Computable and Operationally Meaningful Multipartite Entanglement Measures

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Multipartite entanglement is an essential resource for quantum communication, quantum computing, quantum sensing, and quantum networks. The utility of a quantum state $|\psi\rangle$ for these applications is often directly related to the degree or type of entanglement present in $|\psi\rangle$. Therefore, efficiently quantifying and characterizing multipartite entanglement is of paramount importance. In this work, we introduce a family of multipartite entanglement measures, called concentratable entanglements. Several well-known entanglement measures are recovered as special cases of our family of measures, and hence we provide a general framework for quantifying multipartite entanglement. We prove that the entire family does not increase, on average, under local operations and classical communications. We also provide an operational meaning for these measures in terms of probabilistic concentration of entanglement into Bell pairs. Finally, we show that these quantities can be efficiently estimated on a quantum computer by implementing a parallelized SWAP test, opening up a research direction for measuring multipartite entanglement on quantum devices.

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Introduction.—The presence of entanglement in quantum states is widely recognized as one of, if not the, defining property of quantum mechanics [1]. Since the development of quantum information theory [2,3] it was realized that entanglement is a fundamental resource [4,5] for quantum communications [6–9], quantum cryptography [10,11], and quantum computing [12–14]. Recent advances in quantum control technologies have made it possible to harness the power of entanglement for quantum-enhanced sensing [15–17] and communications [18–20], and for showing quantum advantage using near-term quantum computers [21]. While the ubiquity of entangled quantum states as a resource is clear, their utility for these applications often depends on the degree of entanglement in the quantum state.

The nature of quantum entanglement is well understood for bipartite pure quantum states [2,13,22]. However, the same cannot be said for the multipartite entanglement of pure states [23], where the complexity of entanglement scales exponentially with the number of parties. In fact, already for a system of 3 qubits there exist two different, and inequivalent, types of genuine tripartite entanglement, such that states of the two different kinds cannot be exactly transformed onto the other via the action of local operations and classical communications (LOCC) [24]. While the study of multipartite entanglement has received considerable attention [25–33], there does not exist a single unambiguous way to detect, quantify, and characterize multipartite entanglement. Hence, improving our knowledge on the nature of the entanglement between multiple parties is not only crucial to better understanding the underlying structure of quantum mechanics, but it is also a fundamental step toward enhancing emergent technologies such as distributed quantum sensing [34], longer baseline telescopes [35], and various quantum Internet applications [36–38].

The advent of quantum computing technologies brings forth the possibility of verifying and characterizing the multipartite entanglement present in states prepared on these near-term quantum devices. In this context, entanglement measures that are not only theoretically relevant, but that can also be estimated via quantum algorithms [31, 39–44], become particularly attractive as characterization tools. For instance, it was shown [29,45] that given an n-qubit state $|\psi\rangle$, the linear entropies $\frac{1}{2}(1-\mathrm{Tr}\rho_j^2)$ of the single-qubit reduced states ρ_j can be used to study the entanglement in $|\psi\rangle$. Moreover, since the SWAP test [46–50] can be used to compute linear entropies, these measures can be efficiently estimated on quantum computers or optical quantum devices.

In this work, we introduce a family of quantities, called *concentratable entanglements*, which characterize and quantify the multipartite entanglement in an arbitrary n-qubit pure state $|\psi\rangle$. We first prove that each concentratable entanglement does not increase, on average, under LOCC operations, and hence forms an entanglement monotone.

Then, we show that by combining concentratable entanglements one can obtain several quantities of interest, which can quantify properties such as the entanglement in, and between, subsystems, as well as the total entanglement in $|\psi\rangle$. We then discuss how these quantities can be efficiently estimated on a quantum computer given two copies of $|\psi\rangle$, and employing constant-depth n-qubit parallelized SWAP tests. Finally, we discuss the operational meaning of the concentratable entanglement as the probability of obtaining Bell pairs between qubits in the different copies of $|\psi\rangle$.

Our results generalize previous results in the literature in the sense that (i) several entanglement measures correspond to a special case of the concentratable entanglements [28,29,32,45,51] and (ii) we prove a conjecture in Ref. [42], where it was hypothesized that the parallelized SWAP test can provide the basis for constructing a pure state multipartite entanglement monotone. Finally, the broader implication of our work is to promote a research direction of studying multipartite entanglement using quantum devices, such as cloud-based quantum computers.

Concentratable entanglement.—Consider an n-qubit pure quantum state $|\psi\rangle$. We denote $\mathcal{S} = \{1, 2, \ldots, n\}$ as the set of labels for each qubit, and $\mathcal{P}(\mathcal{S})$ as its power set (i.e., the set of subsets, with cardinality $|\mathcal{S}| = 2^n$). We introduce the concentratable entanglements as a family of entanglement monotones that characterize and quantify the multipartite entanglement in $|\psi\rangle$.

