Nonequilibrium Dynamics and Weakly Broken Integrability

Joseph Durnin,[1] M. J. Bhaseen,[2] and Benjamin Doyon[1]
[1]*Department of Mathematics, King's College London, Strand WC2R 2LS, United Kingdom*
[2]*Department of Physics, King's College London, Strand WC2R 2LS, United Kingdom*

(Received 2 June 2020; revised 12 May 2021; accepted 2 July 2021; published 20 September 2021)

Motivated by dynamical experiments on cold atomic gases, we develop a quantum kinetic approach to weakly perturbed integrable models out of equilibrium. Using the exact matrix elements of the underlying integrable model, we establish an analytical approach to real-time dynamics. The method addresses a broad range of timescales, from the intermediate regime of prethermalization to late-time thermalization. Predictions are given for the time evolution of physical quantities, including effective temperatures and thermalization rates. The approach provides conceptual links between perturbed quantum many-body dynamics and classical Kolmogorov-Arnold-Moser theory. In particular, we identify a family of perturbations which do not cause thermalization in the weakly perturbed regime.

DOI: 10.1103/PhysRevLett.127.130601

Conservation laws play a ubiquitous role in constraining the dynamics of complex many-body systems. This is especially true in low-dimensional integrable systems, where their proliferation gives rise to rich phenomena. A striking example is provided by the quantum Newton's cradle experiment [1], which shows the absence of thermalization over long timescales. The impact of conservation laws in this so-called prethermalization regime is directly encoded via a generalized Gibbs ensemble (GGE) [2–7]: Each conserved quantity is associated with its own effective temperature, leading to anomalous thermalization. This has stimulated a wealth of theoretical activity, including the recent extension of hydrodynamics [8–12] to integrable systems [13–15] and its application to experiment [16,17]. For recent reviews exploring the exotic dynamics of isolated quantum integrable systems, see [18–25].

Despite recent advances in the understanding of integrable systems, real physical systems always contain perturbations. These may influence and destabilize the integrable dynamics, but their effect is hard to quantify. In the classical domain, the effect of weak perturbations is encoded in Kolmogorov-Arnold-Moser (KAM) theory [26], which describes the persistence of quasiperiodic orbits under small perturbations. In the quantum many-body domain, the scenario of prethermalization followed by slow thermalization has been widely studied in this context [27–42]; for recent reviews, see [43,44]. However, insights analogous to KAM theory have been hard to establish, and many experimentally and conceptually relevant questions remain. To what extent does quantum integrability survive in the presence of weak perturbations? How can we quantify and organize the dynamical effects of integrability-destroying interactions? What are the relevant timescales?

In this Letter, we address these questions by developing a quantum kinetic approach to weakly perturbed integrable models out of equilibrium. We show that the dynamics of physical observables from short to long timescales can be described using the exact matrix elements of the underlying integrable model. Our findings are illustrated by numerical evaluation of the key formulas, including the time evolution of the average densities, quasiparticle distributions, and effective temperatures. Embedding the kinetic approach into a general theory, we identify dynamical response functions which encode the timescales of thermalization. We also find a family of integrability-breaking, KAM-like perturbations, which do not lead to thermalization in the weakly coupled regime. A notable insight which emerges from our analysis is that, in one spatial dimension, thermalization and hydrodynamic diffusion are controlled by distinct families of processes, which we characterize. Our findings also provide the integrability-destroying corrections to the Euler hydrodynamics of integrable systems.

Setup.—We consider the general scenario in which a spatially homogeneous one-dimensional integrable system, described by Hamiltonian H_0, is perturbed by an extensive integrability-destroying term $V = \int dx v(x)$. The resulting Hamiltonian is given by $H = H_0 + \lambda V$, where λ controls the strength of the perturbation. The Hamiltonian H_0 is characterized by an infinite number of mutually commuting conserved quantities Q_i, $i = 0, 1, 2, \ldots$, including the momentum $P = Q_1$ and the Hamiltonian $H_0 = Q_2$. In the perturbed system, only two conserved quantities remain: the total energy H and the total momentum P.

In order to explore the dynamics of the nonintegrable Hamiltonian H, we consider a quantum quench from an initial state which is stationary under H_0 but which evolves under the dynamics of H. In light of the integrability of H_0, it is natural to take a GGE as the initial state, whose density matrix is given by $\rho_0 = Z^{-1} e^{-\sum_i \beta_i Q_i}$.

