String Dual to Free $\mathcal{N} = 4$ Supersymmetric Yang-Mills Theory

Matthias R. Gaberdiel[1,*] and Rajesh Gopakumar[2,†]
[1]*Institut für Theoretische Physik, ETH Zürich, Wolfgang-Pauli-Straße 27, 8093 Zürich, Switzerland*
[2]*International Centre for Theoretical Sciences (ICTS-TIFR), Shivakote, Hesaraghatta Hobli, Bangalore North 560 089, India*

(Received 1 June 2021; revised 2 August 2021; accepted 30 August 2021; published 23 September 2021)

We propose a worldsheet description for the $\mathrm{AdS}_5 \times \mathrm{S}^5$ string theory dual to large N, free $\mathcal{N} = 4$ supersymmetric Yang-Mills theory in four dimensions. The worldsheet theory is a natural generalization of the recently investigated tensionless string on $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathbb{T}^4$. As in the case of AdS_3 it has a free field description, with spectrally flowed sectors, and is closely related to an (ambi-)twistor string theory. Here, however, we view it as a critical $N = 4$ (closed) string background. We argue that the corresponding worldsheet gauge constraints reduce the degrees of freedom to a finite number of oscillators (string bits) in each spectrally flowed sector. Imposing a set of residual gauge constraints on this reduced oscillator Fock space then determines the physical spectrum of the string theory. Quite remarkably, we find that this prescription reproduces precisely the entire planar spectrum—of single trace operators—of the free supersymmetric Yang-Mills theory.

DOI: 10.1103/PhysRevLett.127.131601

Introduction.—The idea that gauge theories might be equivalent to string theories is around half a century old. However, it was only after the AdS/CFT correspondence that we have had examples of string theories describing a class of large N, 4D (supersymmetric) gauge theories, as well as a dictionary between observables on both sides [1–3]. This remarkable connection between gravity and gauge theory has been the engine powering many advances in theoretical physics in the last couple of decades.

One major limitation in these developments has been the intractability of the string theory side of the correspondence beyond the large radius or supergravity limit. This is due to the presence of Ramond-Ramond flux in the background, which is difficult to incorporate in the conventional Ramond-Neveu-Schwarz quantization of strings. Thus, in the canonical example of type IIB string theory on $\mathrm{AdS}_5 \times \mathrm{S}^5$ dual to 4D $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) theory, most attempts to go beyond the supergravity limit have employed the Green-Schwarz formalism. But it has been difficult to quantize this theory even in a physical light-cone gauge despite the presence of integrability (see [4] for an overview). One exception has been the influential Berenstein-Maldacena-Nastase (BMN) approach [5], which takes a particular plane wave limit of the anti–de Sitter (AdS) background where the worldsheet theory becomes free.

In this Letter, we will propose a worldsheet description of string theory on $\mathrm{AdS}_5 \times \mathrm{S}^5$ in the tensionless limit where it is believed to be dual to free $\mathcal{N} = 4$ supersymmetric Yang-Mills theory [6]. Our proposal is a logical continuation of our previous investigations [7,8] into the tensionless limit of $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathbb{T}^4$, but it is also structurally very natural in its own right. In particular, the organization of the resulting physical string spectrum is a covariant generalization of the light-cone gauge spectrum of BMN away from any large spin limit. Furthermore, our worldsheet theory is a close cousin of the ambitwistorial open string formulation, due to Berkovits [9], for describing scattering amplitudes of perturbative $\mathcal{N} = 4$ supersymmetric Yang-Mills theory. In fact, the connection with twistors had already appeared in the $\mathrm{AdS}_3 \times \mathrm{S}^3$ case [8] and was a direct motivation for the present proposal.

The equivalence of tensionless strings on $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathbb{T}^4$ with the dual free symmetric product 2D conformal field theory (CFT) has been established (at least at the level of correlators) in [10–12]. The worldsheet description, in that case, consists of two free fermions and bosons and their conjugates (all with conformal weight $\frac{1}{2}$ and first order kinetic terms), realizing the AdS_3 supergroup $\mathfrak{psu}(1,1|2)_1$ [7,8]. The generalization here is to a theory of four such free fermions (ψ^a) and four bosons ($\lambda^\alpha, \mu^{\dot{\alpha}}$) with their canonical conjugates, which give a free field realization of $\mathfrak{psu}(2,2|4)_1$. Geometrically, one can view our proposal as a closed string sigma model with the target space being the twistor space of $\mathrm{AdS}_5 \times \mathrm{S}^5$ [13]. As in the AdS_3 case, we will consider "spectrally flowed" sectors of these fields—see Eqs. (4)–(7)—labeled by a positive integer w.