Definition 1.—For any set of qubit labels $s \in \mathcal{P}(\mathcal{S})\backslash\{\emptyset\}$, the concentratable entanglement is defined as

$$\mathcal{C}_{|\psi\rangle}(s) = 1 - \frac{1}{2^{c(s)}} \sum_{\alpha\in\mathcal{P}(s)} \mathrm{Tr}\rho_\alpha^2, \tag{1}$$

where $c(s)$ is the cardinality of the set s, and $\mathcal{P}(s)$ its power set. Here we denote by ρ_α the joint reduced state, associated to $|\psi\rangle$, of the subsystems labeled by the elements in α (with $\alpha = \emptyset$ leading to $\rho_\alpha := 1$).

Equation (1) shows that each $\mathcal{C}_{|\psi\rangle}(s)$ is the average of the entanglement between the subsets of qubits with labels in s and the rest of the system. This means that different concentratable entanglements can measure both bipartite and multipartite entanglements according to how s is defined. For instance taking the smallest set possible, i.e., $s = \{j\}$ with $j = 1, \ldots, n$, one finds $\mathcal{C}_{|\psi\rangle}(\{j\}) = \frac{1}{2}(1 - \mathrm{Tr}\rho_j^2)$. Thus, when averaged over $\{j\}$, one recovers the measures in [29,45] which quantify the bipartite entanglement between the jth qubit and the rest. On the other hand, taking the largest set possible, i.e., $s = \mathcal{S}$, $\mathcal{C}_{|\psi\rangle}(\mathcal{S})$ quantifies the overall entanglement in $|\psi\rangle$ across all cuts, and as discussed below, this case corresponds to the entanglement measure conjectured in [42]. Moreover, in this case we also recover the entanglement measure of [51] as a special case of the concentratable entanglements. Including these extremal cases, there are a total of $2^n - 1$ concentratable entanglements according to Definition 1.

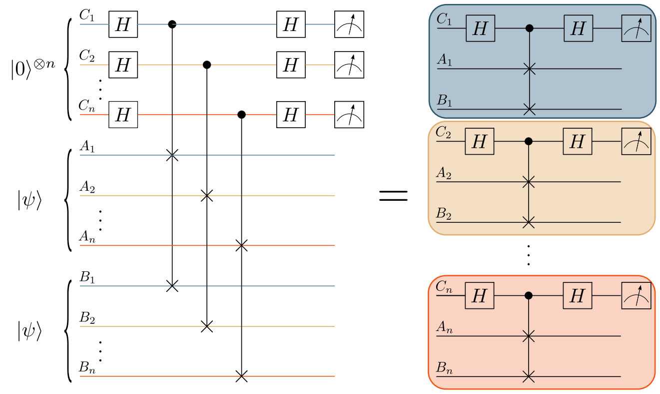

FIG. 1. Circuit for the n-qubit parallelized SWAP test. Given two copies of the quantum state $|\psi\rangle$and n ancilla qubits, the n-qubit parallelized SWAP test consists of employing the kth ancilla to perform a controlled swap test on the kth qubit of each copy of $|\psi\rangle$. Since the n SWAP test can be factorized, one can perform them in parallel, leading to a constant-depth circuit.

Efficient Computation.—A fundamental aspect of the concentratable entanglements is that they can be efficiently estimated on a quantum computer. While each purity, $\mathrm{Tr}[\rho_\alpha^2]$, in Eq. (1) can be computed via an overlap test [49], one can also use two copies of the state $|\psi\rangle$ and n ancilla qubits to employ the n-qubit parallelized SWAP test depicted in Fig. 1 (see Supplemental Material [52] for a discussion on the SWAP test). From Fig. 1, it is clear that the kth ancilla qubit is used to perform a controlled SWAP test on the kth qubit of each copy of $|\psi\rangle$. The tests are independent and thus factorizable. This implies that the n-qubit parallelized SWAP test has a constant circuit depth for any number of qubits.

Given the n-qubit parallelized SWAP test, we define the following relevant quantities. First, let $p(z)$ be the probability of measuring the z bitstring on the n control qubits, and let $\mathcal{Z} = \{0, 1\}^n$ be the set of all such bitstrings. Then, the following proposition (proved in the Supplemental Material [52]) holds.

Proposition 1.—The concentratable entanglement can be computed from the outcomes of the n-qubit parallelized SWAP test as

$$\mathcal{C}_{|\psi\rangle}(s) = 1 - \sum_{z\in\mathcal{Z}_0(s)} p(z), \tag{2}$$

where $\mathcal{Z}_0(s)$ is the set of all bitstrings with 0's on all indices in s.

Proposition 1 shows that $\mathcal{C}_{|\psi\rangle}(s)$ can be computed by performing the parallelized SWAP test on all qubits and adding the probabilities where the control qubits with indices in s are measured in the $|0\rangle$ state. Since this corresponds to a conditional probability, one can also perform SWAP tests only on the qubits with indexes in s [requiring just $c(s)$ ancillary qubits] and express the concentratable entanglement as $\mathcal{C}_{|\psi\rangle}(s) = 1 - p(\mathbf{0}_x)$. Here $p(\mathbf{0}_x) = \sum_{z\in\mathcal{Z}_0(s)} p(z)$ denotes the probability of obtaining the all-zero result from the SWAP test on the qubits with labels in s.

