Double Copy Relation in AdS Space

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We present a double copy relation in AdS_5 that relates tree-level four-point amplitudes of supergravity, super Yang-Mills, and bi-adjoint scalars.

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Introduction.—Scattering amplitudes in flat space exhibit surprising properties that encode deep lessons for quantum field theories and gravity. While we believe many curved-spacetime generalizations exist, explicit realizations are far from obvious to find. Recently, there has been a lot of activity trying to extend two remarkable flat-space properties, color-kinematic duality [1] and the double copy relation [2], to the simplest curved background: the anti–de Sitter (AdS) space [3–8] [9]. The flat-space relations relate gauge theory and gravity amplitudes and have numerous applications in modern amplitude research [12]. Since AdS/CFT maps AdS amplitudes to conformal field theory (CFT) correlators, generalizations to AdS are especially interesting. While color-kinematic duality has been observed for four points [5–8], AdS double copy so far has only worked for three-point functions [3,4]. In fact, it was not clear if the flat-space relation has to be drastically modified at higher points. In this Letter, we present an AdS generalization that realizes the double copy construction in four-point amplitudes for the first time. We relate tree-level amplitudes in $AdS_5 \times S^5$ IIB supergravity, $AdS_5 \times S^3$ supersymmetric Yang-Mills (SYM), and nonsupersymmetric $AdS_5 \times S^1$ bi-adjoint scalars in a simple way that mirrors the flat-space relation. Moreover, our AdS relation works for *all* amplitudes in these theories, applying to massless and massive particles alike.

We will use the Mellin representation for CFT correlators [14,15]. AdS amplitudes become Mellin amplitudes and enjoy a simple analytic structure resembling the flat-space one. Tree-level Mellin amplitudes of AdS supergravity and super gauge theories in various spacetime dimensions were systematically studied in [7,16–23], and a Mellin color-kinematic relation similar to the flat-space one was pointed out in [7]. Unfortunately, applying the flat-space double copy prescription led to no sensible amplitudes. In this Letter, we revisit these results. We will focus on AdS_5 and take advantage of supersymmetry, which allows us to reduce the Mellin amplitudes to simpler "reduced" Mellin amplitudes. We find that it is in these reduced objects that color-kinematic duality and the double copy relation are naturally realized.

Schematically, we will write the reduced amplitude of AdS_5 super gluons with $\mathcal{N} = 2$ superconformal symmetry as a finite sum labeled by integers i, j:

$$\widetilde{\mathcal{M}} \sim \sum_{i,j} \frac{n_s^{i,j} c_s}{s - s_{i,j}} + \frac{n_t^{i,j} c_t}{t - t_{i,j}} + \frac{n_u^{i,j} c_u}{u - u_{i,j}},$$

where the number of terms is determined by the external masses. $c_{s,t,u}$ are standard color factors satisfying $c_s + c_t + c_u = 0$. The kinematic factors $n_{s,t,u}^{i,j}$ turn out to obey the same relation $n_s^{i,j} + n_t^{i,j} + n_u^{i,j} = 0$, giving rise to an AdS color-kinematic duality. Replacing $c_{s,t,u}$ with $n_{s,t,u}^{i,j}$, we recover precisely super graviton reduced amplitudes of $AdS_5 \times S^5$ IIB supergravity [16,17]. On the other hand, replacing $n_{s,t,u}^{i,j}$ by $c_{s,t,u}$ leads to Mellin amplitudes of conformally coupled bi-adjoint scalars on $AdS_5 \times S^1$, which were not studied in the literature. We will prove this fact by direct calculation. The AdS_5 double copy relation presented here relates theories with varying $\mathcal{N} = 0$, 2, 4 superconformal symmetry. However, it also implies that purely bosonic theories of Einstein gravity, Yang-Mills, and bi-adjoint scalars on AdS_5 should be related by double copy, as we will briefly discuss at the end.