The main upshot of our proposal is as follows: after gauge fixing, only a *finite* number (w) of physical worldsheet degrees of freedom (generalized zero modes) survive in each w spectrally flowed sector. These are the "wedge modes" Φ_r, i.e., modes with $\{-[(w-1)/2] \leq r \leq [(w-1)/2]\}$ of the four worldsheet bosons or fermions (and their conjugates), collectively denoted here by Φ [14]. Half of these wedge modes act as creation operators on the spectrally flowed vacuum state $|0\rangle_w$, defined below in Eq. (7), and generate a Fock space of states. On this Fock space we then impose two residual gauge constraints: (1) Every physical state must satisfy $\mathcal{C}_n|\text{phys}\rangle = 0$ for $n \geq 0$, where $\mathcal{C}$ is a $\mathfrak{u}(1)$ current [defined below, Eq. (1)] that needs to be gauged. Here, the (integer valued) modes $\mathcal{C}_n$ act as $[\mathcal{C}_n, \Phi_r] = \pm\frac{1}{2}\Phi_{n+r}$, with Φ_r any of the above wedge modes; the modes $(\lambda^\alpha, \mu^{\dot\alpha}, \psi^a)$ carry a $(-)$ charge, while their canonical conjugates [Eq. (1)] carry a $(+)$ charge; (2) the (generalized) mass-shell condition $L_0 = 0 (\text{mod}\, w)$ on physical states, where L_0 counts, as usual, the mode number, i.e., $[L_n, \Phi_r] = [-(n/2) - r]\Phi_{n+r}$ [15].

Our central claim is this: The resulting physical sector of the oscillator Fock space, for each w, is identical to the space of all the single trace operators built from w supersymmetric Yang-Mills fields and their derivatives.

The oscillators, together with the above constraints capture the cyclicity of the trace, the null relations due to the free equations of motion, and give the right set of $\mathfrak{psu}(2,2|4)$ representations present in the free supersymmetric Yang-Mills spectrum. A general proof of the agreement is given in [16]; we have also checked it explicitly at low levels; see below and [16] for more details.

Worldsheet theory.—The matter fields of our worldsheet theory comprise the weight $\frac{1}{2}$ conjugate pairs of boson fields $(\lambda^\alpha, \mu^\dagger_\alpha)$ and $(\mu^{\dot\alpha}, \lambda^\dagger_{\dot\alpha})$, with $\alpha, \dot\alpha \in \{1,2\}$, as well as four weight $\frac{1}{2}$ complex fermions $(\psi^a, \psi^\dagger_a)$, with $a \in \{1,2,3,4\}$ [17]. Here, α and $\dot\alpha$ are spinor indices with respect to two different $\mathfrak{su}(2)$'s, while ψ^a transforms in the fundamental representation of $\mathfrak{su}(4)$. There is also a corresponding right-moving sector. The modes of these fields obey the commutation relations

$$[\lambda^\alpha_r, (\mu^\dagger_\beta)_s] = \delta^\alpha_\beta \delta_{r,-s}, \qquad [\mu^{\dot\alpha}_r, (\lambda^\dagger_{\dot\beta})_s] = \delta^{\dot\alpha}_{\dot\beta}\delta_{r,-s}, \quad (1)$$

$$\{\psi^a_r, (\psi^\dagger_b)_s\} = \delta^a_b \delta_{r,-s}, \quad (2)$$

where (at least initially) $r, s \in \mathbb{Z} + \frac{1}{2}$. We can view these as components of ambitwistor fields $Y_I = (\mu^\dagger_\alpha, \lambda^\dagger_{\dot\alpha}, \psi^\dagger_a)$ and $Z^I = (\lambda^\alpha, \mu^{\dot\alpha}, \psi^a)$, employing the notation of [9]. Note that for each fixed r, the generators with mode numbers $(r, -r)$ give rise to two copies of the usual oscillator construction of the $\mathfrak{psu}(2,2|4)$ superconformal algebra in 4D [18]; some relevant expressions are given in the Supplemental Material [19].

