Genuine Multipartite Nonlocality with Causal-Diagram Postselection

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The generation and verification of genuine multipartite nonlocality (GMN) is of central interest for both fundamental research and quantum technological applications, such as quantum privacy. To demonstrate GMN in measurement data, the statistics are commonly postselected by neglecting undesired data. Until now, valid postselection strategies have been restricted to local postselection. A general postselection that is decided after communication between parties can mimic nonlocality, even though the complete data are local. Here, we establish conditions under which GMN is demonstrable even if observations are postselected collectively. Intriguingly, certain postselection strategies that require communication among several parties still offer a demonstration of GMN shared between all parties. The results are derived using the causal structure of the experiment and the no-signaling condition imposed by relativity. Finally, we apply our results to show that genuine three-partite nonlocality can be created with independent particle sources.

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Introduction.—Bell nonlocality [1,2] is one of the most intriguing discoveries in modern physics. Besides its heavily discussed fundamental significance and implications [3–5], several technological applications have been developed in fields such as communication [6,7], quantum cryptography [8–12], certified random number generation [13,14], and quantum computation [15,16]. Growing interest is experienced by the field of multipartite nonlocality [3,17–23]. Here, genuine multipartite nonlocality (GMN), a subclass of multipartite nonlocality, plays a central role. Genuinely multipartite nonlocal correlations cannot be described by nonlocal correlations confined to different groups of subsystems but require collective nonlocal correlations between all subsystems [18–20]. This stronger form of nonlocality is the key ingredient for many future quantum technologies such as the quantum internet [24–26] and device-independent multipartite quantum key distributions [17,21–23], and serves as a detection of genuine multipartite entanglement [27–35].

Imagine a group of n experimental parties that have performed an experiment together. Now they want to examine if their observed results demonstrate the presence of GMN by the violation of a Bell inequality [1,36], cf. Fig. 1. A first test of the Bell inequalities using the complete measurement statistics does not yield any violation. It is known that a common postselection strategy that can be decided locally, i.e., each party knows whether to keep or neglect its measurement result without knowledge of other parties, can be used to verify Bell nonlocality [37]. Say that even a local postselection of results does not suffice for a violation of the Bell inequalities. Generally, the more data is ignored, the more Bell inequalities can be violated, cf. Figs. 1(b) and 1(c). However, the correlations could be created by the postselection bias [38]: a postselection that is decided collectively by the experimental parties can potentially mimic nonlocal behavior even if the underlying statistics can be described by local hidden-variable models [39–42]. Can the parties employ strategies beyond local postselection to verify genuine multipartite Bell nonlocality of their correlations?

An instance of this problem affects a proposal by Yurke and Stoler (YS) [43] to create Einstein-Podolsky-Rosen (EPR) effects [44] of the Greenberger-Horne-Zeilinger (GHZ) type [45,46] from independent particle sources. Unlike common approaches to create nonlocal behavior or entanglement by an interaction between subsystems that are then send to the parties, YS make use of the (bosonic or fermionic) Hong-Ou-Mandel effect [47] to create nonlocality [43,48]. A generalization of Ref. [43] became a standard method to create optical GHZ states [45,49–51].

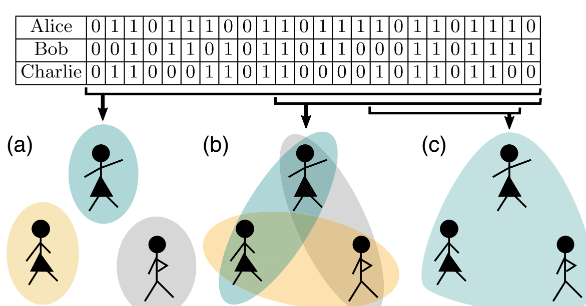

FIG. 1. After postselecting their experimental observations, three parties could observe that they share (a) no nonlocal correlations, (b) multipartite (but not genuine multipartite) nonlocality, or (c) genuine multipartite nonlocality (GMN). When can they be sure that their postselection did not create fake correlations?