Four-point correlators: No supersymmetry.—Let us start with the nonsupersymmetric case. We consider the correlator of four scalar operators $\mathcal{O}_{k_i}$ with conformal dimensions k_i [24]:

$$G_{k_1k_2k_3k_4} = \langle \mathcal{O}_{k_1}\mathcal{O}_{k_2}\mathcal{O}_{k_3}\mathcal{O}_{k_4} \rangle. \tag{1}$$

In Mellin space, correlators are represented as [14,15]

$$G_{k_1k_2k_3k_4} = \int_{-i\infty}^{i\infty} [dsdt] K(x_{ij}^2; s, t, u) \mathcal{M}_{k_1k_2k_3k_4} \Gamma_{\{k_i\}}(s, t, u), \tag{2}$$

where $[dsdt] = [dsdt/(4\pi i)^2]$, and $K(x_{ij}^2; s,t,u)$ is a factor containing all spacetime dependence:

$$K(x_{ij}^2; s,t,u) = (x_{12}^2)^{\frac{s-k_1-k_2}{2}}(x_{34}^2)^{\frac{s-k_3-k_4}{2}}(x_{14}^2)^{\frac{t-k_1-k_4}{2}}(x_{23}^2)^{\frac{t-k_2-k_3}{2}}(x_{13}^2)^{\frac{u-k_1-k_3}{2}}(x_{24}^2)^{\frac{u-k_2-k_4}{2}}.$$

Here, $x_{ij} = x_i - x_j$, and s, t, u are Mandelstam variables satisfying $s+t+u = \sum_{i=1}^4 k_i \equiv \Sigma$ [25]. We have also extracted a factor of Gamma functions,

$$\Gamma_{\{k_i\}}(s,t,u) = \Gamma\left[\frac{k_1+k_2-s}{2}\right]\Gamma\left[\frac{k_3+k_4-s}{2}\right]\Gamma\left[\frac{k_1+k_4-t}{2}\right]\Gamma\left[\frac{k_2+k_3-t}{2}\right]\Gamma\left[\frac{k_1+k_3-u}{2}\right]\Gamma\left[\frac{k_2+k_4-u}{2}\right], \quad (3)$$

that captures the contribution of double-trace operators universally present in the holographic limit [15]. All dynamic information is contained in $\mathcal{M}_{k_1k_2k_3k_4}$, known as the "Mellin amplitude." The four-point function $G_{k_1k_2k_3k_4}$ obeys Bose symmetry, which permutes operators. Bose symmetry acts on the Mellin amplitude by interchanging k_i, as well as permuting the Mandelstam variables s, t, u in the same way it acts on a flat-space amplitude.

Four-point correlators: $\mathcal{N}=2$ superconformal symmetry.—We now consider CFTs with $\mathcal{N}=2$ superconformal symmetry, focusing on the $\frac{1}{2}$–Bogomol'nyi-Prasad-Sommerfield ($\frac{1}{2}$-BPS) operators. These operators are of the form $\mathcal{O}_k^{a_1\ldots a_k}$, where $a_i = 1, 2$ are indices of the R-symmetry group $SU(2)_R$ [26]. The operator $\mathcal{O}_k^{a_1\ldots a_k}$ transforms in the spin $j_R = (k/2)$ representation of $SU(2)_R$ and has conformal dimensions $k = 2, 3, \ldots$. To conveniently keep track of the $SU(2)_R$ indices, we contract them with auxiliary two-component spinors v^a:

$$\mathcal{O}_k(x,v) = \mathcal{O}_k^{a_1\ldots a_k} v^{b_1}\ldots v^{b_k}\epsilon_{a_1b_1}\ldots\epsilon_{a_kb_k}. \quad (4)$$

We then consider their four-point functions [Eq. (1)] and define the Mellin amplitude $\mathcal{M}^{\mathcal{N}=2}_{k_1k_2k_3k_4}$ via Eq. (2).