These matter fields have a vanishing net central charge, and the bilinears $Y_I Z^J$ generate the current algebra $\mathfrak{u}(2,2|4)_1$. The overall $\mathfrak{u}(1)$ generator $\mathcal{C} = \frac{1}{2}Y_I Z^I$ needs to be set to zero (the "ambitwistor constraint") in order to restrict to $\mathfrak{psu}(2,2|4)$; it will play an important role, as already indicated in the introduction. We also note that an open string theory of these ambitwistor fields (Y_I, Z^J) appeared in the construction of [9] with worldsheet action

$$S = \int d^2z (Y_I(\bar\partial + A_{\bar z})Z^I + c.c.), \quad (3)$$

which, together with a current algebra term for the Yang-Mills gauge symmetry, denoted by S_G in [[9], Eq. (6)], reproduces tree level scattering amplitudes of $\mathcal{N} = 4$ supersymmetric Yang-Mills theory. Equation (3) also describes the action for our system, where the nondynamical $\mathfrak{u}(1)$ gauge field A imposes the vanishing of the current $\mathcal{C} = \frac{1}{2}Y_I Z^I$. However, for us Eq. (3) will define a closed superstring theory, i.e., we will not impose any boundary conditions on the free fields as in [[9], Eq. (9)], and it will not have the additional current algebra term S_G of [9]. Nevertheless, the fact that the target space is the same twistor space additionally motivates our worldsheet proposal.

In a sense, all the nontrivial aspects of the worldsheet theory come from the spectrally flowed representations. Recall that these are nonhighest weight affine representations that play a crucial role in noncompact Wess-Zumino-Witten worldsheet theories like those of $\mathfrak{sl}(2,\mathbb{R})$ [20] or $\mathfrak{psu}(1,1|2)$, for strings on AdS_3. These sectors are in one-to-one correspondence with the single cycle states of different length of the dual symmetric product orbifold CFT [7]. It is therefore natural to expect them to play a role in the $\mathfrak{psu}(2,2|4)_1$ theory at hand. Indeed, as mentioned, we will find these sectors to correspond to single trace operators in the dual SYM theory of size w.

Let us therefore define the w spectrally flowed fields (with tildes) as [21]

$$(\tilde\lambda^\alpha)_r = (\lambda^\alpha)_{r-w/2}, \qquad (\tilde\lambda^\dagger_{\dot\alpha})_r = (\lambda^\dagger_{\dot\alpha})_{r-w/2}, \quad (4)$$

$$(\tilde\mu^{\dot\alpha})_r = (\mu^{\dot\alpha})_{r+w/2}, \qquad (\tilde\mu^\dagger_\alpha)_r = (\mu^\dagger_\alpha)_{r+w/2}, \quad (5)$$

$$(\tilde\psi^a_r) = \psi^a_{r-w/2}, \qquad (\tilde\psi^\dagger_a)_r = (\psi^\dagger_a)_{r+w/2} (a = 1,2), \quad (6)$$

$$(\tilde\psi^b_r) = \psi^b_{r+w/2}, \qquad (\tilde\psi^\dagger_b)_r = (\psi^\dagger_b)_{r-w/2} (b = 3,4). \quad (7)$$

The tilde modes have half-integer mode numbers, and thus the untilde modes can be either integer or half-integer valued. The precise form of the spectral flow is a natural generalization of the one defined in [7,8]. In that case, the mode numbers were shifted by $(J^3_0 - K^3_0)$, where J^3_0 and K^3_0 are the spacetime dilatation and R-symmetry current generators, respectively. Here, the modes are analogously shifted by $(\mathcal{D}_0 - \mathcal{R}_0)$. It is obvious that Eqs. (4)–(7) preserve the free field algebra [Eq. (1)], and hence the bilinears of the tilde modes also obey the same commutation relations of $\mathfrak{u}(2,2|4)_1$. We can therefore use these tilde modes to realize new representations of this affine algebra.

