Multiple Series Representations of N-fold Mellin-Barnes Integrals

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Mellin-Barnes (MB) integrals are well-known objects appearing in many branches of mathematics and physics, ranging from hypergeometric functions theory to quantum field theory, solid-state physics, asymptotic theory, etc. Although MB integrals have been studied for more than one century, until now there has been no systematic computational technique of the multiple series representations of N-fold MB integrals for $N > 2$. Relying on a simple geometrical analysis based on conic hulls, we show here a solution to this important problem. Our method can be applied to resonant (i.e., logarithmic) and nonresonant cases and, depending on the form of the MB integrand, it gives rise to convergent series representations or diverging asymptotic ones. When convergent series are obtained, the method also allows, in general, the determination of a single "master series" for each series representation, which considerably simplifies convergence studies and/or numerical checks. We provide, along with this Letter, a *Mathematica* implementation of our technique with examples of applications. Among them, we present the first evaluation of the hexagon and double box conformal Feynman integrals with unit propagator powers.

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Introduction.—N-fold Mellin-Barnes (MB) integrals are defined as

$$I(x_1,x_2,\ldots,x_N) = \int_{-i\infty}^{+i\infty}\frac{dz_1}{2\pi i}\cdots\int_{-i\infty}^{+i\infty}\frac{dz_N}{2\pi i}\frac{\prod_{i=1}^{k}\Gamma^{a_i}[s_i(\mathbf{z})+g_i]}{\prod_{j=1}^{l}\Gamma^{b_j}[t_j(\mathbf{z})+h_j]}x_1^{z_1}\ldots x_N^{z_N}, \tag{1}$$

where a_i, b_j, k, l, and N are positive integers (with $k \geq N$ after possible cancellations due to the denominator), $\mathbf{z} = (z_1,\ldots,z_N)$ and where we have defined $s_i(\mathbf{z}) \doteq \mathbf{e}_i\cdot\mathbf{z}$ and $t_j(\mathbf{z}) \doteq \mathbf{f}_j\cdot\mathbf{z}$ for a later purpose. The vectors $\mathbf{e}_i$, $\mathbf{f}_j$ and the scalars g_i, h_j are reals, while $\mathbf{x} = (x_1,\ldots,x_N)$ can be complex, and the contours of integration, which avoid the poles of the gamma functions that belong to the numerator of the MB integrand, have to be specified. In the present Letter, we focus on the common situation where the set of poles of each of these gamma functions is not split in different subsets by the contours.

The importance of MB integrals cannot be overstated, as they appear in domains as diverse as hypergeometric functions theory [1–3], electromagnetic wave propagation in turbulence [4], asymptotics [5], quantum field theory (QFT) [6], etc. In QFT, which is of particular interest for the authors, an impressive array of publications of the last decades may be mentioned (see [6] for a complementary list). Early studies can be found in [7–9], followed by classical works [10–23] highlighting the relevance of MB integrals in QFT. These motivated the automatization of some of the computational steps of the MB technique [24–28]. Numerous applications were guided by the needs of particle physics phenomenology, e.g., [29–40] but also by more formal motivations [41–54]. Recently, MB integrals and the Mellin transform entered the conformal bootstrap, see e.g., [55,56] and references therein. Other recent and diverse applications exist as, for instance, in option pricing [57], detector physics [58] or Ruderman-Kittel-Kasuya-Yosida interaction in condensed matter [59].

Even though MB integrals have been thoroughly studied for several decades in theoretical physics, and in fact for more than one century in the mathematical literature—from the pioneering works [60–62] to the most recent advances (see e.g., [63] and references therein)–it has been recently emphasized in [64,65] that there is still no systematic computational technique for the extraction of their multiple series representations in the N-fold case when $N > 2$ (for the $N = 2$ case with straight contours see [49,66,67]).

We present here the first solution to this important problem, which, in addition to its own interest in the theory of MB integrals, can potentially lead to many new results in the fields mentioned above. A *Mathematica* implementation of our method is given in the Supplemental Material [68] to this Letter, along with important specific examples of application of our method. The code is used, among others, to obtain the first evaluation of two highly nontrivial resonant cases in QFT: the hexagon and double box conformal Feynman integrals with unit propagator powers (see [53] for the nonresonant generic propagator powers cases).