The $\mathcal{N}=2$ superconformal symmetry imposes extra constraints on the form of correlators via the superconformal Ward identities [27]. Solving them leads to

$$G^{\mathcal{N}=2}_{k_1k_2k_3k_4} = G^{\mathcal{N}=2}_{0,k_1k_2k_3k_4} + R^{(2)}H^{\mathcal{N}=2}_{k_1k_2k_3k_4}, \quad (5)$$

where $G^{\mathcal{N}=2}_{0,k_1k_2k_3k_4}$ is the protected part of the correlator independent of marginal couplings. The factor $R^{(2)}$ is crossing symmetric and is fixed by superconformal symmetry to be

$$R^{(2)} = (v_1\cdot v_2)^2(v_3\cdot v_4)^2 x_{13}^2 x_{24}^2 (1-z\alpha)(1-\bar{z}\alpha). \quad (6)$$

Here, $v_i\cdot v_j = v_i^a v_j^b \epsilon_{ab}$, $\alpha = [(v_1\cdot v_3)(v_2\cdot v_4)]/[(v_1\cdot v_2)(v_3\cdot v_4)]$ is the $SU(2)_R$ cross ratio, and z, $\bar{z}$ are conformal cross ratios given by

$$z\bar{z} = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} = U, \qquad (1-z)(1-\bar{z}) = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2} = V. \quad (7)$$

All the dynamical information is contained in the simpler "reduced" correlator $H^{\mathcal{N}=2}_{k_1k_2k_3k_4}$, which can be viewed as a correlator of operators with *shifted* conformal dimensions $k_i + 1$ and shifted $SU(2)_R$ spins $(k_i/2) - 1$.

In the regime to be considered, corresponding to AdS tree level, the reduced correlator in fact captures *all* the information. To make it more precise, let us define a reduced Mellin amplitude via the reduced correlator

$$H^{\mathcal{N}=2}_{k_1k_2k_3k_4} = \int_{-i\infty}^{i\infty}[dsdt]K(x_{ij}^2; s,t,\tilde{u})\widetilde{\mathcal{M}}^{\mathcal{N}=2}_{k_1k_2k_3k_4}\Gamma_{\{k_i\}}(s,t,\tilde{u}).$$

Note that it is important to shift the u variable to $\tilde{u} = u - 2$ so that $s + t + \tilde{u} = \Sigma - 2$. The shift is to compensate the nonzero weights of the factor $R^{(2)}$ under conformal transformations. As a consequence, Bose symmetry acts differently in the full and reduced Mellin amplitudes as

$$\mathcal{M}^{\mathcal{N}=2}_{k_1k_2k_3k_4}: \text{ permuting } s,t,u,$$
$$\widetilde{\mathcal{M}}^{\mathcal{N}=2}_{k_1k_2k_3k_4}: \text{ permuting } s,t,\tilde{u}. \quad (8)$$

In the tree-level regime, the protected part $G^{\mathcal{N}=2}_{0,k_1k_2k_3k_4}$ does *not* contribute to the Mellin amplitude [7]. Rather it is generated by a contour pinching mechanism described in [17]. Therefore, full amplitudes are *completely* determined by reduced amplitudes, with the precise relation given by translating both sides of Eq. (5) into Mellin space

$$\mathcal{M}^{\mathcal{N}=2}_{k_1k_2k_3k_4} = \mathbb{R}^{(2)}\circ\widetilde{\mathcal{M}}^{\mathcal{N}=2}_{k_1k_2k_3k_4}. \quad (9)$$

The factor $R^{(2)}$ now becomes a difference operator $\mathbb{R}^{(2)}$ [7]. To obtain it, we interpret each monomial U^mV^n in

$$\frac{R^{(2)}}{(v_1\cdot v_2)^2(v_3\cdot v_4)^2 x_{13}^2 x_{24}^2} \quad (10)$$

as a difference operator $U^mV^n \to \mathbb{O}^{(2)}_{m,n}$ that acts on functions $f(s,t)$ according to

$$\mathbb{O}^{(\mathcal{N})}_{m,n}\circ f(s,t)=\frac{\Gamma_{\{k_i\}}(s-2m,t-2n,\Sigma-\mathcal{N}-s-t+2m+2n)}{\Gamma_{\{k_i\}}(s,t,\Sigma-s-t)}\times f(s-2m,t-2n). \quad (11)$$