Let us denote by $|0\rangle_w$ the spectrally flowed vacuum state, i.e., the state that is annihilated by all the tilde modes with a positive (half-integer) mode number. In terms of the original modes (without tildes), this means

$$\mu_r^{\dot\alpha}, \quad (\mu_{\dot\alpha}^\dagger)_r, \quad (\psi_{1,2}^\dagger)_r, \quad \psi_r^{3,4}, \quad \left(-\frac{w-1}{2} \le r \le \frac{w-1}{2}\right) \tag{8}$$

now act as creation operators on $|0\rangle_w$. In addition, the modes

$$\lambda_r^{\alpha}, \quad (\lambda_{\dot\alpha}^\dagger)_r, \quad (\psi^{1,2})_r, \quad (\psi_{3,4}^\dagger)_r, \tag{9}$$

with $r \le -[(w+1)/2]$, also do not annihilate $|0\rangle_w$ (and altogether they generate the full Fock space). We will denote the former (and their canonical conjugates) as wedge modes, and the latter (and their conjugates with $r \ge [(w+1)/2]$) as "out-of-the-wedge" modes.

In terms of the original (untilde) modes, the resulting representations are not highest weight since some of the positive (affine) modes do not annihilate the ground state. In analogy to the situation for AdS_3, we propose that the worldsheet spectrum consists of the familiar free field Neveu-Schwarz sector (where all free field modes are half-integer moded), together with the w spectrally flowed images of this Neveu-Schwarz sector, where $w \in \mathbb{N}$ acts simultaneously on both left and right movers.

The spectral flow also has a nontrivial effect on the dilatation operator $\mathcal{D}_0$ and the Cartan generators of $\mathfrak{su}(4)$, and we find explicitly

$$\tilde{\mathcal{D}}_n = \mathcal{D}_n - w\delta_{n,0}, \qquad \tilde{\mathcal{R}}_n = \mathcal{R}_n - w\delta_{n,0}, \tag{10}$$

where $\mathcal{R}_n = \frac{1}{2}[-(\mathcal{R}_1^1)_n - (\mathcal{R}_2^2)_n + (\mathcal{R}_3^3)_n + (\mathcal{R}_4^4)_n]$, and $\tilde{\mathcal{D}}_0|0\rangle_w = \tilde{\mathcal{R}}_0|0\rangle_w = 0$; see the Supplemental Material [19] for details. The spectral flow also shifts the worldsheet Virasoro generator as in the AdS_3 case

$$\tilde{L}_n = L_n - w(\mathcal{D}_n - \mathcal{R}_n). \tag{11}$$

Note that the combination $(\mathcal{D}_0 - \mathcal{R}_0)$ is the BMN-like light-cone Hamiltonian, which vanishes on the Virasoro highest weight half–Bogomol'nyi-Prasad-Sommerfield (half-BPS) state $|0\rangle_w$. It might seem that we are breaking the $\mathfrak{su}(4)$ invariance with our choice of spectral flow on the fermions—see Eqs. (6)–(7)—but this prescription only picks out a specific $\mathfrak{su}(4)$ highest weight state, and the physical states lie in complete $\mathfrak{psu}(2,2|4)$ representations, as will become clear momentarily. Note that although $|0\rangle_w$ is not a highest weight affine primary, Eq. (11) ensures that it is a Virasoro primary.

Worldsheet spectrum.—We now motivate the quantization of our closed string theory, again as a generalization of the $\mathrm{AdS}_3 \times \mathrm{S}^3$ case. There, we have half as many bosons and fermions, and a similar $\mathfrak{u}(1)$ generator, which has to be quotiented out in order to get the $\mathfrak{psu}(1,1|2)$ spacetime current algebra [7,8]. (There is also a topologically twisted $\mathbb{T}^4$ sector in that case, which is absent for $\mathrm{AdS}_5 \times \mathrm{S}^5$.) Following [22,23], we expect that this worldsheet theory is quantized through its embedding in the $N = 2$ string, with the above $\mathfrak{u}(1)$ constraint being implemented by the topologically twisted $N = 2$ algebra [24]. This would then lead to two bosonic constraints (arising from the Virasoro and the $\mathfrak{u}(1)$ current of the $N = 2$), as well as the two fermionic ones (from the two supercurrents). Locally, this removes four bosonic and four fermionic physical degrees of freedom. As seen in [7], we are left with an essentially topological theory of the (generalized) zero modes on $\mathrm{AdS}_3 \times \mathrm{S}^3$, together with the physical $\mathbb{T}^4$ excitations. We will comment more on this later.