Here we remark that Eqs. (1) and (2) are complimentary in the sense that the number of terms in the summations are inversely proportional. That is, the summation in Eq. (1) contains $2^{c(s)}$ terms, while that of Eq. (2) contains $2^{n-c(s)}$ terms. Hence, we remark that it is preferable to employ Eq. (2) when analyzing multipartite entanglement, as this avoids potentially having to compute a prohibitively large number of purities as those required in other entanglement measures [59]. For instance, if $[n - c(s)] \in \mathcal{O}[\log(n)]$ then Eq. (2) only contains the number of terms in $\mathcal{O}[poly(n)]$. For the purpose of analyzing multipartite entanglement we henceforth focus on Eq. (2).

Finally, we remark, that, as shown in the Supplemental Material [52], the SWAP test still works when the two copies of $|\psi\rangle$ are not exactly the same. Specifically, let $|\psi\rangle$ and $|\psi'\rangle$ be two faulty copies of the state with $\||\psi\rangle\langle\psi| - |\psi'\rangle\langle\psi'|\|_1 \leqslant \varepsilon$; then, we find that the error in the concentratable entanglement is upper bounded by $\mathcal{O}(\varepsilon^2)$, indicating that small errors in the state lead to a small concentratable entanglement difference.

Properties of $\mathcal{C}_{|\psi\rangle}(s)$.—We now present our main results which provide properties and additional insight for the concentratable entanglements. The proofs of these results are provided in the Supplemental Material [52].

Theorem 1.—The concentratable entanglement has the following properties:

(1): $\mathcal{C}_{|\psi\rangle}(s)$ is nonincreasing, on average, under LOCC operations and hence is a well-defined pure state entanglement measure. (2): If $|\psi\rangle$ is a separable state of the form $|\psi\rangle = \bigotimes_{j=1}^{n} |\phi_j\rangle$, then $\mathcal{C}_{|\psi\rangle}(s) = 0$ for all $s \in \mathcal{P}(\mathcal{S})\backslash\{\emptyset\}$. (3): $\mathcal{C}_{|\psi\rangle}(s') \leqslant \mathcal{C}_{|\psi\rangle}(s)$ if $s' \subseteq s$. (4): Subadditivity, $\mathcal{C}_{|\psi\rangle}(s \cup s') \leqslant \mathcal{C}_{|\psi\rangle}(s) + \mathcal{C}_{|\psi\rangle}(s')$ for $s \cap s' = \emptyset$. (5): Continuity, let $|\psi\rangle$ and $|\phi\rangle$ be two states such that $\||\psi\rangle\langle\psi| - |\phi\rangle\langle\phi|\|_1 \leqslant \varepsilon$; then $|\mathcal{C}_{|\psi\rangle}(s) - \mathcal{C}_{|\phi\rangle}(s)| \leqslant 2\varepsilon$.

Here, property (3) guarantees that the concentratable entanglement always measures less entanglement in any subsystem of s. In addition, we remark that, by combining properties (3) and (4), we have $\{\mathcal{C}_{|\psi\rangle}(s), \mathcal{C}_{|\psi\rangle}(s')\} \leqslant \mathcal{C}_{|\psi\rangle}(s \cup s') \leqslant \mathcal{C}_{|\psi\rangle}(s) + \mathcal{C}_{|\psi\rangle}(s')$ for $s \cap s' = \emptyset$.

To further understand how the concentratable entanglements measure entanglement, we provide additional details on the probabilities $p(z)$. First, consider the following explicit formula for the probabilities $p(z)$.

Proposition 2.—Given the expansion of the state $|\psi\rangle = \sum_{\boldsymbol{i}} c_{\boldsymbol{i}} |i_1\rangle|i_2\rangle, \ldots, |i_n\rangle$, the probability $p(z)$ for any $z \in \mathcal{Z}$ is given by

$$p(z) = \frac{1}{2^n} \sum_{\boldsymbol{i},\boldsymbol{i}',\boldsymbol{j},\boldsymbol{j}'} c_{\boldsymbol{i}} c_{\boldsymbol{i}'} c_{\boldsymbol{j}}^* c_{\boldsymbol{j}'}^* T_{\boldsymbol{i}\boldsymbol{i}'\boldsymbol{j}\boldsymbol{j}'}(z), \tag{3}$$

where $T_{\boldsymbol{i}\boldsymbol{i}'\boldsymbol{j}\boldsymbol{j}'}(z) = \prod_k [\delta_{i_k j_k}\delta_{i'_k j'_k} + (-1)^{z_k}\delta_{i_k j'_k}\delta_{i'_k j_k}]$, and where z_k denotes the kth bit in z.

Alternatively, one can also express $p(z)$ as a function of purities of reduced states of $|\psi\rangle$. Let us define as $w(z)$ the Hamming weight of z, and let $\mathcal{S}_1 \subseteq \mathcal{S}$ be the set of labels for the bits in z that are equal to 1 [with $|\mathcal{S}_1| = w(z)$]. Finally, let c_{hs} be the cardinality of $\mathcal{S}_h \cap s$. One finds

$$p(z) = \frac{1}{2^n} \sum_{s\in\mathcal{P}(\mathcal{S})} (-1)^{c_{hs}} \mathrm{Tr}\rho_x^2. \tag{4}$$

Equation (4) leads to the following proposition.