Four-point correlators: $\mathcal{N}=4$ superconformal symmetry.—The kinematics of $\mathcal{N}=4$ is similar. The $\frac{1}{2}$-BPS operator, labeled by an integer $k=2,3,\ldots$, transforms in the rank-k symmetric traceless representation of the $SO(6)_R$ R-symmetry group and has dimension k. We keep track of the R-symmetry indices by using null $SO(6)$ vectors t^r [28]:

$$O_k(x,t)=O^{r_1\ldots r_k}(x)t^{r_1}\ldots t^{r_k},\qquad r_i=1,\ldots,6, \quad (12)$$

where $t\cdot t=0$. The $\mathcal{N}=4$ superconformal symmetry dictates that the four-point function is of the form [27,29]

$$G^{\mathcal{N}=4}_{k_1k_2k_3k_4}=G^{\mathcal{N}=4}_{0,k_1k_2k_3k_4}+R^{(4)}H^{\mathcal{N}=4}_{k_1k_2k_3k_4}, \quad (13)$$

where $G^{\mathcal{N}=4}_{0,k_1k_2k_3k_4}$ is the protected part, and $H^{\mathcal{N}=4}_{k_1k_2k_3k_4}$ is the reduced correlator. Note that the reduced correlator also has shifted quantum numbers, with dimensions k_i+2 and $SO(6)$ spin k_i-2 for each operator. The factor $R^{(4)}$ is determined by supersymmetry

$$R^{(4)}=t_{12}^2t_{34}^2x_{13}^4x_{24}^4(1-z\alpha)(1-\bar{z}\alpha)(1-z\bar{\alpha})(1-\bar{z}\bar{\alpha}) \quad (14)$$

and doubles the $\mathcal{N}=2$ factor [Eq. (6)]. Here $t_{ij}=t_i\cdot t_j$, and

$$\alpha\bar{\alpha}=\frac{t_{13}t_{24}}{t_{12}t_{34}}=\sigma,\qquad(1-\alpha)(1-\bar{\alpha})=\frac{t_{14}t_{23}}{t_{12}t_{34}}=\tau. \quad (15)$$

The full correlator $G^{\mathcal{N}=4}_{k_1k_2k_3k_4}$ gives rise to the full amplitude $\mathcal{M}^{\mathcal{N}=4}_{k_1k_2k_3k_4}$ via Eq. (2). The $\mathcal{N}=4$ reduced amplitude is similarly given by

$$H^{\mathcal{N}=4}_{k_1k_2k_3k_4}=\int_{-i\infty}^{i\infty}[dsdt]K(x_{ij}^2;s,t,\tilde{u})\widetilde{\mathcal{M}}^{\mathcal{N}=4}_{k_1k_2k_3k_4}\Gamma_{\{k_i\}}(s,t,\tilde{u}).$$

But note here that the shift in $\tilde{u}$ is by 4, i.e., $\tilde{u}=u-4$. The greater shift is due to the higher conformal weights of $R^{(4)}$. Bose symmetry again permutes s, t, u in $\mathcal{M}^{\mathcal{N}=4}_{k_1k_2k_3k_4}$ and s, t, $\tilde{u}$ in $\widetilde{\mathcal{M}}^{\mathcal{N}=4}_{k_1k_2k_3k_4}$. At AdS tree level, the protected part again does not contribute to the Mellin amplitude [16,17]. Therefore, the full amplitudes are determined by the reduced amplitudes via

$$\mathcal{M}^{\mathcal{N}=4}_{k_1k_2k_3k_4}=\mathbb{R}^{(4)}\circ\widetilde{\mathcal{M}}^{\mathcal{N}=4}_{k_1k_2k_3k_4}, \quad (16)$$

where we have promoted $R^{(4)}$ into a difference operator $\mathbb{R}^{(4)}$ [16,17]. The action of each monomial U^mV^n in $R^{(4)}/[(t_{12})^2(t_{34})^2x_{13}^4x_{24}^4]$ is given by Eq. (11) with $\mathcal{N}=4$.