Here, $Z = \mathrm{Tr}(e^{-\sum_i \beta_i Q_i})$ and the β_i are the inverse effective temperatures associated with each conserved quantity Q_i. These are the most general states that maximize entropy with respect to all of the extensive conserved quantities of H_0; they, therefore, provide natural initial states for studying the dynamics of perturbed integrable systems.

The quench setup described above is well suited to studying thermalization. At long times, it is expected that expectation values of local observables $\langle \mathcal{O}(x,t)\rangle = \mathrm{Tr}[\rho_0 e^{iHt}\mathcal{O}(x)e^{-iHt}]$ tend to the value they would take in a boosted thermal ensemble described by H and P. Explicitly, $\lim_{t\to\infty}\langle \mathcal{O}(x,t)\rangle = Z_s^{-1}\mathrm{Tr}[e^{-\beta_s(H-\nu_s P)}\mathcal{O}(x)]$, where the stationary values β_s and ν_s are uniquely fixed by $\langle H\rangle$ and $\langle P\rangle$. Thermalization is proven rigorously in various situations [45–48], and if it occurs, it does so for any perturbation strength λ. From a physical perspective, however, the most important questions are to what extent integrability still plays a role at finite times and how the system reaches thermalization. For small perturbations, it could be expected that integrability strongly influences these processes and constrains the dominant physics.

Dynamics of charges.—To see the effects of the integrability-breaking term, it is instructive to examine the time evolution of the charges Q_i under the Hamiltonian H. To lowest order in λ, the time evolution of the corresponding charge densities $q_i(x,t)$ can be computed within second-order perturbation theory:

$$\partial_t\langle q_i(0,t)\rangle = \lambda^2\int_{-t}^{t} ds\langle[V^0(s),Q_i]v(0)\rangle^c, \qquad (1)$$

where here and throughout we set $\hbar = k_B = 1$ and we denote the connected correlation function by $\langle\ldots\rangle^c$. Time evolution on the left-hand side is with respect to the nonintegrable Hamiltonian H, while time evolution on the right-hand side is with respect to the integrable H_0, with $V^0(s) = e^{iH_0 s}Ve^{-iH_0 s}$; see Supplemental Material [49].

A key feature of this perturbative approach is that it can describe both the rapid onset of prethermalization and the slower process of thermalization. As prethermalization builds up on a λ-independent timescale, the state changes abruptly, but the conserved densities receive only small corrections of the order of λ^2, as follows from Eq. (1). As a result, the prethermalized state is nonthermal and is, in fact, close to a new GGE for the unperturbed Hamiltonian H_0. Afterward, the dynamics occurs over timescales of the order of $1/\lambda^2$. We will refer to this as the Boltzmann regime. It is accessed by the formal $t\to\infty$ limit of Eq. (1), with $\bar{t} = \lambda^2 t$ held fixed. In this limit, the unperturbed energy density is stationary, while the $\bar{t}$ derivatives of other observables take finite, nonzero values, which satisfy the GGE equations of state. Proofs of these statements can be found in Refs. [40,53]. Thus, in the Boltzmann regime, the GGE continues to evolve slowly with time. The final stationary regime is expected to occur for $t \gg 1/\lambda^2$, which requires going beyond the perturbative result (1); see [31]. Nonetheless, for weakly broken integrability, the Boltzmann regime is very long in comparison with experimental timescales. Moreover, its physical properties are fully accessible using integrability, as we now demonstrate.

Form factors.—As the right-hand side of Eq. (1) involves time evolution under the integrable Hamiltonian H_0, powerful techniques are available for its evaluation. The principal idea is that the matrix elements of the perturbing operator v can be computed by means of a spectral decomposition, in terms of a suitable basis of eigenstates of H_0. For example, the initial GGE can be represented by a state $|\rho_p\rangle$, with $Z^{-1}\mathrm{Tr}(\rho\mathcal{O}) = \langle\rho_p|\mathcal{O}|\rho_p\rangle$. Here, the quasiparticle density $\rho_p(\theta)$, as a function of the rapidity θ, is fixed by the thermodynamic Bethe ansatz [54–56]. Excited states $|\rho_p;\boldsymbol{p},\boldsymbol{h}\rangle$ involve particle and hole excitations on top of this [57–62], where $\boldsymbol{p}$ and $\boldsymbol{h}$ indicate their respective sets of rapidities. These diagonalize the momentum Q_1, energy Q_2, and other conserved quantities Q_i, with one-particle eigenvalues given by $\kappa(\theta)$, $\varepsilon(\theta)$, and $\eta_i(\theta)$, respectively. Performing the spectral decomposition on Eq. (1) yields