The method.—The type of series representations that can be derived from Eq. (1) strongly depends on the N-dimensional vector $\mathbf{\Delta} = \sum_i a_i \mathbf{e}_i - \sum_j b_j \mathbf{f}_j$. If $\mathbf{\Delta}$ is null, which is the case we focus on in the present Letter, this corresponds to a degenerate situation [66,67] where there exist several convergent series representations for the MB integral, converging in different regions of the $\mathbf{x}$ parameter space. These series are analytic continuations of one another if the quantity $\alpha \doteq \mathrm{Min}_{||\mathbf{y}||=1}(\sum_i a_i |\mathbf{e}_i \cdot \mathbf{y}| - \sum_j b_j |\mathbf{f}_j \cdot \mathbf{y}|)$ is positive [66].

The question, now, is how to derive these series representations. To ease the reading of the presentation of our method, which rests on a simple geometric analysis, we focus here on the nonresonant case where there is no point in the $\mathbf{z}$ space at which more than N singular (hyper) planes (associated with the gamma functions in the numerator of the integrand of the N-fold MB integral) intersect. The poles of the MB integrand are thus of order one, thereby avoiding a discussion on the technical aspects of multivariate residues computations [69,70] because only nonlogarithmic series representations can appear. Resonant, i.e., logarithmic cases, are discussed in the Supplemental Material [68], as well as in [54].

To illustrate the different steps of the method, we propose to consider the simple paradigmatical example of the Appell F_1 double hypergeometric function whose MB representation reads [1]

$$\begin{aligned} &F_1(a, b_1, b_2; c; u_1, u_2) \\ &\quad = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b_1)\Gamma(b_2)} \\ &\qquad \times \int_{-i\infty}^{+i\infty} \frac{dz_1}{2\pi i} \int_{-i\infty}^{+i\infty} \frac{dz_2}{2\pi i} (-u_1)^{z_1} (-u_2)^{z_2} \Gamma(-z_1)\Gamma(-z_2) \\ &\qquad \times \frac{\Gamma(a + z_1 + z_2)\Gamma(b_1 + z_1)\Gamma(b_2 + z_2)}{\Gamma(c + z_1 + z_2)}, \end{aligned} \tag{2}$$

where the contours of integration are such that they separate the sets of poles of $\Gamma(-z_1)$ and $\Gamma(-z_2)$ from those of the other gamma functions in the numerator of the MB integrand. To avoid resonant situations, we choose generic values for the parameters a, b_1, b_2, and c. It can be seen from Eq. (2) that $\mathbf{\Delta} = (0, 0)$, which means that this is a degenerate case, and a simple analysis shows that $\alpha = 2$. Therefore, as mentioned above, one can conclude that the different series representations of the twofold MB integral that we will derive are analytic continuations of one another, converging in different regions of the (u_1, u_2) space.

In the general MB case, each of the series representations that we look for is a particular linear combination of some multiple series. In the nonresonant case, such a linear combination is obtained as a sum of terms suitably extracted from a set S of what we call "building blocks" in the following. The latter are thus nothing but the multiple series dressed with their overall coefficient and sign.

The key point of our method (in the nonresonant case) is that each of these building blocks is associated with one N-combination of gamma functions in the numerator of the MB integrand and with one conic hull, and that specific intersections of these conic hulls are in one-to-one correspondence with the sums of building blocks that form the different series representations of the MB integral under study (in the resonant case, the same intersections give birth to series representations, which are, however, not made of building blocks).

Let us look at this in more detail. For each possible N-combination of gamma functions in the numerator of the MB integrand, let us consider the pointed conic hull, built from the vectors $\mathbf{e}_i$ of the gamma functions that belong to the N-combination. An N-combination whose associated conic hull is N-dimensional is retained, while the N-combinations yielding lower-dimensional objects are discarded. Finding all relevant N-combinations, one therefore obtains a set of corresponding conic hulls, which we call S', where $\mathrm{card}(S') = \mathrm{card}(S)$.