Super gluon amplitudes.—We are now ready to discuss holographic correlators in specific theories. We start with super gluons in AdS_5 preserving $\mathcal{N}=2$ superconformal symmetry, which can be realized as D3 branes probing F theory singularities [30,31], or as $\mathcal{N}=4$ SYM with probe flavor D7 branes [32]. In both case, there is an $\mathrm{AdS}_5\times\mathrm{S}^3$ subspace in the holographic description, on which live localized degrees of freedom transforming in the adjoint representation of a color group G_F. These degrees of freedom form a vector multiplet, and its Kaluza-Klein reduction gives infinite towers of $\frac{1}{2}$-BPS superconformal multiplets. We refer to the $\frac{1}{2}$-BPS superprimaries as super gluons. At large central charge, gravity decouples and one has only a spin-1 gauge theory. Note S^3 has isometry $SO(4)=SU(2)_R\times SU(2)_L$. The first factor is identified with the $\mathcal{N}=2$ R-symmetry group, while the second $SU(2)_L$ is a global symmetry suppressed in the above discussion. The operator $\mathcal{O}_k$ has spin $[(k-2)/2]$ under $SU(2)_L$ [31]. We can similarly contract the indices with $k-2$ $SU(2)_L$ spinors $\bar{v}^{\bar{a}}$, $\bar{a}=1,2$. In reduced correlators, v and $\bar{v}$ further recombine into null vectors of $SO(4)$ via Pauli matrices and appear only as polynomials of t_{ij} [7]:

$$t^{r'}=\sigma^{r'}_{a\bar{a}}v^a\bar{v}^{\bar{a}},\qquad r'=1,\ldots,4,\qquad t\cdot t=0. \quad (17)$$

To write down the super gluon amplitudes, let us choose, without loss of generality, the ordering $k_1\le k_2\le k_3\le k_4$, and distinguish two cases:

$$k_1+k_4\ge k_2+k_3(\text{case I}),\qquad k_1+k_4<k_2+k_3(\text{case II}).$$

To measure the deviation from the equal weight case $k_i=(\Sigma/4)$, it is useful to introduce the following parameters:

$$\kappa_s=|k_3+k_4-k_1-k_2|,\qquad \kappa_t=|k_1+k_4-k_2-k_3|,$$
$$\kappa_u=|k_1+k_3-k_2-k_4|. \quad (18)$$

The reduced Mellin amplitudes are given by [7,33]

$$\widetilde{\mathcal{M}}^{\mathcal{N}=2}_{k_1k_2k_3k_4}=\sum_{\substack{i+j+k=\mathcal{E}-2\\0\le i,j,k\le\mathcal{E}-2}}\frac{\sigma^i\tau^j}{i!j!k!(\frac{2i+\kappa_u}{2})!(\frac{2j+\kappa_t}{2})!(\frac{2k+\kappa_s}{2})!}\times\left[\frac{n_s^{i,j}c_s}{s-s_M+2k}+\frac{n_t^{i,j}c_t}{t-t_M+2j}+\frac{n_u^{i,j}c_u}{\tilde{u}-u_M+2i}\right]\times I(t_{ab}),$$

which has been rewritten to manifest Bose symmetry. Let us unpack this expression a bit. Here,

$$\mathcal{E}=\frac{k_1+k_2+k_3-k_4}{2}\,(\text{case I}),\qquad \mathcal{E}=k_1(\text{case II})$$

is the "extremality," which determines the complexity of the amplitude. After extracting a factor in t_{ab},

$$I(t_{ab})=t_{34}^{\frac{\kappa_s}{2}}t_{24}^{\frac{\kappa_u}{2}}(t_{12}t_{34})^{-\mathcal{E}+2}\times\begin{cases}t_{14}^{\frac{\kappa_t}{2}} & (\text{case I})\\ t_{23}^{\frac{\kappa_t}{2}} & (\text{case II})\end{cases},\qquad (19)$$