$$\partial_{\bar{t}}\langle q_i(0,t)\rangle = 2\int d\tilde{\boldsymbol{p}}d\tilde{\boldsymbol{h}}\eta_i\delta(\kappa)\frac{\sin\varepsilon t}{\varepsilon}|\langle\rho_p;\boldsymbol{p},\boldsymbol{h}|v|\rho_p\rangle|^2, \qquad (2)$$

as shown in Supplemental Material [49]. The integrand $d\tilde{p} = dp\rho_h(\boldsymbol{p})$ includes the factor $\rho_h(\boldsymbol{p}) = \prod_{\theta\in\boldsymbol{p}}\rho_h(\theta)$. This describes the accessible "phase space" given by the density of holes $\rho_h(\theta)$, and likewise for $\rho_p(h)$ in terms of $\rho_p(\theta)$. Here, $\kappa = \sum_{\theta\in\boldsymbol{p}}\kappa(\theta) - \sum_{\gamma\in\boldsymbol{h}}\kappa(\gamma)$, and similarly for ε and η_i.

The expression (2) has a simple interpretation: In accordance with Refs. [63,64], particles and holes are in and out states of scattering processes. The change in the charge density $\langle q_i(0,t)\rangle$ is given by a weighted sum over all the momentum-conserving processes, with transition rates given by the form factors squared $|\langle\rho_p;\boldsymbol{p},\boldsymbol{h}|v|\rho_p\rangle|^2$ of the perturbing operator, in conformity with Fermi's golden rule. By evaluating these matrix elements, one can obtain a quantitative description of the thermalization process, from short to long timescales.

Prethermalization.—The form factor approach gives a quantitative approach to prethermalization which is consistent with previous results. For example, after an interaction quench, the charge densities undergo fast initial dynamics, followed by an oscillatory power-law approach to a quasistationary regime which persists for long times. This can be verified by applying a small ϕ^4 perturbation to a free massive scalar field, whose form factors can be evaluated using the methods of Ref. [59]. The results are provided in Supplemental Material [49]; similar numerical results are obtained in Ref. [65].

Boltzmann regime.—After prethermalization, the approximate GGE continues to evolve in accordance with Eq. (1). In the Boltzmann regime, the time evolution of the state is slow, varying over long timescales of the order of $1/\lambda^2$. As such, the change in the state can be large, with the power-law tails describing the approach to the instantaneous GGE giving perturbatively small corrections. In this regime, the evolution is toward an (approximate) boosted thermal state for the final Hamiltonian, in accordance with thermalization. Taking $t \to \infty$, the evolution equations in this regime are given by

$$\partial_{\bar{t}} \langle q_i \rangle_{\boldsymbol{\beta}(\bar{t})} = \int_{-\infty}^{\infty} ds \langle [V^0(s), Q_i] v(0) \rangle^c_{\boldsymbol{\beta}(\bar{t})}, \quad (3)$$

where the subscript $\boldsymbol{\beta}(\bar{t})$ indicates that the expectation value is taken in the instantaneous GGE. As we demonstrate in Supplemental Material [49], a general H theorem shows that Eq. (3) is consistent with thermalization.