Proposition 3.—If z has odd Hamming weight [if $w(z)$ is odd], then $p(z) = 0$.

Proposition 3 has several implications. First, one can see that by performing the n-qubit parallelized SWAP test, one can never measure a bitstring with an odd number of ones. Then, the formula for the concentratable entanglements in Proposition 1 can be expressed as

$$\mathcal{C}_{|\psi\rangle}(s) = \sum_{z\in\mathcal{Z}_1^{\mathrm{even}}(s)} p(z), \tag{5}$$

where we recall that $\mathcal{Z}_0(s)$ was defined as the set of all bitstrings with 0's on all indices in s, and where we respectively define $\mathcal{Z}_1^{\mathrm{even}}(s)$ and $\mathcal{Z}_1^{\mathrm{odd}}(s)$ as the compliments of $\mathcal{Z}_0(s)$ with even and odd Hamming weight, such that $\mathcal{Z}_0(s) \cup \mathcal{Z}_1^{\mathrm{even}}(s) \cup \mathcal{Z}_1^{\mathrm{odd}}(s) = \mathcal{Z}$. Simply said, $\mathcal{Z}_1^{\mathrm{even}}(s)$ is the set of bitstrings with even Hamming weight and with at least a 1 in an index in s. For instance, if $s = \mathcal{S}$ (i.e., when the concentratable entanglement measures all the correlations in $|\psi\rangle$), then $\mathcal{C}_{|\psi\rangle}(\mathcal{S}) = 1 - p(\mathbf{0}) = \sum_{z:w(z)\mathrm{even}} p(z)$, and we recover exactly the conjectured measure of entanglement of [42].

Equation (5) shows that the information of the multipartite entanglement in $|\psi\rangle$ is encoded in the probabilistic outcomes of the n-qubit parallelized SWAP test when an even number of control qubits are measured in the $|1\rangle$ state. For instance, the probability of measuring a bitstring with Hamming weight $w(z) = 2$, where $z_k = z_{k'} = 1$ contains information regarding the bipartite entanglement between qubits k and k'. Specifically, the following proposition holds.

Proposition 4.—Let $|\psi\rangle$ be a biseparable state $|\psi\rangle = |\psi\rangle_A \otimes |\psi\rangle_B$. Then for any bitstring z of Hamming weight $w(z) = 2$, where $z_k = z_{k'} = 1$ we have $p(z) = 0$ if qubit k is in subsystem A, and qubit k' is in subsystem B.

Proposition 4 can be generalized to show that the probability of measuring a bitstring with Hamming weight

$w(z)$ contains information regarding the entanglement between the qubits with labels in $\mathcal{S}_1$. That is, one can prove that $p(z)$ is equal to zero if the qubits in $\mathcal{S}_1$ belong to nonentangled partitions of $|\psi\rangle$.

Here we remark that while the $p(z)$ contain information regarding the multipartite entanglement in $|\psi\rangle$, these probabilities are generally not entanglement monotones. The exception is $p(\mathbf{1})$ when n is even, i.e., the probability of measuring all the control qubits in the $|1\rangle$ state. For this special case we find the following.

Proposition 5.—If n is even, then $p(\mathbf{1})$ is an entanglement monotone. Moreover, in this case $p(\mathbf{1}) = \tau_{(n)}/2^n$. where $\tau_{(n)}$ is the n tangle.

The n tangle was introduced in [28] as a measure of multipartite entanglement in n qubits states that generalizes the concurrence [60,61]. The n tangle of a pure state $|\psi\rangle$ is $\tau_{(n)} = |\langle\psi|\tilde{\psi}\rangle|^2$, with $|\tilde{\psi}\rangle = \sigma_y^{\otimes n}|\psi^*\rangle$ where σ_y is the Pauli-Y operator, and $|\psi^*\rangle$ is the conjugate of $|\psi\rangle$. Hence, for the special case of 2 qubits one finds that $C_{|\psi\rangle}(s) = \tau_{(2)}/4 = C^2/4$ for all $s \in \mathcal{P}(\mathcal{S})$, where here C denotes the concurrence [60]. In general, we see from Proposition 5 that the n tangle is always one of the terms in the summation of Eq. (5), and hence is included in the concentratable entanglements.

Interestingly, $p(z)$ can also be interpreted as the probability of concentrating the entanglement in the two copies of $|\psi\rangle$ and "distilling" Bell pairs.

Proposition 6.—Given two copies of $|\psi\rangle$, if the kth control qubit of the n-qubit parallelized SWAP test was measured in the state $|1\rangle$, then the joint postmeasured state of the kth qubits of each copy of $|\psi\rangle$ is the Bell state $|\Phi^-\rangle = (1/\sqrt{2})(|01\rangle - |10\rangle)$.

Proposition 6 shows that when one measures [with probability $p(z)$] a bitstring z with (even) Hamming weight $w(z)$, then one has produced $w(z)$ Bell pairs between qubits in the different copies of $|\psi\rangle$ with indices in $\mathcal{S}_h$. This protocol is schematically shown in Fig. 2. In addition, Proposition 6 also sheds additional light on the concentratable entanglement $\mathcal{C}_{|\psi\rangle}(s)$ as the probability of obtaining any of the qubit pairs with labels in s in a Bell pair when performing a SWAP test.