the reduced Mellin amplitudes are degree-$(\mathcal{E}-2)$ polynomials in σ and τ defined in Eq. (15). The color dependence is captured by the color factors

$$c_s=f^{I_1I_2J}f^{JI_3I_4},\qquad c_t=f^{I_1I_4J}f^{JI_2I_3},\qquad c_u=f^{I_1I_3J}f^{JI_2I_4},$$

where f^{IJK} are the structure constants of the color group G_F. Thanks to the Jacobi identity, they satisfy $c_s+c_t+c_u=0$. The kinematic factors $n^{i,j}_{s,t,u}$ are given by

$$\begin{aligned}n_s^{i,j}&=\frac{1}{t-t_M+2j}-\frac{1}{\tilde{u}-u_M+2i},\\ n_t^{i,j}&=\frac{1}{\tilde{u}-u_M+2i}-\frac{1}{s-s_M+2k},\\ n_u^{i,j}&=\frac{1}{s-s_M+2k}-\frac{1}{t-t_M+2j}.\end{aligned}\qquad (20)$$

The nonlocality of these expressions is only superficial and should not raise any alarm. In fact, a similar phenomenon occurs in flat space [34]. Evidently, $n^{i,j}_{s,t,u}$ obey

$$n_s^{i,j}+n_t^{i,j}+n_u^{i,j}=0,\qquad (21)$$

which gives rise to a realization of the "color-kinematic duality" [1] in AdS. In contrast to the duality pointed out in [7], this new realization has the *same* form for both massless ($k_i=2$) and massive ($k_i>2$) super gluons. Finally, the remaining parameters are given by

$$\begin{aligned}s_M&=\min\{k_1+k_2,k_3+k_4\}-2,\\ t_M&=\min\{k_1+k_4,k_2+k_3\}-2,\\ u_M&=\min\{k_1+k_3,k_2+k_4\}-2.\end{aligned}\qquad (22)$$

Super graviton amplitudes.—Let us now take a further step with the color-kinematic duality [Eq. (21)] and replace color factors $c_{s,t,u}$ in each monomial $\sigma^i\tau^j$ by kinematic factors $n^{i,j}_{s,t,u}$. The result is

$$\widetilde{\mathcal{M}}^{\mathcal{N}=2\otimes\mathcal{N}=2}_{k_1k_2k_3k_4}=\sum_{\substack{i+j+k=\mathcal{E}-2\\0\le i,j,k\le\mathcal{E}-2}}\frac{\sigma^i\tau^j}{i!j!k!(\frac{2i+\kappa_u}{2})!(\frac{2j+\kappa_t}{2})!(\frac{2k+\kappa_s}{2})!}\times\frac{-9I(t_{ab})}{(s-s_M+2k)(t-t_M+2j)(\tilde{u}-u_M+2i)}.$$

To interpret it as $\mathcal{N}=4$ reduced amplitudes, we need to *replace* the $\tilde{u}$ variable with the $\mathcal{N}=4$ one, as required by Bose symmetry of $\widetilde{\mathcal{M}}^{\mathcal{N}=4}_{k_1k_2k_3k_4}$. Furthermore, we replace the $SO(4)$ vectors $t^{r'}$ by $SO(6)$ null vectors [35]. Remarkably, it gives all the super graviton reduced Mellin amplitudes of IIB supergravity on $\mathrm{AdS}_5\times\mathrm{S}^5$ [16,17]

$$\widetilde{\mathcal{M}}^{\mathcal{N}=4}_{k_1k_2k_3k_4}=\sqrt{k_1k_2k_3k_4}\times\widetilde{\mathcal{M}}^{\mathcal{N}=2\otimes\mathcal{N}=2}_{k_1k_2k_3k_4}\qquad (23)$$

up to an overall factor [36]. This generalizes the "double copy relation" [2] into AdS space for four-point functions [38]. In fact, redefining the super gravitons by $\mathcal{O}_k\to\mathcal{O}_k/\sqrt{k}$ gets rid of the normalization factor and gives the super graviton three-point functions also as the square of the super gluon ones [39].