The spectral decomposition (2) available for integrable systems allows us to recast Eq. (3) as a Boltzmann-type kinetic equation. This sums over energy- and momentum-conserving scattering processes with arbitrary numbers of particles. This generalizes approaches based on the kinetics of free models [16,34,66–73] to interacting integrable systems. Reexpressing Eq. (3) in terms of the time-dependent quasiparticle density $\rho_p(\theta)$, which represents the time-evolving GGE (see Supplemental Material [49]), one obtains

$$\begin{aligned} \partial_{\bar{t}} \rho_p(\theta) &= I[\rho_p](\theta) \\ &:= \int d\boldsymbol{p} d\boldsymbol{h} K(\theta, \boldsymbol{p}) B(\boldsymbol{p} \to \boldsymbol{h}) \\ &\quad \times [\rho_h(\boldsymbol{p}) \rho_p(\boldsymbol{h}) - \rho_p(\boldsymbol{p}) \rho_h(\boldsymbol{h})], \end{aligned} \quad (4)$$

where

$$B(\boldsymbol{p} \to \boldsymbol{h}) = 2\pi \delta(\kappa) \delta(\varepsilon) |\langle \rho_p | v | \rho_p; \boldsymbol{p}, \boldsymbol{h} \rangle|^2 = B(\boldsymbol{h} \to \boldsymbol{p}) \quad (5)$$

is the matrix element for particle-hole scattering processes. In the special case of perturbations of free models, $K(\theta, \boldsymbol{p}) = \sum_{\Phi \in \boldsymbol{p}} \delta(\theta - \Phi)$, and we have a generalization of the quantum Boltzmann equation to include higher-order scattering processes. If the unperturbed Hamiltonian H_0 is interacting, then K also describes the effect of indirect processes where a particle of rapidity θ is created or destroyed in the interacting background in response to a scattering event. In this case,

$$K(\theta, \boldsymbol{p}) = \sum_{\Phi \in \boldsymbol{p}} K(\theta, \Phi), \quad (6)$$

where

$$K(\theta, \Phi) = \delta(\theta - \Phi) + \frac{\partial}{\partial \Phi} \left[\frac{F(\theta, \Phi) \rho_p(\Phi)}{\rho_p(\Phi) + \rho_h(\Phi)} \right]. \quad (7)$$

Here, $F(\theta, \Phi)$ is the backflow function representing the effect of adding an excitation to the interacting background; see Supplemental Material [49]. Here, we assume particle-hole symmetry, in accordance with the usual microscopic reversibility condition of the Boltzmann scattering kernel (5).

We show in Supplemental Material [49] that an arbitrary boosted thermal state is a fixed point of the time evolution given in Eq. (4), confirming the general H theorem presented there. We note finally that Eq. (4) is an expansion in the number of excitations, which can often be recast as a low-density expansion. This is analogous to the LeClair-Mussardo series for equilibrium expectation values [74], which is observed to converge quickly.

Multiparticle scattering.—The kinetic equation (4) generically contains infinitely many scattering processes with arbitrarily large numbers of particles $p \to h$. In the absence of internal degrees of freedom, the $2 \to 2$ scattering processes do not contribute: These preserve momenta by $1+1$-dimensional kinematics, so the term in square brackets in Eq. (4) vanishes. This is consistent with the notion that thermalization requires the nontrivial rearrangement of momenta. In generic integrable models, the higher-particle form factors are typically nonzero, thereby leading to thermalization via Eq. (4). The ϕ^4 theory considered above is special, as these higher-particle form factors vanish. As such, it does not thermalize in the Boltzmann regime in $1+1$ dimensions, in agreement with three-loop results for correlation functions [75,76]. For the ϕ^6 perturbation, the $2 \to 4$ and $3 \to 3$ processes contribute. In Fig. 1, we show the time evolution of the rapidity distribution $n(\theta) = 2\pi \rho_p(\theta)/\cosh(\theta)$, and the first few effective temperatures, in a ϕ^6 quench. The results are consistent with thermalization and illustrate how effective temperatures may exhibit nonmonotonic dynamics.

In order to expose the relevant physics, we have concentrated for simplicity on a perturbation of a free model. However, an important aspect of this work is that it applies equally well to the case of an interacting integrable model. As an example, we consider the experimentally relevant case of two Lieb-Liniger gases perturbed by a density-density coupling. We consider arbitrary interaction strengths in the low-density regime. In this case, the two degrees of freedom allow for a nontrivial $2 \to 2$ contribution in the Boltzmann regime; for further details, see Supplemental Material [49].