To see this in our F_1 example, let us label each of the five gamma functions of the integrand's numerator of Eq. (2) by $i = 1, \ldots, 5$ to keep track of them and display them in a tabular form (see Table I) along with their corresponding normal vector $\mathbf{e}_i$ and what we call their singular factor $s_i(\mathbf{z})$, defined in Eq. (1). Now, since the MB integral is twofold, one has to consider all possible 2-combinations (i_1, i_2) of these gamma functions and their associated conic hulls C_{i_1, i_2}, where i_1 and i_2 are the labels, given in the first column of Table I, of the gamma functions that belong to a given 2-combination. There are $\binom{5}{2} = 10$ possible 2-combinations, out of which only eight are retained, as for the two 2-combinations (1,4) and (2,5) the associated conic hulls are of lower dimension than the fold of the MB integral.

This way, the set of conic hulls associated with the retained 2-combinations is

TABLE I. List of gamma functions in the numerator of the integrand in Eq. (2) and their associated normal vectors and singular factors.

i	Γ function	$\mathbf{e}_i$	$s_i(\mathbf{z})$
1	$\Gamma(-z_1)$	$(-1,0)$	$-z_1$
2	$\Gamma(-z_2)$	$(0,-1)$	$-z_2$
3	$\Gamma(a+z_1+z_2)$	(1,1)	z_1+z_2
4	$\Gamma(b_1+z_1)$	(1,0)	z_1
5	$\Gamma(b_2+z_2)$	(0,1)	z_2

$$S' = \{C_{1,2}, C_{1,3}, C_{1,5}, C_{2,3}, C_{2,4}, C_{3,4}, C_{3,5}, C_{4,5}\}. \quad (3)$$

As an example, the conic hull $C_{1,3}$ associated with (1,3), whose edges are along the vectors $\mathbf{e}_1 = (-1,0)$ and $\mathbf{e}_3 = (1,1)$, is shown in Fig. 1 (top left). $C_{3,5}$ (respectively, $C_{4,5}$) is shown in the top middle (respectively, top right).

As mentioned above, one can now associate with each retained 2-combination (i_1, i_2) a building block, denoted by B_{i_1,i_2}. Consequently, S simply reads

$$S = \{B_{1,2}, B_{1,3}, B_{1,5}, B_{2,3}, B_{2,4}, B_{3,4}, B_{3,5}, B_{4,5}\}. \quad (4)$$

We now have to compute explicitly the expressions of each of these building blocks and find the series representations that can be built from them. Note that it is, of course, possible to perform these two steps in reverse order because our method does not rest on the convergence properties of the involved multiple series.

In the general case, to each retained N-combination, there is a corresponding set of poles located at the intersections of exactly N singular (hyper)planes (those

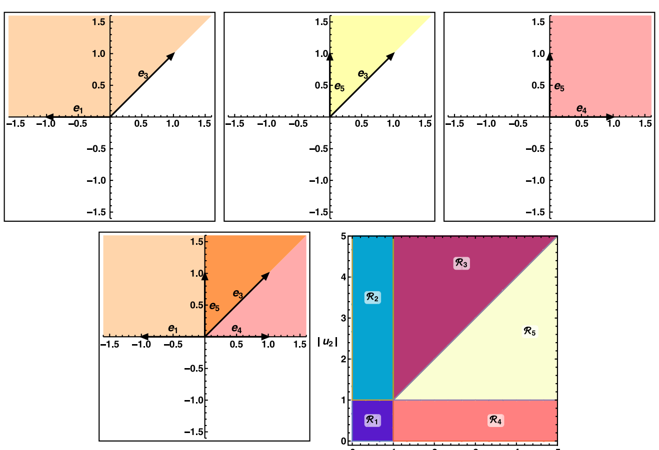

FIG. 1. Conic hulls $C_{1,3}$ (top left), $C_{3,5}$ (top middle), and $C_{4,5}$ (top right) and their intersection (orange region, bottom left) with edges along $\mathbf{e}_3$ and $\mathbf{e}_5$ which corresponds in fact to $C_{3,5}$. Bottom right: convergence regions, given in Eq. (7), of the five series representations of the MB integral in Eq. (2). The well-known Appell F_1 double hypergeometric series converges in $\mathcal{R}_1$, its four analytic continuations in the other regions.