Given that there are twice as many fermions and bosons, we expect that we can quantize the present $\mathfrak{psu}(2,2|4)_1$ theory by an analogous embedding into the small $N = 4$ string where the $\mathfrak{u}(1)$ constraint is again being implemented by the topologically twisted small $N = 4$ algebra. We note that the ghost system associated to this topologically twisted $N = 4$ string has a vanishing central charge so that the net central charge, after adding the matter piece, is still zero; see also [25]. The bosonic generators of the $N = 4$ algebra consist of a triplet of currents and the Virasoro generator, while we have now four fermionic supercurrents. We therefore expect that imposing the $N = 4$ constraints will locally remove eight bosonic and eight fermionic degrees of freedom, leaving only a topological sector behind. More specifically, we postulate that the $N = 4$ conditions remove all eight bosonic and fermionic out-of-the-wedge modes, and we thus retain only the wedge modes $\{-[(w-1)/2] \le r \le [(w-1)/2]\}$ of Eq. (8).

The Fock space generated by these modes is the tensor product of w copies of that generated by a single set of these oscillators. The latter generate what is called the singleton representation of $\mathfrak{psu}(2,2|4)$ once one imposes the condition of vanishing central charge $\mathcal{C}$ (see, e.g., [26] and the Supplemental Material [19]). It is natural that physical states should be defined by requiring the residual constraints $\mathcal{C}_n|\mathrm{phys}\rangle = 0$ (for $n = 0, \ldots, w-1$). Similarly, we would expect that translation invariance along the discretized worldsheet made up by the w string bits would be guaranteed by requiring the generalized mass-shell condition $L_0 = 0 (\mathrm{mod}\, w)$. These are the two residual constraints mentioned in the introduction that determine the physical string spectrum. Since $\mathcal{C}_n|0\rangle_w = L_0|0\rangle_w = 0$ for

$n \geq 0$, the spectrally flowed vacuum state $|0\rangle_w$ is a physical state by this criterion. We shall presently identify it with the half-BPS BMN vacuum.

Matching with the SYM spectrum.—We give a general argument for the matching of the string spectrum defined above with the planar spectrum of free $\mathcal{N} = 4$ supersymmetric Yang-Mills theory in [16]. In the following, we will get a sense for how this works in practice by looking at special cases. Let us first define the generators:

$$(S_I{}^J)_m = \sum_{r=m-\frac{w-1}{2}}^{\frac{w-1}{2}} (Y_I)_r (Z^J)_{m-r}, \tag{12}$$

where Y_I and Z^J were introduced before; see the paragraph below Eq. (1). For $m \geq 0$, these generators map our Fock space to itself, and they commute with $\mathcal{C}_n$ on this Fock space. The zero modes $(S_I^J)_0$ map physical states to physical states, and they furnish, in fact, a realization of the global $\mathfrak{u}(2,2|4)$ symmetry; this shows, in particular, that the physical states fall into representations of the global $\mathfrak{psu}(2,2|4)$ algebra.

For example, starting from the physical state $|0\rangle_w$ in the w spectrally flowed sector, the $(S_I^J)_0$ generate the full $\mathfrak{psu}(2,2|4)$ half-BPS multiplet with quantum numbers $(0,0;[0,w,0])_w$; here, the first two entries are the 4D spins with respect to the two $\mathfrak{su}(2)$ subalgebras of $\mathfrak{psu}(2,2|4)$, while $[0,w,0]$ are the Dynkin labels of $\mathfrak{su}(4)$, and the subscript denotes the eigenvalue of $\mathcal{D}_0$ [27]. [The nontrivial quantum numbers are a consequence of the shifts in Eq. (10).] In fact, we have checked for small values of w that this BPS multiplet accounts for all the physical states in the sector with $L_0 = 0$. (For $w = 2$, this also follows from our more general argument below.) Our construction therefore gives rise to a covariant version of the light-cone multiplet built on the BMN vacuum.