Examples.—Let us now showcase how the probabilities $p(z)$ and the concentratable entanglement can be used to characterize and quantify the multipartite entanglement in n-qubit W and GHZ states. First, let us consider the W state $|W\rangle = (1/\sqrt{n})\sum_{\mathbf{x}:w(\mathbf{x})=1}|\mathbf{x}\rangle$, i.e., the equal superposition of all states with Hamming weight equal to 1. A direct calculation shows that $p(z) = (1/n^2)$ for all z with $w(z) = 2$ and $p(z) = 0$ for all z with $w(z) > 2$. That is, in Eq. (5) one can only have terms where z has only two 1s. Concomitantly, when employing the n-qubit parallelized SWAP test one cannot concentrate the multipartite entanglement in $|W\rangle$ to simultaneously produce more than two Bell pairs. Then, noting that for a given s there are $\sum_{\mu=1}^{c(s)}\binom{n-\mu}{1} = c(s)[2n - c(s) - 1]/2$ nonzero terms in Eq. (5), one finds $\mathcal{C}_{|W\rangle}(s) = c(s)[2n - c(s) - 1]/2n^2$.

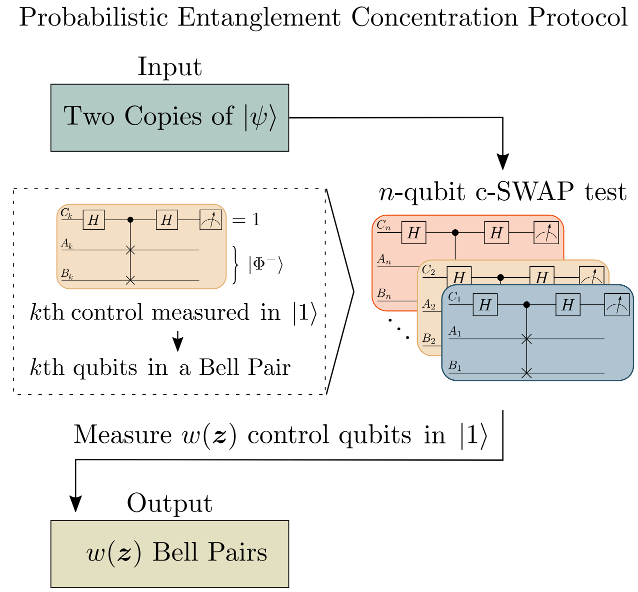

FIG. 2. Protocol for concentrating entanglement. Given two copies of $|\psi\rangle$, one can employ the n-qubit parallelized SWAP test to prepare Bell pairs between qubits in the different copies of $|\psi\rangle$. Specifically, measuring the kth control qubit in the state $|1\rangle$ implies that the joint state of the kth qubit of each copy of $|\psi\rangle$ is the Bell state $|\Phi^-\rangle = (1/\sqrt{2})(|01\rangle - |10\rangle)$. Hence, a single run of the n-qubit parallelized SWAP test has a probability $p(z)$ of concentrating the multipartite entanglement in the copies of $|\psi\rangle$ and producing $w(z)$ Bell pairs.

On the other hand, consider the GHZ state $|GHZ\rangle = (1/\sqrt{2})(|\mathbf{0}\rangle + |\mathbf{1}\rangle)$. We now find $p(z) = (1/2^n)$ for all z with (even) Hamming weight $w(z) \geqslant 2$. Unlike the W state, when employing the n-qubit parallelized SWAP test one can obtain up to n simultaneous Bell pairs. In this case, given s, there are $\sum_{\mu=1}^{c(s)}\sum_{\nu=1}^{(n-\mu+1)/2}\binom{n-\mu}{2\nu-1}$ nonzero terms in Eq. (5), leading to $\mathcal{C}_{|GHZ\rangle}(s) = \frac{1}{2}\{1 - 1/[2^{c(s)-\delta_{c(s)n}}]\}$, where the $\delta_{c(s)n}$ arises from the fact that $c(n) = n$ and $c(n) = n - 1$ have the same number of terms. Note that, as expected, both $\mathcal{C}_{|W\rangle}(s)$ and $\mathcal{C}_{|GHZ\rangle}(s)$ only depend on the cardinality of s and not on the actual indices in the set, as both states are invariant under permutations of the qubits.

We can now show that when $s = \{j\}$ $[c(s) = 1]$, then $\mathcal{C}_{|W\rangle}(\{j\}) = [(n-1)/n^2]$ and $\mathcal{C}_{|GHZ\rangle}(\{j\}) = \frac{1}{4}$. This implies that the bipartite entanglement of a single qubit in $|W\rangle$ decreases with n, while on the other hand, it is constant for any n-qubit $|GHZ\rangle$ state. Moreover, if $s = \mathcal{S}$ $[c(s) = n]$, then $\mathcal{C}_{|W\rangle}(\mathcal{S}) = [(n-1)/2n]$ and $\mathcal{C}_{|GHZ\rangle}(\mathcal{S}) = \frac{1}{2} - \frac{1}{2^n}$, and we recover the results in [42]. Note that for both cases considered one finds that $\mathcal{C}_{|GHZ\rangle}(s) > \mathcal{C}_{|W\rangle}(s)$, and hence that the concentrable