of the gamma functions in the N-combination), which, by a straightforward residue calculation, gives the corresponding building block in S. Following [49], one begins by bringing the singularity to the origin using appropriate changes of variables on the MB integrand and one applies the generalized reflection formula $\Gamma(z-n) = [\Gamma(1+z)\Gamma(1-z)(-1)^n]/[z\Gamma(n+1-z)]$ $(n \in \mathbb{Z})$, on each of the singular gamma functions so that their singular part appears explicitly. It then remains, in order to get the residue, to divide the obtained expression by $|\det A|$, where $A = (A_{rs})_{1\le r\le N, 1\le s\le N}$ with $A_{rs} = (\mathbf{e}_{i_r})_s$, to remove the N singular factors in the denominator and to put the z_i, $i = (1, \ldots, N)$ to zero. Summing over all residues, one then obtains the expression of the desired building block.

Let us show how this works by considering, for instance, in Eq. (4), the case of $B_{1,3}$, which is the sum of residues of the poles associated with (1,3), located at $(n_1, -a-n_1-n_2)$ for $n_i \in \mathbb{N}$ $(i = 1, 2)$.

One first brings the singularity to the origin using the changes of variable $z_1 \to z_1 + n_1$ and $z_2 \to z_2 - a - n_1 - n_2$. Then, applying the reflection formula on the singular gamma functions, the MB integrand becomes

$$\begin{aligned}
&(-u_1)^{z_1+n_1}(-u_2)^{z_2-a-n_1-n_2}\frac{\Gamma(1-z_1)\Gamma(1+z_1)(-1)^{n_1}}{(-z_1)\Gamma(n_1+1+z_1)} \\
&\quad\times\frac{\Gamma(-z_2+a+n_1+n_2)\Gamma(1+z_1+z_2)\Gamma(1-z_1-z_2)}{(z_1+z_2)\Gamma(n_2+1-z_1-z_2)} \\
&\quad\times(-1)^{n_2}\frac{\Gamma(b_1+z_1+n_1)\Gamma(b_2+z_2-a-n_1-n_2)}{\Gamma(c+z_1+n_1+z_2-a-n_1-n_2)}.
\end{aligned}$$

Now, since $|\det A|$, where $A = \left(\begin{smallmatrix}-1 & 0\\ 1 & 1\end{smallmatrix}\right)$, gives one, it remains to remove the singular factors $s_1 = -z_1$ and $s_3 = z_1 + z_2$ from the denominator and to put $z_1 = z_2 = 0$. Multiplying by the overall prefactor [ratio of gamma functions in Eq. (2)] and summing over n_1 and n_2, one then obtains the expression of the building block

$$\begin{aligned}
B_{1,3} &= \frac{\Gamma(c)}{\Gamma(a)\Gamma(b_1)\Gamma(b_2)}(-u_2)^{-a}\sum_{n_1,n_2=0}^{\infty}\left(-\frac{u_1}{u_2}\right)^{n_1}\left(\frac{1}{u_2}\right)^{n_2} \\
&\quad\times\frac{\Gamma(a+n_1+n_2)\Gamma(b_1+n_1)\Gamma(-a+b_2-n_1-n_2)}{\Gamma(n_1+1)\Gamma(n_2+1)\Gamma(-a+c-n_2)} \\
&= \frac{\Gamma(c)\Gamma(b_2-a)}{\Gamma(b_2)\Gamma(c-a)}(-u_2)^{-a} \\
&\quad\times F_1\left(a, b_1, a-c+1; a-b_2+1; \frac{u_1}{u_2}, \frac{1}{u_2}\right). \quad (5)
\end{aligned}$$

A similar analysis yields

$$\begin{aligned}
B_{1,5} &= \frac{\Gamma(c)\Gamma(a-b_2)}{\Gamma(a)\Gamma(c-b_2)}(-u_2)^{-b_2} \\
&\quad\times G_2\left(b_1, b_2, b_2-c+1; a-b_2; -u_1, -\frac{1}{u_2}\right), \quad (6)
\end{aligned}$$

where G_2 is one of the Horn double hypergeometric series [71]. It is thus straightforward, from similar calculations, to derive the explicit form of each of the building blocks of Eq. (4).