A relatively elementary counting argument suggests that there should be no physical states with $L_0 > 0$, and we have also verified this explicitly for some simple cases. This leaves us with the physical states with $L_0 = -mw$ $(m > 0)$, which have, in general, a quite complicated structure. We have confirmed by explicit computations (in particular we have checked all states with $w, \mathcal{D}_0 \leq 4$) that they reproduce the intricate set of non-BPS multiplets (built from w free fields) of the single trace spectrum of $\mathcal{N} = 4$ SYM, enumerated, for example, in [28–30]. We will give more details of the comparison in [16].

In the following, we shall concentrate on the case $w = 2$ where we can be more explicit. In this case, the modes in Eq. (8) have mode numbers $\pm\frac{1}{2}$. To describe the physical states with $L_0 = -2m$, we observe that we can reduce any physical state by the application of suitable $(S_I^J)_0$ generators to one that has the smallest number $(4m)$ of oscillators, all with mode number $(+\frac{1}{2})$. (In particular, if we choose I and J to both correspond to the modes in Eq. (9), this reduces the number of modes by 2.) Furthermore, all such states can be mapped into one another by the action of the $(S_I^J)_0$ zero modes [where we now take one index to correspond to Eq. (9), and one to Eq. (8)]. This shows that all the physical states at $L_0 = -2m$ lie in an irreducible representation of $\mathfrak{psu}(2,2|4)$.

We can actually identify these irreducible representations explicitly. For $L_0 = 0$, the physical states sit in the BPS multiplet $(0,0;[0,2,0])_2$ that is generated from $|0\rangle_2$ by the $(S_I^J)_0$ modes. For $L_0 = -2$, the physical state with smallest $\mathcal{D}_0$ eigenvalue is obtained by applying the four fermionic $+\frac{1}{2}$ modes to $|0\rangle_2$; this leads to the highest weight state of the $\mathfrak{psu}(2,2|4)$ representation $(0,0;[0,0,0])_2$—the non-BPS Konishi multiplet. This construction generalizes to $L_0 = -2m$, for which the states with lowest $\mathcal{D}_0$ eigenvalue are

$$\prod_{i=1}^{2m-2} [(\mu^\dagger_{\alpha_i})_{\frac{1}{2}} \mu^{\dot{\alpha}_i}_{\frac{1}{2}}] (\psi^\dagger_1)_{\frac{1}{2}} (\psi^\dagger_2)_{\frac{1}{2}} \psi^3_{\frac{1}{2}} \psi^4_{\frac{1}{2}} |0\rangle_2, \tag{13}$$

and they generate the $\mathfrak{psu}(2,2|4)$ representation

$$L_0 = -2m\colon\ (m-1, m-1; [0,0,0])_{2m} \tag{14}$$

since the indices $\{\alpha_i\}$ and $\{\dot{\alpha}_i\}$ are totally symmetrized. These are the higher spin conserved currents (bilinears) of the free theory with the lowest component having spin $(2m-2)$. This then precisely reproduces the first line of [[30], Eq. (3.8)]—the representation in Eq. (14) is what is called $\mathcal{V}_{2m}$ there. These states, summed over m, combine into a single higher spin multiplet—the symmetrized doubleton—of the higher spin algebra $\mathfrak{hs}(2,2|4)$.

Let us also note that for $w = 0$ there are no physical states other than the Neveu-Schwarz vacuum that corresponds to the identity operator. For $w = 1$, the only physical states are in the $L_0 = 0$ sector (Ramond vacuum) and correspond to the singleton representation: a single copy of the supersymmetric Yang-Mills fields; see the Supplemental Material [19] for details. As usual, this is present only in a $\mathrm{U}(N)$ gauge theory and can be projected out. Thus, the first nontrivial states arise from the above $w = 2$ sector.

We can see that the organization of the spectrum is very similar to that of BMN: we have a BPS "vacuum," from which the full Fock space is generated by oscillator excitations that carry worldsheet momentum—the wedge modes can be thought of as being a discrete Fourier transform of w string bit position operators; see [16] for more details. However, our oscillators are twistorial rather than the light-cone Green-Schwarz ones of BMN [5,31]. In fact, what we have is a worldsheet realization of a picture proposed for weakly coupled supersymmetric Yang-Mills theory in [32].

