in three dimensions [Fig. 1(a)], and lines in two dimensions [Figs. 1(b) and 1(c)].

Remarkably, the space $\mathcal{M}_1 \cap \mathcal{M}_2$ of EP3s correspond to a fold of $\mathcal{M}_1$. Similar singular points to that shown in Figs. 1(b) and 1(c) were found in other models [18,19]. This suggests a quite fundamental property of EP3s, that can be addressed within catastrophe theory which classifies the possible kinks of curves and thus establishes their topological stability [35]. Intuitively, such a singular shape can be traced back to the third-order characteristic polynomial $P_{\boldsymbol{\lambda}}(E) = 0$, since cubic curves are known as a basic example that displays such a *catastrophe* when the three roots coincide. To characterize more precisely this singularity, we use the machinery of catastrophe theory and compute a list of invariants (called *corank*, *codimension*, and *determinacy*) that classify the catastrophe. Following Refs. [35,36], we find a corank of 1, a codimension of 1, and a determinacy of 3, which identifies EP3s as a *fold* of $\mathcal{M}_1$. This is the simplest fundamental catastrophe in the classification of catastrophe theory.

Before characterizing in more details the robustness of the EP3, let us first comment on the original physical behaviors revealed in Fig. 1. Indeed, $\mathcal{M}_1$ indicates a change of sign of the discriminant, and thus, according to Table I, it denotes a transition between complex eigenvalues ($\Delta > 0$) and purely imaginary eigenvalues ($\Delta < 0$), which, in our case, all have the same sign which is fixed by $\mathrm{sgn}(\gamma)$. The eigenvalue spectrum of Eq. (8) is shown in Fig. 2. It follows that the $\Delta < 0$ domain corresponds to a regime where all the eigenmodes are fully evanescent (for $\gamma > 0$) and thus where propagation is prohibited. Quite remarkably, while a first *propagating* $\rightarrow$ *nonpropagating* transition happens when increasing friction, an even more striking *nonpropagating* $\rightarrow$ *propagating* second transition occurs when increasing dissipation furthermore. In those phase diagrams, the EP3 appears as the threshold beyond which such transitions exist.

One can characterize the stability of symmetry protected EPs furthermore by using the topological properties of the *resultant vector* $\boldsymbol{\mathcal{R}} \in \mathbb{R}^{\mu-1}$ introduced above. For an EP3, it defines a map $\boldsymbol{\mathcal{R}}/|\boldsymbol{\mathcal{R}}| : \mathbb{R}^2 \backslash \{\boldsymbol{\lambda}_0\} \rightarrow S^1$ whose homotopy properties are encoded into the integer-valued winding number

$$W_3 = \frac{1}{2\pi} \oint_{\mathcal{C}_\lambda} \frac{1}{\|\boldsymbol{\mathcal{R}}\|^2} \left(\mathcal{R}_1 \frac{\partial \mathcal{R}_2}{\partial \lambda_\alpha} - \mathcal{R}_2 \frac{\partial \mathcal{R}_1}{\partial \lambda_\alpha} \right) \mathrm{d}\lambda_\alpha, \quad (9)$$

where the close circuit $\mathcal{C}_\lambda$ surrounds the EP3 and where an implicit sum is taken over $\alpha = \{1, 2\}$. It is worth noticing that, in lattice systems, $\mathcal{M}_1$ and $\mathcal{M}_2$ are necessarily closed loops in the 2D Brillouin zone, so that their intersections come by pairs with opposite W_3's, which can be seen by a straightforward extension of the doubling theorem by

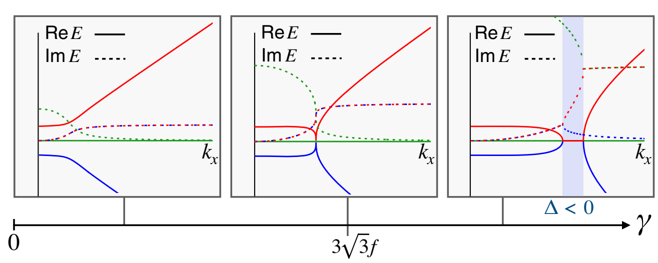

FIG. 2. Complex dispersion relation $E(k)$ of the shallow water model (8) for different values of γ. An EP3 appears at the critical value $\gamma_0 = 3\sqrt{3} f$ for an arbitrary f and its eigenfrequency is found to be $E_0 = \mathrm{i}\gamma_0/3$. Beyond this point, the discriminant Δ of the characteristic polynomial of $\tilde{H}$ becomes negative over a finite domain of wavebnumbers, where the three eigenmodes are fully damped.