Nearly integrable perturbations.—Perturbations that break integrability yet do not lead to thermalization in the Boltzmann regime can be seen as "nearly integrable perturbations," in analogy with the concept from KAM theory [26]. The ϕ^4 perturbation of the free massive scalar field discussed above is such an example. We show that such perturbations exist generically. To see this, we rewrite the time evolution (3) as

$$\partial_{\bar{t}} \langle q_i \rangle = ([v, Q_i], v), \quad (8)$$

where (a, b) is a suitable inner product [77], defined by

$$(a, b) = \int dt dx \langle (1 - \mathbb{P}) [a^0(x, t)]^\dagger b(0, 0) \rangle^c. \quad (9)$$

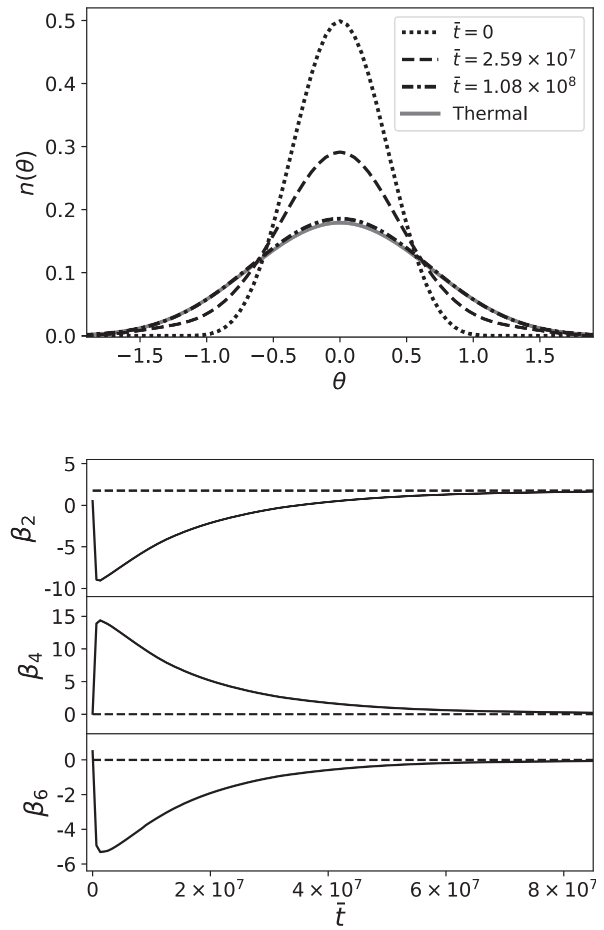

FIG. 1. Upper panel: time evolution of the rapidity distribution $n(\theta)$ in the Boltzmann regime, for the free scalar field theory with unit mass perturbed by $\lambda/(6!)\int dx\phi^6(x)$, following a quantum quench from $\lambda = 0$ to $\lambda > 0$. The initial distribution is the GGE with $\beta_2 = \beta_6 = 0.5$, $\beta_4 = 0.1$, and all other $\beta_i = 0$. At late times, $n(\theta)$ approaches a thermal distribution as indicated by the gray solid line. Lower panel: time evolution of the first three effective inverse temperatures for the same quench, showing a nonmonotonic approach to thermalization. The large values of $\bar{t}$ reflects the standard normalization conventions for the scalar field theory, which effectively reduces the strength of the ϕ^6 perturbation.

Here, $a^0(x,t) = e^{iH_0t}a(x)e^{-iH_0t}$, and $\mathbb{P}$ is the projector onto the space of charges Q_i; see Supplemental Material [49]. We show in Supplemental Material [49] that current operators j_k, satisfying $\partial_t q_k + \partial_x j_k = 0$, commute with the conserved charges under the inner product: $([j_k, Q_i], a) = 0$ for all a, i. According to Eq. (8), under a perturbation $v = j_k$, the state remains constant throughout the Boltzmann regime. Therefore, *current operators are nearly integrable perturbations*. This extends the notion of perturbed integrable models which preserve integrability in equilibrium [78–83]. For example, there exist families of integrable models, $H = H_0 + V_\lambda$, which correspond to perturbations by current operators, $V_\lambda = \lambda\int dx j_k(x) + O(\lambda^2)$, at leading order [84]. A similar relationship holds between the sine-Gordon model [85] and the ϕ^4 perturbation of the scalar field. The observation here is that thermalization is absent at leading order, despite these models not being integrable.