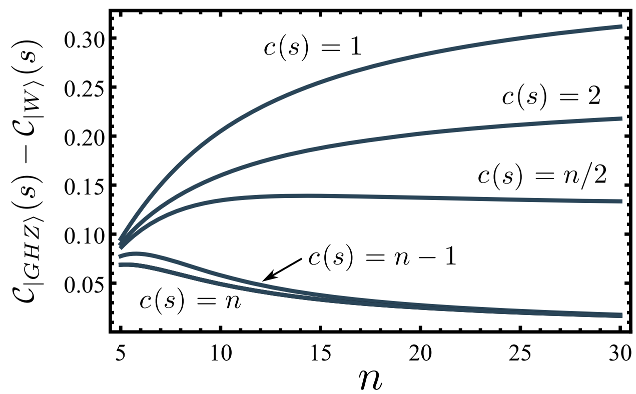

FIG. 3. Comparison of the concentratable entanglements for the GHZ and W states. In the figure we show the difference $\mathcal{C}_{|GHZ\rangle}(s) - \mathcal{C}_{|W\rangle}(s)$ versus the number of qubits n for different sets s with cardinalities $c(s) = 1, 2, n/2, n-1, n$. In all cases we find $\mathcal{C}_{|GHZ\rangle}(s) > \mathcal{C}_{|W\rangle}(s)$.

entanglements detect more multipartite entanglement in $|GHZ\rangle$ than in $|W\rangle$. For $s = \mathcal{S}$, however, in the limit of $n \to \infty$ both $\mathcal{C}_{|GHZ\rangle}(s)$ and $\mathcal{C}_{|W\rangle}(s)$ tend to the same value of $\frac{1}{2}$. Lastly, we note that a property of W states is that if 1 qubit is measured and projected out of the state, one can still measure entanglement in the ensuing state. Specifically, if one measures a qubit, then the concentratable entanglement will be equal to $c(s)[2n - c(s) - 3]/2(n-1)^2$ (to 0) with probability $1 - 1/n^2$ ($1/n^2$), as this corresponds to measuring the qubit in the zero (one) state. However, for GHZ states, projecting out just 1 qubit always yields a state with zero concentratable entanglement —confirming the well-known fact that while the W state is less entangled than the GHZ, it is more robust to noise.

In Fig. 3 we further analyze the difference $\Delta\mathcal{C} = \mathcal{C}_{|GHZ\rangle}(s) - \mathcal{C}_{|W\rangle}(s)$ for different cardinalities of s. Here we see that for $c(s) \leqslant n/2$, $\Delta\mathcal{C}$ increases (or remains constant) as n increases implying that, again, small subsystems of qubits in $|W\rangle$ contain less multipartite entanglement than those in $|GHZ\rangle$. For $c(s) \sim n$, the difference $\Delta\mathcal{C}$ decreases as n increases, showing that the total multipartite entanglement is asymptotically the same for the two states.

Conclusion.—In this work, we introduced a computable and operationally meaningful family of entanglement monotones called the concentratable entanglements. For a pure state $|\psi\rangle$, these quantities can be estimated on a quantum computer given two copies of $|\psi\rangle$ via a parallelized SWAP test. We showed that they quantify and characterize the entanglement in and between subsystems of the composite quantum state in addition to quantifying global entanglement. We derived their operational meaning in terms of the probability of obtaining Bell pairs via the parallelized SWAP test. We also showed that well-known entanglement measures such as the n tangle, concurrence, and linear entropy of entanglement are recovered as special cases of concentratable entanglements. As a special case of our results, we proved a conjecture from Ref. [42], which claimed that the parallelized SWAP test could be used to quantify and categorize pure state multipartite entanglement.

An important future direction will be to experimentally observe our entanglement measures on real quantum devices (e.g., using a quantum optical Fredkin gate [62]). A detailed analysis of the impact of hardware noise on the parallelized SWAP test will be useful for such implementations. As noise can turn pure states into mixed states, an additional important direction will be to generalize our entanglement measures (and their operational meaning) to mixed states. Finally, one could also analyze if a SWAP test with more copies of $|\psi\rangle$ can provide further information beyond that in the concentratable entanglement.

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[1] A. Einstein, B. Podolsky, and N. Rosen, Can quantum-mechanical description of physical reality be considered complete?, Phys. Rev. **47**, 777 (1935).
[2] M. M. Wilde, *Quantum Information Theory* (Cambridge University Press, Cambridge, England, 2013).
[3] M. B. Plenio and V. Vedral, Entanglement in quantum information theory, Contemp. Phys. **39**, 431 (1998).
[4] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, Quantum entanglement, Rev. Mod. Phys. **81**, 865 (2009).
[5] N. Gigena, M. Di Tullio, and R. Rossignoli, One-body entanglement as a quantum resource in fermionic systems, Phys. Rev. A **102**, 042410 (2020).
[6] C. H. Bennett, G. Brassard, C. Crépeau, R. Jozsa, A. Peres, and W. K. Wootters, Teleporting an Unknown Quantum State Via Dual Classical and Einstein-Podolsky-Rosen Channels, Phys. Rev. Lett. **70**, 1895 (1993).
[7] J. Barrett, Nonsequential positive-operator-valued measurements on entangled mixed states do not always violate a Bell inequality, Phys. Rev. A **65**, 042302 (2002).