Let us now explain how to build the various series representations of the N-fold MB integral without any convergence analysis, which is among the significant features of this Letter. We observe that there is a one-to-one correspondence between these series representations and the subsets of conic hulls of S' whose intersection is nonempty, with the important constraint that if a subset of conic hulls satisfying the nonempty intersection condition is included in a bigger subset that also satisfies it, then the former does not correspond to a series representation. In order to write down the expression of the series representation associated with a given subset, one simply has to add the building blocks in S that correspond to each of the conic hulls of the subset. Every subset of conic hulls in S' satisfying the nonempty intersection condition will then lead to one distinct series representation of the MB integral.

In the case of Eq. (2), a straightforward geometrical analysis yields five subsets, which therefore leads to five series representations that are analytic continuations of one another. The subsets are $\{C_{1,2}\}, \{C_{1,3}, C_{1,5}\}$, $\{C_{1,3}, C_{3,5}, C_{4,5}\}, \{C_{2,3}, C_{2,4}\}$, and $\{C_{2,3}, C_{3,4}, C_{4,5}\}$. As an example, we have shown the intersection corresponding to the third subset in Fig. 1 (bottom left).

One therefore obtains

$$F_1(a; b_1, b_2, c; u_1, u_2) = \begin{cases} B^*_{1,2} & \text{for } |u_1| < 1 \cap |u_2| < 1 \quad (\mathcal{R}_1) \\ B_{1,3} + B^*_{1,5} & \text{for } |u_1| < 1 \cap |\frac{1}{u_2}| < 1 \quad (\mathcal{R}_2) \\ B_{1,3} + B^*_{3,5} + B_{4,5} & \text{for } |\frac{1}{u_1}| < 1 \cap |\frac{u_1}{u_2}| < 1 \quad (\mathcal{R}_3) \\ B_{2,3} + B^*_{2,4} & \text{for } |\frac{1}{u_1}| < 1 \cap |u_2| < 1 \quad (\mathcal{R}_4) \\ B_{2,3} + B^*_{3,4} + B_{4,5} & \text{for } |\frac{u_2}{u_1}| < 1 \cap |\frac{1}{u_2}| < 1 \quad (\mathcal{R}_5) \end{cases} \tag{7}$$

where the * on a building block is a notation which indicates that it is the master series (see below) associated with that series representation. The series representation $B_{1,3} + B_{1,5}$ and $B_{2,3} + B_{2,4}$ coincide with Eq. (17) of [72], while $B_{1,3} + B_{3,5} + B_{4,5}$ and $B_{2,3} + B_{3,4} + B_{4,5}$ match with Eq. (22) of the same reference.

Obviously the last two series representations of Eq. (7) could be deduced from the second and third ones by using the permutation symmetry $F_1(a, b_1, b_2; c; u_1, u_2) = F_1(a, b_2, b_1; c; u_2, u_1)$.

Master series.—Until here, we did not discuss convergence issues, because our method does not need to solve for the latter in order to extract the different series representations from the MB integral. However, once obtained, one may need to know the convergence regions of the series. We will see now that by introducing master series, this task can be greatly simplified.

In the degenerate case, the convergence region of a particular series representation of the MB integral is given by the intersection of the convergence regions of each of the series of which the series representation is built. Therefore, one way to find the convergence region of a series representation is to find the convergence region of each of these terms. Beyond triple or even double series, these convergence issues can be difficult open problems. Moreover, the higher k and/or N in Eq. (1) are, the more the linear combinations that constitute the series representations each have a large number of terms with different convergence properties. This also increases the complexity of the convergence analysis.

The alternative strategy that we propose is to find a set of poles that can parametrize, up to a change of variables, all the poles associated with the considered series representation. We call this set the “master set”. One can then construct from the master set a single series, which we name the master series, and we conjecture that its convergence region will either coincide or be a subset of the convergence region of the series representation under consideration. In the former case, which happens when there is no gamma function in the denominator of the MB integrand (or when there is at most a finite number of cancellations of poles by the gamma functions in the denominator), this considerably simplifies the task to that of finding the region of convergence of only this series (this is the case for our F_1 example); whereas in the latter case, although not explicitly giving the convergence region of the series representation, this is of precious help to facilitate the numerical checks. Note that even when the convergence region of the master series is too complicated to be derived, it is of great utility because it is sufficient to find a single set of numerical values that make it converge, to have the whole series representation also converging for the same set of values (this point is clearly illustrated in the study of the resonant double box and hexagon Feynman integrals performed in the Supplemental Material [68]).