The EPR effects in Ref. [43] suffice to exclude local hidden-variable models. However, the necessary postselection to demonstrate GMN cannot be decided locally. Thus, the YS scheme did not show GMN until now.

In this work, we introduce postselection strategies beyond local postselection that can be employed to demonstrate GMN. Our analysis of the multipartite Bell scenario provides sufficient conditions such that a collective postselection is valid. Particularly, in the n-partite case, a postselection is valid if it can be decided even after exclusion of any $\lfloor n/2 \rfloor$ parties from the decision, where $\lfloor . \rfloor$ is the floor function. This implies that, somewhat surprisingly, statistics that were postselected through communication of several parties can still serve as a certification of GMN among all parties. In the three-partite case, the postselection condition simplifies to an all-but-one principle similar to Ref. [52], where causal diagrams are used to safely postselect statistics to verify general (nongenuine) multipartite nonlocality. Here, we use causal diagrams for hybrid local-nonlocal hidden-variable models (instead of local hidden-variable models [52]) to prove valid postselections for GMN. In addition, we explicitly use the no-signaling principle dictated by relativity. The analysis is performed using causal diagrams [38,52] that extend the language of statistics to include a causal examination of multivariate data. We emphasize that in contrast to the common use of causal diagrams as an explanatory tool in quantum physics, the current work and Ref. [52] show that causal diagrams can be exploited to derive new theorems. We finally apply our results to the YS setup [43] to show that genuine three-partite nonlocality can be created from independent particle sources.

Main results.—Bell nonlocality can be certified if the measured statistics violate a Bell inequality that was derived assuming local realism and free will. By the latter assumptions, the general joint probability distribution of experimental results can be written as a local hidden-variable model [1,36], yielding conditions on the statistics in the form of Bell inequalities. To derive Bell inequalities that distinguish between partial nonlocality and GMN, the local hidden-variable model is replaced by a hybrid local-nonlocal hidden-variable model [18,20,53,54]. A violation of these inequalities then demonstrates the presence of GMN.

Commonly, measured statistics are postselected to obtain statistics that violate Bell inequalities. However, postselection potentially creates additional correlations due to the postselection bias [38,52,57] which can mimic nonlocal behavior. In the following, we introduce postselection strategies that are valid to demonstrate GMN.

First, consider a three-partite Bell scenario. The hybrid hidden-variable model asserts that, given that the three parties Alice, Bob, and Charlie measure observables x, y, and z, the probability for outcomes a, b, and c is given by [18,20,53]

$$P_{abc|xyz} = \sum_{\lambda_1 \in \Lambda_1} P_{\lambda_1} P_{bc|yz\lambda_1} P_{a|x\lambda_1} + \sum_{\lambda_2 \in \Lambda_2} P_{\lambda_2} P_{ac|xz\lambda_2} P_{b|y\lambda_2} + \sum_{\lambda_3 \in \Lambda_3} P_{\lambda_3} P_{ab|xy\lambda_3} P_{c|z\lambda_3}, \tag{1}$$

where $\sum_{\lambda \in \Lambda} P_\lambda = 1$, $\Lambda = \Lambda_1 \cup \Lambda_2 \cup \Lambda_3$. Here, we divided the hidden variables Λ into subsets Λ_i indicating which two parties share nonlocal correlations for a given λ. The free will assumption was used to write $P_{\lambda|xyz} = P_\lambda$, i.e., measurement choices are independent of the hidden variables. Furthermore, the correlations in Eq. (1) must fulfill the no-signaling principle [20], e.g.,