[8] R. Cleve and H. Buhrman, Substituting quantum entanglement for communication, Phys. Rev. A **56**, 1201 (1997).
[9] N. Gigena and R. Rossignoli, Bipartite entanglement in fermion systems, Phys. Rev. A **95**, 062320 (2017).
[10] A. K. Ekert, Quantum Cryptography Based on Bell's Theorem, Phys. Rev. Lett. **67**, 661 (1991).
[11] N. Gisin, G. Ribordy, W. Tittel, and H. Zbinden, Quantum cryptography, Rev. Mod. Phys. **74**, 145 (2002).
[12] A. Ekert and R. Jozsa, Quantum algorithms: Entanglement–enhanced information processing, Phil. Trans. R. Soc. A **356**, 1769 (1998).
[13] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, England, 2000).
[14] A. Datta, S. T. Flammia, and C. M. Caves, Entanglement and the power of one qubit, Phys. Rev. A **72**, 042316 (2005).
[15] T. Chalopin, C. Bouazza, A. Evrard, V. Makhalov, D. Dreon, J. Dalibard, L. A. Sidorenkov, and S. Nascimbene, Quantum-enhanced sensing using non-classical spin states of a highly magnetic atom, Nat. Commun. **9**, 4955 (2018).
[16] R. C. Pooser, N. Savino, E. Batson, J. L. Beckey, J. Garcia, and B. J. Lawrie, Truncated Nonlinear Interferometry for Quantum-Enhanced Atomic Force Microscopy, Phys. Rev. Lett. **124**, 230504 (2020).
[17] L. Pezzè, Entanglement-enhanced sensor networks, Nat. Photonics **15**, 74 (2021).
[18] S.-K. Liao, W.-Q. Cai, W.-Y. Liu, L. Zhang, Y. Li, J.-G. Ren, J. Yin, Q. Shen, Y. Cao, Z.-P. Li *et al.*, Satellite-to-ground quantum key distribution, Nature (London) **549**, 43 (2017).
[19] X.-S. Ma, T. Herbst, T. Scheidl, D. Wang, S. Kropatschek, W. Naylor, B. Wittmann, A. Mech, J. Kofler, E. Anisimova *et al.*, Quantum teleportation over 143 kilometres using active feed-forward, Nature (London) **489**, 269 (2012).
[20] R. Valivarthi, S. I. Davis, C. Peña, S. Xie, N. Lauk, L. Narváez, J. P. Allmaras, A. D. Beyer, Y. Gim, M. Hussein *et al.*, Teleportation systems toward a quantum internet, PRX Quantum **1**, 020317 (2020).
[21] F. Arute, K. Arya, R. Babbush, D. Bacon *et al.*, Quantum supremacy using a programmable superconducting processor, Nature (London) **574**, 505 (2019).
[22] K. Zyczkowski and I. Bengtsson, *Geometry of quantum states* (Cambridge University Press, Cambridge, England, 2006).
[23] M. Walter, D. Gross, and J. Eisert, Multipartite entanglement, in *Quantum Information: From Foundations to Quantum Technology Applications* (Wiley-VCH, Hoboken, New Jersey, 2016), pp. 293–330.
[24] W. Dür, G. Vidal, and J. I. Cirac, Three qubits can be entangled in two inequivalent ways, Phys. Rev. A **62**, 062314 (2000).
[25] V. Coffman, J. Kundu, and W. K. Wootters, Distributed entanglement, Phys. Rev. A **61**, 052306 (2000).
[26] A. Acín, A. Andrianov, L. Costa, E. Jané, J. I. Latorre, and R. Tarrach, Generalized Schmidt Decomposition and Classification of Three-Quantum-Bit States, Phys. Rev. Lett. **85**, 1560 (2000).
[27] H. Barnum and N. Linden, Monotones and invariants for multi-particle quantum states, J. Phys. A **34**, 6787 (2001).
[28] A. Wong and N. Christensen, Potential multiparticle entanglement measure, Phys. Rev. A **63**, 044301 (2001).
[29] D. A. Meyer and N. R. Wallach, Global entanglement in multiparticle systems, J. Math. Phys. (N.Y.) **43**, 4273 (2002).
[30] A. Miyake, Classification of multipartite entangled states by multidimensional determinants, Phys. Rev. A **67**, 012108 (2003).
[31] G. Tóth and O. Gühne, Detecting Genuine Multipartite Entanglement with Two Local Measurements, Phys. Rev. Lett. **94**, 060501 (2005).
[32] M. Walter, B. Doran, D. Gross, and M. Christandl, Entanglement polytopes: Multiparticle entanglement from single-particle information, Science **340**, 1205 (2013).
[33] A. Sawicki, M. Oszmaniec, and M. Kuś, Critical sets of the total variance can detect all stochastic local operations and classical communication classes of multiparticle entanglement, Phys. Rev. A **86**, 040304(R) (2012).
[34] X. Guo, C. R. Breum, J. Borregaard, S. Izumi, M. V. Larsen, T. Gehring, M. Christandl, J. S Neergaard-Nielsen, and U. L. Andersen, Distributed quantum sensing in a continuous-variable entangled network, Nat. Phys. **16**, 281 (2020).
[35] D. Gottesman, T. Jennewein, and S. Croke, Longer-baseline telescopes using quantum repeaters, Phys. Rev. Lett. **109**, 070503 (2012).
[36] H. J. Kimble, The quantum internet, Nature (London) **453**, 1023 (2008).
[37] M. Navascués, E. Wolfe, D. Rosset, and A. Pozas-Kerstjens, Genuine Network Multipartite Entanglement, Phys. Rev. Lett. **125**, 240505 (2020).
[38] S. Wehner, D. Elkouss, and R. Hanson, Quantum internet: A vision for the road ahead, Science **362** (2018), https://doi.org/10.1126/science.aam9288.
[39] M. Cerezo, A. Arrasmith, R. Babbush, S. C. Benjamin, S. Endo, K. Fujii, J. R. McClean, K. Mitarai, X. Yuan, L. Cincio, and P. J. Coles, Variational quantum algorithms, arXiv:2012.09265.
[40] C. M. Alves and D. Jaksch, Multipartite Entanglement Detection in Bosons, Phys. Rev. Lett. **93**, 110501 (2004).
[41] G. Smith, J. A. Smolin, X. Yuan, Q. Zhao, D. Girolami, and X. Ma, Quantifying coherence and entanglement via simple measurements, arXiv:1707.09928.
[42] S. Foulds, V. Kendon, and T. P. Spiller, The controlled swap test for determining quantum entanglement, Quantum Science and Technology **6**, 035002 (2021).
[43] P. Hayden, K. Milner, and M. M. Wilde, Two-message quantum interactive proofs and the quantum separability problem, in *2013 IEEE Conference on Computational Complexity* (IEEE, 2013), pp. 156–167.
[44] G. Gutoski, P. Hayden, K. Milner, and M. M. Wilde, Quantum interactive proofs and the complexity of separability testing, arXiv:1308.5788.
[45] G. K. Brennen, An observable measure of entanglement for pure states of multi-qubit systems, arXiv:quant-ph/0305094.
[46] H. Buhrman, R. Cleve, J. Watrous, and R. de Wolf, Quantum Fingerprinting, Phys. Rev. Lett. **87**, 167902 (2001).
[47] A. W. Harrow and A. Montanaro, Testing product states, quantum merlin-arthur games and tensor optimization, J. ACM **60**, 1 (2013).