The discussion above gives a natural classification of perturbations and an associated classification of scattering processes. Indeed, under the inner product (9), local operators form a Hilbert space $\mathcal{H}''$ [77]. This admits an orthogonal decomposition $\mathcal{H}'' = \mathcal{H}_N \oplus \mathcal{H}_B$, where $\mathcal{H}_N$ is the nearly integrable subsector that commutes with Q_i within $\mathcal{H}''$, and $\mathcal{H}_B$ is the thermalizing Boltzmann subsector. In the kinetic description, operators in $\mathcal{H}_N$ couple only to $2 \to 2$ scattering processes. These, as explained above, do not lead to thermalization. It was shown in Refs. [63,64] that such processes lead to hydrodynamic diffusion instead, as they fully determine the Onsager matrix $\mathcal{L}_{ij} = (j_i, j_j)$ [77,86]. Thus, there is a separation between processes leading to hydrodynamic diffusion, associated with $\mathcal{H}_N$, and those leading to thermalization, associated with $\mathcal{H}_B$.

Thermalization and entropy production.—The late-time dynamics near the final, stationary state is obtained by linearizing the evolution operator [87]. In terms of the inverse effective temperatures β_i, this gives

$$\sum_j \mathsf{C}_{ij}\partial_{\bar{t}}\beta_j = -\sum_j \mathcal{B}_{ij}\beta_j; \qquad (10)$$

see Supplemental Material [49]. Here, we define the *Boltzmann matrix* $\mathcal{B}_{ij} = ([v, Q_i], [v, Q_j])$, while the static covariance matrix is $\mathsf{C}_{ik} = \partial\langle q_i\rangle/\partial\beta_k$; both are non-negative and evaluated in the stationary state. A similar evolution equation also holds for the small deviations of the conserved densities $\delta q_i = \langle q_i\rangle - \langle q_i\rangle_s$. As $\mathcal{B}_{ij} = 0$ for either $i, j = 1$ or 2, the spectrum of $\Gamma = \mathcal{B}\mathsf{C}^{-1}$ always contains the eigenvalue 0, corresponding to the conserved modes of the Boltzmann dynamics. The rest of the spectrum controls the rate of approach to thermalization: If it extends continuously to 0, then the approach is polynomial, whereas if there is a gap of size $\gamma > 0$, it is exponential $\delta q_i \propto e^{-t/\tau}$ with $\tau = \lambda^{-2}\gamma^{-1}$ [87–89]. It is notable that the timescale τ is determined solely by the final state, with the conserved energy and momentum densities containing the only information about the initial state.

In Fig. 2, we show numerical results consistent with an exponential approach to thermalization for the ϕ^6 perturbation. Therefore, for the ϕ^6 perturbation, the spectrum of the Boltzmann matrix has a gap $\gamma > 0$. At high temperatures, we find an increasing thermalization timescale $\tau \sim T^\alpha$ with $\alpha \approx 3/2$, corresponding to an effectively gapless regime. In contrast, at low temperatures, we observe Arrhenius behavior with $\tau \sim e^{3m/T}$, corresponding to the three-body collisions in the ϕ^6 theory; see Supplemental Material [49].

The Boltzmann matrix determines the late-time dynamics of all physical quantities. Notably, the production of entropy near the final stationary state takes the form

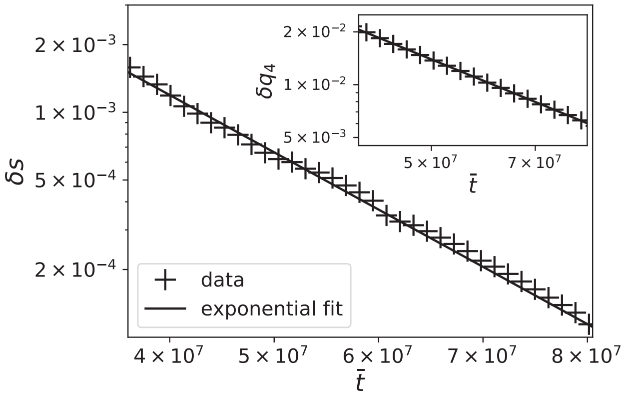

FIG. 2. Exponential approach of the entropy and (inset) higher-order charge q_4 to their stationary values, at times $\bar{t} \gg 1$, for the same quench protocol as in Fig. 1. The timescale $\gamma^{-1} \approx 3.58 \times 10^7$ is found for δq_4, and 1.71×10^7 for δs, in agreement with the theoretical value $\gamma^{-1}/2$.