$$P_{a|xyz} = P_{a|x}. \tag{2}$$

This ensures that no party can send information to other parties instantaneously by choice of the measurement setting. While valid Bell inequalities for genuine nonlocality can be derived without demanding the no-signaling principle [18], stronger Bell inequalities can be proven including no-signaling (or one-way signaling) conditions [20]. Our results only hold if the hybrid hidden-variable model fulfills the no-signaling (or one-way signaling) condition. A diagram describing all possible causal relations of the different random variables is shown in Fig. 2(a). Each variable is represented as a capital letter while their possible values are denoted as lowercase letters. Solid arrows describe possible causal influences along the arrows' directions [38]. We emphasize that some causal influences are restricted by the no-signaling conditions and cannot be described by any classical causal model without fine-tuning [58,59]. For instance, while there might be a causal influence from Y to B and from B to A, there is no causal influence from Y to A, cf. Eq. (2). By conditioning on a particular Λ_i, the causal diagram can be restricted, see Fig. 2(b) for Λ_3. In the following, boxes around variables in the diagram indicate that the variables are conditioned on.

Consider now a postselection (represented as a binary variable K) of the observed results. We expand the postselected statistics in terms of the local hidden variables,

$$P_{abc|xyzk} = \sum_{\lambda} P_{\lambda|xyzk} P_{abc|xyz\lambda k}. \tag{3}$$

This exposes that a sufficient condition that $P_{abc|xyzk}$ fulfills a Bell inequality derived from Eq. (1), is that $P_{abc|xyzk}$ factorizes in a similar way: if

$$\text{(I)}\ P_{\lambda|xyzk} = P_{\lambda|k},$$

$$\text{(IIc)}\ P_{abc|xyz\lambda_3 k} = P_{ab|xy\lambda_3 k} P_{c|z\lambda_3 k},$$

with similar conditions (IIa) and (IIb) for Λ_1 and Λ_2, the postselected statistics are valid to test for a violation of the

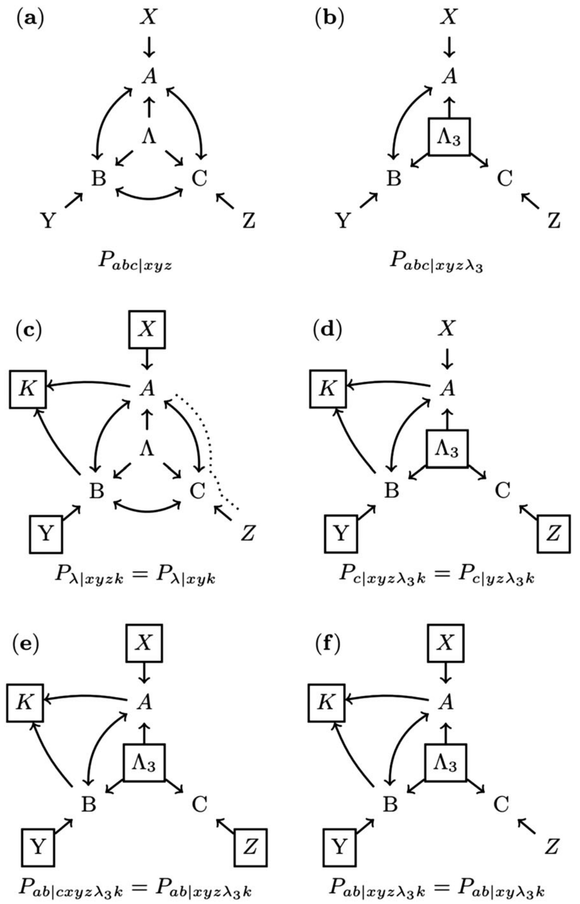

FIG. 2. (a) Causal diagram of the three-partite hybrid local-nonlocal hidden-variable model (1). The causal relations are subject to the no-signaling conditions (2). (b) Causal diagram of the subensemble Λ_3 of the hybrid model, allowing for correlations between A and B. (c)–(f) Causal diagrams representing different steps of the proof of safe postselection if the postselection (K) is decided by two parties. Solid arrows represent possible causal influences between variables. Variables that are conditioned on are marked with a box. In (c), we indicate a fine-tuning condition due to the no-signaling principle as a dotted line.

Bell inequality. Note that one could also require that the postselected statistics fulfill the no-signaling principle (2) if the no-signaling conditions are needed to derive the Bell inequality of interest [20].