Bi-adjoint scalar amplitudes.—In flat space, one can also replace kinematic factors by color factors and obtain amplitudes of bi-adjoint scalars. We show that the same happens in AdS, and it serves as a nontrivial check. Note that in the above example the superconformal factor $R^{(2)}$ was doubled to $R^{(4)}$ [c.f. Eqs. (6) and (14)]. Going in the opposite direction, we expect $R^{(0)}=1$, i.e., the resulting theory has no supersymmetry. Moreover, since the internal spaces changed from S^3 to S^5, a reasonable guess is that this sequence starts with S^1, which will soon be confirmed. The symmetry groups are therefore $SO(\mathcal{N}+2)$, and we recall that operators in the reduced amplitudes transform in the rank-(k_i-2) symmetric traceless representation.

Note that for $\mathcal{N}=0$, the null polarization vectors are two-component. Since we can rescale the null vectors, we are left with two inequivalent choices:

$$t_\pm=\frac{1}{\sqrt{2}}(1,\pm i).\qquad (24)$$

The dimension k operator $\mathcal{O}^\pm_k\equiv\mathcal{O}_k(x,t_\pm)$ has $\pm(k-2)$ charges under $U(1)=SO(2)$, depending on the polarization chosen. Moreover, we assume the scalar interactions are only cubic. Then, $U(1)$ charge conservation dictates that at least one of the κ_s, κ_t, κ_u parameters in Eq. (18) is zero. For the chosen ordering $k_1\le k_2\le k_3\le k_4$, we must impose the condition $\kappa_t=0$. This leaves

$$\langle\mathcal{O}^+_{k_1}\mathcal{O}^-_{k_2}\mathcal{O}^-_{k_3}\mathcal{O}^+_{k_4}\rangle,\quad\text{or}\quad\langle\mathcal{O}^-_{k_1}\mathcal{O}^+_{k_2}\mathcal{O}^+_{k_3}\mathcal{O}^-_{k_4}\rangle,\qquad (25)$$

which have identical amplitudes [41]. Noting

$$\sigma = 1, \qquad \tau = 0, \qquad \mathrm{I}(t_{ab}) = 1 \tag{26}$$

and replacing $n^{ij}_{s,t,u}$ with the color factors $c'_{s,t,u}$ for another color group G'_F, we find

$$\mathcal{M}^{\mathcal{N}=0}_{k_1k_2k_3k_4} = \sum_{\substack{i+k=\mathcal{E}-2\\ 0\le i,k\le \mathcal{E}-2}} \frac{-2\mathcal{N}_{k_1k_2k_3k_4}}{i!k!(\frac{2i+\kappa_u}{2})!(\frac{2k+\kappa_s}{2})!} \times \left[\frac{c_s c'_s}{s-s_M+2k} + \frac{c_t c'_t}{t-t_M} + \frac{c_u c'_u}{u-u_M+2i}\right]. \tag{27}$$

We dropped the tildes because nonsupersymmetric theories have only full amplitudes [Eq. (2)], and there is no shift in the u variable. We also included a to-be-determined k_i-dependent normalization factor $-2\mathcal{N}_{k_1k_2k_3k_4}$ as in the supergravity case. Remarkably, Eq. (27) can be rewritten as the sum of *three* AdS_5 scalar exchange diagrams:

$$\mathcal{N}_{k_1k_2k_3k_4}\left[\frac{c_s c'_s}{p_s-1}\mathcal{S}^{(s)}_{p_s} + \frac{c_t c'_t}{p_t-1}\mathcal{S}^{(t)}_{p_t} + \frac{c_u c'_u}{p_u-1}\mathcal{S}^{(u)}_{p_u}\right], \tag{28}$$

where $\mathcal{S}^{(s)}_p$ is the amplitude of exchanging a dimension-p scalar in the s-channel (and similarly for the other two channels) [42]:

$$\mathcal{S}^{(s)}_p = \sum_{m=0}^{\infty} \frac{-2(\frac{2+p-k_1-k_2}{2})_m(\frac{2+p-k_3-k_4}{2})_m}{(s-p-2m)m!(p-1)_m\Gamma[\frac{k_1+k_2-p}{2}]\Gamma[\frac{k_3+k_4-p}{2}]} \times \frac{\Gamma[p]}{\Gamma[\frac{k_1-k_2+p}{2}]\Gamma[\frac{k_2-k_1+p}{2}]\Gamma[\frac{k_3-k_4+p}{2}]\Gamma[\frac{k_4-k_3+p}{2}]}.$$