$$\partial_{\bar{t}} s = \sum_{i,j\geq 3} \beta_i \mathcal{B}_{ij} \beta_j = ([v, \log\rho], [v, \log\rho]), \qquad (11)$$

where $\log\rho = -\sum_i \beta_i Q_i$ is the entropy operator; see Supplemental Material [49]. As the right-hand side in Eq. (11) is quadratic in the β_i's, if there is a gap γ, the time evolution of the entropy is also exponential, but with a rate 2γ. This is twice that found in the time evolution of the inverse temperatures and charge densities, which we confirm in Fig. 2.

Exponential decay can also be seen in correlation functions, as they are determined at large times by the conserved quantities. By projection methods, two-point functions at scaled wave numbers $\bar{k} = k/\lambda^2$ in the final state behave as

$$\langle \mathcal{O}_1 \mathcal{O}_2 \rangle^c (\bar{k}, \bar{t}) = -\sum_{ij} \partial_{\langle q_i \rangle} \langle \mathcal{O}_1 \rangle \exp[i\mathsf{A}\bar{k}\,\bar{t} - \Gamma|\bar{t}|]_{ij} \partial_{\beta_j} \langle \mathcal{O}_2 \rangle, \qquad (12)$$

where the matrix $\mathsf{A}_{ij} = \partial_{\langle q_j \rangle} \langle j_i \rangle$ encodes the propagation of the conserved modes and Γ their decay. In particular, this gives the Lorentzian broadening of the Drude peaks associated with the broken charges, $\int d\bar{t} e^{i\bar{\varpi}\bar{t}} \langle j_i j_k \rangle^c (\bar{k} = 0, \bar{t}) = 2[\mathsf{A}(\Gamma^2 + \bar{\varpi}^2)^{-1} \Gamma \mathsf{AC}]_{ik}$; see also [42]. We observe that the singularity in the complex ω plane that is nearest to the real line is at a distance γ. Dynamical correlation functions in the thermal state, therefore, determine the rate of approach toward it. A similar situation also occurs in holographic models, where the eigenvalues of the Boltzmann matrix are analogous to quasinormal modes; see, for example, [90]. As a signature of the integrability of the unperturbed model, this singularity is expected to be a branch point because of the continuum of hydrodynamic modes parametrized by the rapidity θ.

Hydrodynamics.—The kinetic approach developed here is applicable beyond quenches from homogeneous states, to include integrability-destroying perturbations in the hydrodynamic description of integrable models [13,14]. In this context, the effects of integrability breaking on the diffusive scale were recently discussed in Ref. [42]. Here, we stress that the effects of weak perturbations are also manifest on the larger, Euler scale. In the Euler scaling limit $x, t \to \infty$, $\lambda \to 0$ with $\bar{t} = \lambda^2 t$ and $\bar{x} = \lambda^2 x$ held fixed, the entropy increase of local fluid cells occurs on Euler hydrodynamic timescales. The spectral decomposition (4) in the Boltzmann regime adds a generalized collision term $I(\theta)$ to the fluid equations, $\partial_{\bar{t}} \rho_p(\theta) + \partial_{\bar{x}} [v^{\rm eff}(\theta) \rho_p(\theta)] = I(\theta)$, where $v^{\rm eff}$ is given in Refs. [13,14]. This opens the door to future studies of the crossover from integrable to nonintegrable hydrodynamics, including the emergence of shocks, which are absent in the former case [91–93].

Conclusions.—In this work, we have developed a form factor approach to perturbed integrable models out of equilibrium. We have shown that one can address a broad range of timescales, including the approach to thermalization. We have provided analytical and numerical predictions for the time evolution of physical observables, including conserved charges, effective temperatures, and rapidity distributions. We observe that the rate of thermalization for entropy is always exactly twice as large as that for conserved charges. We have also shown that there always exists a family of perturbations that do not thermalize in the weakly perturbed regime. It would be interesting to verify these predictions in experiment.

All of the results contained in this Letter can be obtained from the equations provided. The data corresponding to the figures is available at [94].