We now focus on postselection strategies that can be equivalently decided by any subset of the experimental parties of a certain minimum size. For three parties, we consider a postselection K that can be equivalently decided by any two parties, implying that

$$P^{(AB)}_{abc|xyzk} = P^{(AC)}_{abc|xyzk} = P^{(BC)}_{abc|xyzk} \tag{4}$$

where, e.g., $P^{(AB)}_{abc|xyzk}$ denotes the postselected conditional probability when the postselection is decided by Alice and Bob. Thus, the postselected distribution coincides for whichever two parties reconcile to decide it, and we simply write $P_{abc|xyzk}$ in the following. This decision equivalence requires global conditions on the possible experimental results such that the latter become partially redundant. An example that we will further discuss below are experiments where, for postselection, each party should find a certain number of particles and the total number of particles is conserved.

The central tool in the proof of our results is causal inference and the d-separation tool set [38,52]: given a causal diagram that connects different random variables (nodes) with causal relations (arrows), only certain dependencies between variables are possible [38]. In short, two variables can only be dependent if they are connected by a path. The d-separation rules dictate which paths are blocked when conditioning on other variables of the diagram. The d-separation rules read as follows: (1) a path is blocked if there is a collider in the path, i.e., a variable at which causal arrows collide, (2) conditioning on a non-collider along the path blocks the path, (3) conditioning on a collider (or its descendant) along the path unblocks the path. We can now prove our first main result.

Theorem 1. In the three-partite Bell scenario, a postselection that can be decided by any two (all-but-one) parties is valid for verification of genuine three-partite nonlocality.

Proof.—Assume that the postselection K can be decided by any two parties. Thus, we can add to the causal diagrams of Figs. 2(a) and 2(b) the postselection variable K with causal influences from any two parties, e.g., A and B, see Fig. 2(c). All of the resulting diagrams are valid and can be used in the proof. Condition (I) is proved in three steps. First, we show $P_{\lambda|xyzk} = P_{\lambda|xyk}$. In Fig. 2(c), we condition on X, Y, and K (indicated as boxes) and check for all possible paths between Λ and Z. The direct path $Z \to C \leftarrow \Lambda$ is blocked because C is a collider (that is not conditioned on) along this path. Consider the path $Z \to C \to A \to K \leftarrow B \leftarrow \Lambda$. If there were general causal influences from Z to A, this path would be open because the collider K is conditioned on. However, using the no-signaling condition between Alice and Charlie, Z can have no influence on A so this path is blocked. This restriction is marked as a dotted line in Fig. 2(c), indicating that this path segment is blocked. Similarly, one can reason that all paths between Λ and Z are blocked. We note that, above, the conditioning on X and/or Y can also be removed without unblocking any path. By using similar diagrams as in Fig. 2(c) but with the postselection K decided by A and C (and conditioning on X and K), one shows that $P_{\lambda|xyk} = P_{\lambda|xk}$. Finally, a diagram with K decided by B and C (and conditioning only on K) yields $P_{\lambda|xk} = P_{\lambda|k}$ and condition (I) follows. These steps and their causal diagrams are detailed in Supplemental Material [60]. No-signaling conditions on the postselected statistics such as, e.g., $P_{a|xyz\lambda k} = P_{a|x\lambda k}$, can be proven in a similar fashion. Note that condition (I) even holds true even if the no-signaling condition is replaced by the less restrictive one-way signaling condition [20] (then the order of the three steps matters). To show condition (IIc), we first use the chain rule to write

$$P_{abc|xyz\lambda_3 k} = P_{ab|cxyz\lambda_3 k} P_{c|xyz\lambda_3 k}. \tag{5}$$