Moreover, the weights $p_{s,t,u}$ are precisely those selected by the $U(1)$ charge conservation

$$p_s = k_2 - k_1 + 2, \qquad p_t = k_1 + k_4 - 2, \qquad p_u = k_3 - k_1 + 2.$$

Note that Eq. (27) is equivalent to Eq. (28), is highly nontrivial, and *a priori* does not need to happen. We can further fix the normalization $\mathcal{N}_{k_1k_2k_3k_4}$ by noting $\mathcal{N}_{k_1k_2k_3k_4}/(p_s-1)$, etc., have the interpretation of products of three-point function coefficients $C_{k_1k_2p_s}C_{k_3k_4p_s}$. The solution, up to a k_i-independent overall factor, is

$$C_{k_1k_2k_3} = \frac{1}{\sqrt{(k_1-1)(k_2-1)(k_3-1)}}. \tag{29}$$

Finally, we confirm by direct calculation that the theory is conformally coupled scalars on $\mathrm{AdS}_5\times \mathrm{S}^1$. The conformal mass on this manifold is $M^2_{\mathrm{conf}} = -4$ [43]. Decomposing the scalar field ϕ into S^1 modes $\phi(z,\tau) = \sum_{n=-\infty}^{\infty}\varphi_n(z)e^{in\tau}$, we find each mode has mass $M^2_n = n^2 - 4$. This translates into a conformal dimension $|n|+2$, agreeing with our charge-dimension relation $n = \pm(k-2)$. We can further check three-point functions. A cubic vertex ϕ^3 in $\mathrm{AdS}_5\times \mathrm{S}^1$ gives rise to infinitely many AdS_5 cubic vertices $\sum \varphi_{n_1}\varphi_{n_2}\varphi_{n_3}$, where $\{n_i\}$ conserve the $U(1)$ charge. Using the result of [44], it is straightforward to show that three-point functions are precisely as in Eq. (29). Note that both $C_{k_1k_2k_3}$ and $\mathcal{N}_{k_1k_2k_3k_4}$ can be set to one by redefining $\mathcal{O}_k \to \sqrt{k-1}\mathcal{O}_k$. Then, the double copy relation also holds for three-point functions.

Discussions.—In this Letter, we found an extension of the double copy relation in curved spacetimes that relates all tree-level four-point functions of $\mathrm{AdS}_5\times \mathrm{S}^5$ IIB supergravity, $\mathrm{AdS}_5\times \mathrm{S}^3$ SYM, and $\mathrm{AdS}_5\times \mathrm{S}^1$ bi-adjoint scalars. Although our result is supersymmetric, it has immediate implications on *bosonic* Einstein gravity and Yang-Mills theory in AdS_5 with no internal factor. Thanks to supersymmetry, four-graviton and four-gluon amplitudes can be obtained from the reduced correlators of $k_i = 2$ super gravitons and super gluons by action of differential operators [45]. At tree level, these spinning correlators are identical to the ones in bosonic theories because the exchanged fields are the same [46]. Our result, then, indicates that the bosonic amplitudes should also be related by a double copy construction [47], the details of which we will leave for a future work. Another interesting direction is to extend our results to higher points, although more data on holographic correlators is needed [48]. While the focus here is AdS_5 amplitudes, double copy relations for other backgrounds are also worth exploring. In particular, the AdS_7 case [21] admits similar definitions of reduced amplitudes [50,51]. Finally, it would be interesting to explore extensions at higher genus, where the relevant CFT techniques were developed in [52].

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