Relation to the analysis for AdS_3.—It is instructive to apply our approach of quantizing the system to the case of

$AdS_3 \times S^3 \times \mathbb{T}^4$, i.e., to consider the reduced oscillator Fock space for that case. The relevant wedge modes are then associated to μ^1, $\mu_1^\dagger$, $\psi_1^\dagger$, ψ^3, say, and we need to impose the constraints of $\mathcal{C}_n = 0$, $L_0 = 0 (\mathrm{mod}\, w)$. Interestingly, we find a nice closed subsector of the states of the dual free symmetric product CFT. In some sense, these states are compactification independent and include, for $w = 2$, the higher spin fields of $\mathfrak{hs}(1,1|2)$ [33]. The states for higher w are special matter multiplets of the higher spin symmetry. We describe the details in [16]. The consistency of this result with the different approach of [7] is additional support for our proposed quantization procedure for $AdS_5 \times S^5$.

Comments.—We noted earlier that the worldsheet fields can be viewed as ambitwistor string variables Y_I, Z^I with the constraint $\mathcal{C} = \frac{1}{2} Y_I Z^I = 0$. There is also an analog of spectral flow in that language since one needs to sum over instanton sectors carrying nontrivial U(1) flux. In fact, in the presence of flux these sectors have a space of left-moving zero modes (the holomorphic sections of the nontrivial line bundle) for the twistors [34,35], which can be identified with our wedge modes. Viewing the wedge modes as (generalized) zero modes motivates considering only the chiral (say, left-moving) degrees of freedom, as we have implicitly done above. We should expect the left- and right-moving wedge modes not to be independent degrees of freedom but rather to be related by a reality constraint [36]. It would be interesting to understand the exact relation of our approach to quantizing the theory as an $N = 4$ string to the (ambi-)twistor string approach to supersymmetric Yang-Mills scattering amplitudes [9,37,38].

Discussion.—While we have invoked the superstructure of an embedding into a twisted $N = 4$ string to quantize our worldsheet theory, it is clear that the reduced oscillator construction we have motivated captures the physics of the free Yang-Mills spectrum. We expect correlators of our worldsheet theory to exhibit a similar localization as in the 2D case [8,10]. In fact, Feynman diagram contributions to free Yang-Mills correlators admit a geometric interpretation in terms of covering maps to twistor space [39] and generalize the correspondence with Strebel differentials [40,41] already seen in [42]. It would be interesting to also connect this to the picture of [43]. The free Yang-Mills spectrum exhibits a Hagedorn transition [44–46], and it would be nice to understand its physics in our worldsheet model.

We note that, once our proposed worldsheet theory is fully quantized, it should be possible to systematically consider perturbations away from the free point since there is a corresponding marginal operator on the worldsheet. Given our string bit picture, we expect to see a direct worldsheet reflection of the integrable spin chain Hamiltonians [47] and to be able to derive the AdS/CFT correspondence in a perturbative expansion.

Finally, we must remark that our construction has a truncation, effectively, to the bosonic twistor oscillators $(\mu^{\dot{\alpha}}, \mu_{\dot{\alpha}}^\dagger)$, which then reproduces the planar spectrum of free, pure Yang-Mills. It is also intriguing that the twisted $N = 4$ string embedding admits the $N = 0$ (bosonic) string as a special vacuum, where the worldsheet supersymmetry is spontaneously broken [48,49]. Could this be the route to obtaining the long sought after string dual to planar QCD?

We thank Andrea Dei, Bob Knighton, Pronobesh Maity, Kiarash Naderi, and Vit Sriprachyakul for useful discussions and related collaborations. The research of M. R. G. is supported by a personal grant of the Swiss National Science Foundation and more generally via the NCCR SwissMAP. R. G.'s research is partially supported by a J. C. Bose fellowship of the DST, as well as project RTI4001 of the Department of Atomic Energy, Government of India.

*Corresponding author.
gaberdiel@itp.phys.ethz.ch
†Corresponding author.
rajesh.gopakumar@icts.res.in

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