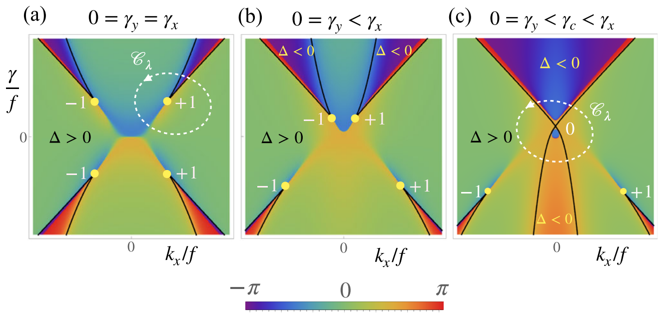

FIG. 3. Color plot of the relative angle between $\mathcal{R}_1$ and $\mathcal{R}_2$ in $(k_x/f, \gamma/f)$ parameter space for (a) no Rayleigh friction, (b) a small and (c) a large anisotropic Rayleigh friction. The winding of this angle along loops surrounding the EP3s indicates 2 pairs of EP3s of charge $W_3 = \pm 1$ until a critical value γ_c of the anisotropy (c) beyond which a new domain with forbidden propagation ($\Delta < 0$) emerges.

Nielsen and Ninomiya [37,38] as recently generalized for non-Hermitian systems without symmetry [16].

The invariant W_3 is the winding number of the relative angle between $\mathcal{R}_1$ and $\mathcal{R}_2$. For the frictional shallow water model, its value is shown in Fig. 3(a). It turns out that in this model, EP3s come by pairs $W_3 = \pm 1$ at $k_x = \pm k_0$. This indicates that such pairs may possibly merge and annihilate each other if one moves them, similarly to Dirac and Weyl points in semimetals [39–42]. Since the position of the EP3s is fixed in the isotropic case, a mechanism involving their motion necessarily breaks rotational symmetry. This can be achieved by considering now $\gamma_x \neq \gamma_y$, as shown in Fig. 3(a) where one pair ($\gamma_N/f > 0$) get closer while the other pair ($\gamma_N/f < 0$) is pulled away. When increasing γ_x furthermore, the points of the pair at $\gamma_N/f > 0$ finally merge. Surprisingly enough, a new domain of nonpropagating waves ($\Delta < 0$) emerges for $\gamma_N < 0$ [see Fig. 3(b)]. A similar analysis can be carried out from the other resultant vector $\tilde{\mathcal{R}}$ introduced above, but its winding number is found to be zero for that model.

More generally, multifold ($\mu \geqslant 3$) symmetry-protected EPs can be characterized by the homotopy properties of their resultant vectors $\mathcal{R}$ (or $\tilde{\mathcal{R}}$). Let us fix $d = 0$, so that the EP is an isolated point $\lambda_0 \in \mathbb{R}^{\mu-1}$, according to Eq. (4). Then the resultant vector defines the map $\mathcal{R}/|\mathcal{R}| : \mathbb{R}^{\mu-1} \backslash \{\lambda_0\} \to S^{\mu-2}$ whose degree [43] $W_\mu \equiv \deg \mathcal{R} \in \mathbb{Z}$ is well defined and reads [44]

$$W_\mu = \sum_{\lambda_i} \mathrm{sgn}\left(\frac{\partial \mathcal{R}_j}{\partial \lambda_p}\right)\Big|_{\mathcal{R}(\lambda_i)=\mathcal{R}_0}, \tag{10}$$

where $\mathcal{R}_0$ is an arbitrary point where the Jacobian matrix $(\partial \mathcal{R}_j/\partial \lambda_p)$ does not vanish. This homotopy invariant is a topological property of multifold symmetry-protected EPs. The expression (9) of the winding number W_3 for EP3s is the simplest example of such invariants.

Our analysis opens various novel perspectives in the control of topological properties of non-Hermitian systems. In future works, it could be interesting to investigate symmetry-protected EP4s that should appear in three-dimensional parameter spaces, and whose topological charge W_4 takes the form of a wrapping number of the resultant vector

$$W_4 = \frac{1}{4\pi} \oint_{\mathcal{S}_\lambda} \frac{\mathcal{R}}{\|\mathcal{R}\|^3} \cdot \left(\frac{\partial \mathcal{R}}{\partial \lambda_\alpha} \times \frac{\partial \mathcal{R}}{\partial \lambda_\beta}\right) d\lambda_\alpha \wedge d\lambda_\beta. \tag{11}$$

In particular, this suggests the existence of stable non-Hermitian versions of 3D Dirac points in sharp contrast with both Hermitian 3D Dirac points and non-Hermitian Weyl points that were both found to be unstable.

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