We have that $P_{c|xyz\lambda_3 k} = P_{c|yz\lambda_3 k}$, see Fig. 2(d), and similarly $P_{c|yz\lambda_3 k} = P_{c|z\lambda_3 k}$. For the other term, we have $P_{ab|cxyz\lambda_3 k} = P_{ab|xyz\lambda_3 k}$ [Fig. 2(e)] and $P_{ab|xy\lambda_3 k}$ [Fig. 2(f)]. This yields condition (IIc). Conditions (IIa) and (IIb) can be shown using the relevant diagrams. Note that, to prove conditions (II), we do not have to use the no-signaling condition. ∎

Now let us turn to conditions for valid postselection for detecting genuine n-partite nonlocality for $n > 3$. First consider the case $n = 4$. The corresponding local-nonlocal hidden-variable model consists of subensembles that allow nonlocal correlations among at most three parties. Similar to above, an all-but-one postselection is valid when applied to subensembles for which three parties share nonlocality and the remaining party is only correlated by a common local hidden variable. However, for subensembles in which two pairs share bipartite nonlocality, an all-but-one postselection (i.e., a postselection decided by three parties) can create a postselection bias: the postselected distribution of these subensembles generally does not factorize into two pairs of parties. This insight is discussed in detail in Supplemental Material [60]. Therefore, for $n = 4$, we can only exclude a postselection bias if the postselection

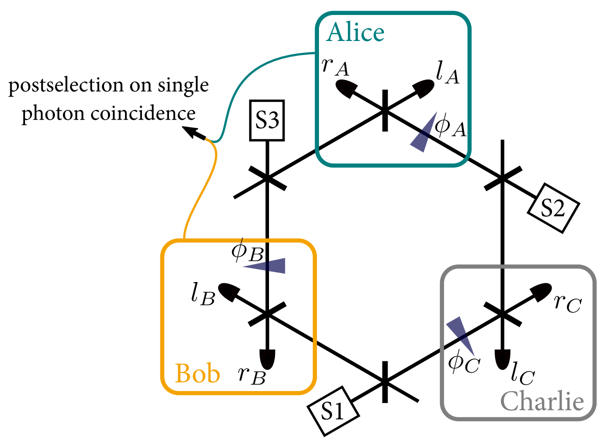

FIG. 3. Proposal by Yurke and Stoler [43]: Three independent photon sources emit photons that are distributed via beam splitters among Alice, Bob, and Charlie. Each party imprints a local phase in one of the incoming arms and measures the two modes after applying a second beam splitter. After postselecting events that show a single photon detection per party, the statistics show genuinely nonlocal (GHZ-like) features. Since the postselection can be decided after exclusion of any party, it represents a valid test for Bell inequality violations.

can be decided by any two parties. Generally, we can prove the following theorem. A detailed proof is given in [60].

Theorem 2. In the n-partite Bell scenario, a postselection that can be decided by any all-but-$\lfloor n/2 \rfloor$ parties is valid for verification of genuine n-partite nonlocality.

Applications.—The above findings for $n = 3$ can be applied to setups where the number of particles is conserved. This is similar to the findings of Refs. [50,52] because undesirable events come in pairs. We now apply our results to the YS proposal of Ref. [43]. The corresponding setup for two parties [48] makes use of local postselection which represents a valid postselection [37,39]. For three parties, a local postselection is not sufficient to violate Bell inequalities.

The setup of Ref. [43] is shown in Fig. 3. Three independent sources ($S1$, $S2$, and $S3$) emit a single photon. Each photon passes a beam splitter whose outcoming modes are directed to two measurement parties. At each party, the two incoming modes pass a second beam splitter after which they are measured with photon-counting detectors. Additionally, each party chooses a measurement setting by imprinting a phase (ϕ_A, ϕ_B, and ϕ_C) in one of the incoming modes. Each party P can either detect no photon (0_P), a single photon in the left (l_P) or right detector (r_P), or two photons in the left (l_P^2) or right detector (r_P^2). For perfectly indistinguishable photon sources, events with a photon detection both detectors destructively interfere [47].