[48] G. Gutoski, P. Hayden, K. Milner, and M. M. Wilde, Quantum interactive proofs and the complexity of separability testing, Theory Comput. **11**, 59 (2015).

[49] L. Cincio, Y. Subaşı, A. T. Sornborger, and P. J. Coles, Learning the quantum algorithm for state overlap, New J. Phys. **20**, 113022 (2018).

[50] Y. Subaşı, L. Cincio, and P. J. Coles, Entanglement spectroscopy with a depth-two quantum circuit, J. Phys. A **52**, 044001 (2019).

[51] A. R. R. Carvalho, F. Mintert, and A. Buchleitner, Decoherence and Multipartite Entanglement, Phys. Rev. Lett. **93**, 230501 (2004).

[52] See Supplemental Information at http://link.aps.org/supplemental/10.1103/PhysRevLett.127.140501 for proofs of our main results, as well as additional details on the SWAP test, which contains Refs. [53–58].

[53] M. A. Nielsen and G. Vidal, Majorization and the interconversion of bipartite states, Quantum Inf. Comput. **1**, 76 (2001).

[54] M. A. Nielsen, Conditions for a Class of Entanglement Transformations, Phys. Rev. Lett. **83**, 436 (1999).

[55] G. Vidal, Entanglement monotones, https://arxiv.org/abs/quant-ph/9807077.

[56] D. Bruss and G. Leuchs, *Quantum Information: From Foundations to Quantum Technology Applications* (Wiley-VCH, New York, 2019).

[57] L. Schatzki, A. Arrasmith, P. J. Coles, and M. Cerezo, arXiv:2109.03400 [Quantum Mach. Learn. (to be published)].

[58] P. J. Coles, M. Cerezo, and L. Cincio, Strong bound between trace distance and hilbert-schmidt distance for low-rank states, Phys. Rev. A **100**, 022103 (2019).

[59] A. J. Scott, Multipartite entanglement, quantum-error-correcting codes, and entangling power of quantum evolutions, Phys. Rev. A **69**, 052330 (2004).

[60] W. K. Wootters, Entanglement of Formation of an Arbitrary State of Two Qubits, Phys. Rev. Lett. **80**, 2245 (1998).

[61] G. Jaeger, A. V. Sergienko, B. E. A. Saleh, and M. C. Teich, Entanglement, mixedness, and spin-flip symmetry in multiple-qubit systems, Phys. Rev. A **68**, 022318 (2003).

[62] G. J. Milburn, Quantum Optical Fredkin Gate, Phys. Rev. Lett. **62**, 2124 (1989).