In the case of higher-fold MB integrals, it is not straightforward to find the master set algebraically. We therefore propose a simpler technique, where we infer the master series from the N-dimensional conic hull (the “master conic hull”) formed by the intersection of the conic hulls associated with the N-combinations from which the series representation is built. First, one obtains the N basis vectors $\mathbf{e}_i$ $(i = 1, \ldots, N)$ of the master conic hull. Then the set of poles resulting from the meeting of the singular (hyper)planes associated with the gamma functions $\Gamma(\mathbf{e}_i \cdot z_i)$, $(i = 1, \ldots, N)$ gives the master set. Although the direction of the basis vectors $\mathbf{e}_i$ is given, their magnitude has to be fixed in such a way that the master set parametrizes all the poles that correspond to the series representation, up to a change of variable. Note that it can happen, in some cases, that the master series built from the master set is in fact one of the building blocks. This is the case for our F_1 example above and it is illustrated in Fig. 1 (bottom left) for the third series of Eq. (7), where it is indeed clear that the plotted intersection is a conic hull that matches with $C_{3,5}$. This means that $B_{3,5}$ is the master series associated with the third series representation of Eq. (7). Therefore, the convergence region of $B_{3,5}$ coincides with the region $\mathcal{R}_3$ (this can be easily checked by explicitly computing the intersection of the convergence regions of $B_{1,3}$, $B_{3,5}$, and $B_{4,5}$). In Fig. 1 (bottom right), we show the convergence regions obtained from a study of the master series, indicated by a star in Eq. (7), of each series representation of Eq. (2).

We close this section by noting that, as far as the master series is concerned, resonant and nonresonant situations are treated in the same way.

Conclusions.—A new, and so far unique, simple and powerful systematic method for deriving series representations of N-fold MB integrals has been presented. It has the great advantage of selecting the different terms that form these series representations without the need of a prior study of the convergence regions of each of these terms. In the degenerate case, for each of the so obtained series representations, our method also allows one, in general, to derive a single master series. We have shown how the latter considerably simplifies the convergence analysis and/or the numerical checks.

We have also shown that our method can be used to deal with resonant (i.e., logarithmic) situations in the Supplemental Material [68] as well as in our recent work [54]. In the latter paper, in addition to showing an interesting interplay between QFT and hypergeometric functions theory, our method has been used to identify spurious contributions of a recent Yangian bootstrap approach used to compute Feynman integrals [73].

To show that investigations in cases with a high number of variables are not an unrealistic goal using our framework, we have applied it to ninefold MB integrals in [53], obtaining recently for the first time some series representations of the hexagon and double box conformal Feynman integrals, for generic powers of the propagators. Although these objects are very complicated, earlier attempts to compute them having failed (see, for instance, [52]), they were easily computable with our approach because their MB representations belong to the nonresonant class. This is due to the fact that the propagator powers of these Feynman integrals are generic. Note that it is generally advised to compute Feynman integrals for generic powers of the propagators with the MB technique (see [6]). The same is true for multiple hypergeometric functions, which are, in general, studied for generic values of their parameters. This gives us one more reason to believe that the efficiency and simplicity of our approach in the nonresonant case will give birth to many new results.

All the examples mentioned until here belong to the so-called degenerate class, where $\mathbf{\Delta} = 0$, but our method can also treat the $\mathbf{\Delta} \neq 0$ case, where diverging asymptotic series representations can be obtained, as it will be shown in a subsequent publication.

We finish here by mentioning that we have provided, in the Supplemental Material [68], the first version of a *Mathematica* implementation of our method. It gave, in less than two minutes of CPU time, a series representation consisting of 26 terms for the hexagon [53]. In contrast the MBsums *Mathematica* package of [28] gives a hardy usable linear combination of 112 368 terms in over 12 h on the same computer. We have also used our code to derive the first series representations of the hexagon and double box in the highly nontrivial resonant case of unit propagator powers.

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