Assuming unit detection efficiency, the observed events can be grouped into two groups: either each party receives a photon, or one party does not detect a photon and one detects two (D). The first group can be further divided into an even (E) or odd (O) number of right detector clicks. Depending on the total phase $\phi = (\phi_A + \phi_B + \phi_C)/2$, the probability $P(e)$ of an event e is $P(e \in E) = \cos^2(\phi)/16$, $P(e \in O) = \sin^2(\phi)/16$, and $P(e \in D) = 1/32$ [43]. To observe genuine three-partite nonlocality, we must postselect events in E and O. This postselection can be decided by any two parties and, according to Theorem 1, is valid to verify GMN. Indeed, say Alice measures two observables x_i ($i = 1, 2$) resulting in outcomes $a = 1$ ($a = -1$) for Alice's observation of r_A (l_A), similarly for Bob and Charlie. Hybrid hidden-variable models (1) fulfill the three-partite-nonlocality-testing Svetlichny inequality [18]

$$\begin{aligned} I = |&\langle x_1 y_1 z_1 \rangle + \langle x_1 y_1 z_2 \rangle + \langle x_1 y_2 z_1 \rangle + \langle x_2 y_1 z_1 \rangle \\ &- \langle x_1 y_2 z_2 \rangle - \langle x_2 y_1 z_2 \rangle - \langle x_2 y_2 z_1 \rangle - \langle x_2 y_2 z_2 \rangle| \le 4, \end{aligned} \tag{6}$$

where $\langle \cdots \rangle$ is a statistical average. Let Alice choose between $\phi_A = 0$ (x_1) and $\phi_A = -\pi/2$ (x_2), Bob between $\phi_B = \pi/4$ (y_1) and $\phi_B = -\pi/4$ (y_2), and Charlie between $\phi_C = 0$ (z_1) and $\phi_C = -\pi/2$ (z_2). Then the Svetlichny inequality (6) is maximally violated by the postselected statistics, $I = 4\sqrt{2}$ [61]. Note that, in the hybrid hidden-variable model, we allow for a source of classical shared randomness among all three parties.

In the case $n > 3$, a conservation of the number of particles is not sufficient that postselection can be decided by all-but-$\lfloor n/2 \rfloor$ parties. There must be further constraints or conservation laws imposed on the possible events.

In experiments, finite detection efficiencies open an additional loophole, the detection loophole [3,5,57]. In schemes such as the YS proposal that fulfill the all-but-one principle for perfect efficiencies, for realistic efficiencies the postselection cannot be decided by all-but-one parties anymore. Commonly, the detection loophole is circumvented by the fair-sampling assumption [3,5] that the detection of incoming particles does not depend on the measurement setting of the detector. The detection loophole can be rigorously closed by sharpening the Bell inequalities [62] or taking into account all observed events [37]. An application of these methods to the YS proposal is beyond the scope of this work.

Conclusions.—We have introduced postselection strategies beyond local postselection such that the postselected statistics can validly be used to examine Bell inequality violation and verification of GMN. In the n-partite Bell scenario, we have shown that postselected statistics represent valid tests of multipartite Bell inequalities if the postselection can be decided by any all-but-$\lfloor n/2 \rfloor$ parties. In other words, certain partially collaborative postselection strategies do not hinder the certification of GMN. Furthermore, the probability of successful postselection can be arbitrarily small as long as it fulfills the above condition. In the three-partite scenario, our results reduce to an all-but-one principle and can be applied to setups where the total number of particles is conserved. Particularly, for the proposal by Yurke and Stoler [43], the postselected statistics are shown to maximally violate the Svetlichny inequality, demonstrating the creation of GMN from independent particle sources. Our results crucially facilitate the development of future quantum technologies due to the key role played by GMN. The explicit use of causal diagrams in the proofs highlights their potential as a new tool in quantum and general physics.

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Correction: Final revisions were not incorporated in the production version of the manuscript and have now been incorporated: A new paragraph was inserted after the paragraph that includes Eq. (3), which includes a new equation [now numbered (4) with subsequent equations renumbered], and a sentence was added at the end of the paragraph containing the new Eq. (6).