We are grateful to Jacopo De Nardis and Balázs Pozsgay for comments and discussions. M. J. B. acknowledges helpful conversations with Marcos Rigol, and B. D. thanks Wojciech De Roeck for enlightening discussions and Isabelle Bouchoule and Jérôme Dubail for collaborations on related subjects. J. D. acknowledges funding from the Engineering and Physical Sciences Research Council Centre for Doctoral Training in Cross-Disciplinary Approaches to Non-Equilibrium Systems (CANES) under Grant No. EP/L015854/1. B. D. acknowledges funding from the Royal Society under a Leverhulme Trust Senior Research Fellowship, "Emergent hydrodynamics in integrable systems: non-equilibrium theory," Ref. No. SRF\R1\180103. B. D. is grateful to the Tokyo Institute of Technology for support and hospitality during an invited professorship, where part of this research was done. This research was further supported in part by the International Centre for Theoretical Sciences (ICTS) during a visit (J. D. and B. D.) for participating in the program Thermalization, Many body localization and Hydrodynamics (Code: ICTS/hydrodynamics2019/11). All authors thank the Centre for Non-Equilibrium Science (CNES) and the Thomas Young Centre (TYC).

[1] T. Kinoshita, T. Wenger, and D. S. Weiss, A quantum Newton's cradle, Nature (London) **440**, 900 (2006).
[2] M. Rigol, V. Dunjko, V. Yurovsky, and M. Olshanii, Relaxation in a Completely Integrable Many-Body Quantum System: An Ab Initio Study of the Dynamics of the Highly Excited States of 1D Lattice Hard-Core Bosons, Phys. Rev. Lett. **98**, 050405 (2007).
[3] M. Rigol, Breakdown of Thermalization in Finite One-Dimensional Systems, Phys. Rev. Lett. **103**, 100403 (2009).
[4] A. C. Cassidy, C. W. Clark, and M. Rigol, Generalized Thermalization in an Integrable Lattice System, Phys. Rev. Lett. **106**, 140405 (2011).
[5] M. Gring, M. Kuhnert, T. Langen, T. Kitagawa, B. Rauer, M. Schreitl, I. Mazets, D. A. Smith, E. Demler, and J. Schmiedmayer, Relaxation and prethermalization in an isolated quantum system, Science **337**, 1318 (2012).
[6] E. Ilievski, J. De Nardis, B. Wouters, J.-S. Caux, F. H. L. Essler, and T. Prosen, Complete Generalized Gibbs Ensembles in an Interacting Theory, Phys. Rev. Lett. **115**, 157201 (2015).
[7] T. Langen, S. Erne, R. Geiger, B. Rauer, T. Schweigler, M. Kuhnert, W. Rohringer, I. E. Mazets, T. Gasenzer, and J. Schmiedmayer, Experimental observation of a generalized Gibbs ensemble, Science **348**, 207 (2015).
[8] P. Nozieres and D. Pines, *The Theory of Quantum Liquids* (Benjamin, New York, 1966).
[9] S. Jeon and L. G. Yaffe, From quantum field theory to hydrodynamics: Transport coefficients and effective kinetic theory, Phys. Rev. D **53**, 5799 (1996).
[10] A. G. Abanov, Hydrodynamics of correlated systems, in *Applications of Random Matrices in Physics*, edited by E. Brézin, V. Kazakov, D. Serban, P. Wiegmann, and A. Zabrodin, NATO Science Series II: Mathematics, Physics and Chemistry (Springer, Dordrecht, 2006), Vol. 221, pp. 139–161.
[11] E. Bettelheim, A. G. Abanov, and P. Wiegmann, Nonlinear Quantum Shock Waves in Fractional Quantum Hall Edge States, Phys. Rev. Lett. **97**, 246401 (2006).
[12] M. J. Bhaseen, B. Doyon, A. Lucas, and K. Schalm, Far from equilibrium energy flow in quantum critical systems, Nat. Phys. **11**, 509 (2015).
[13] O. A. Castro-Alvaredo, B. Doyon, and T. Yoshimura, Emergent Hydrodynamics in Integrable Quantum Systems Out of Equilibrium, Phys. Rev. X **6**, 041065 (2016).
[14] B. Bertini, M. Collura, J. De Nardis, and M. Fagotti, Transport in Out-of-Equilibrium *XXZ* Chains: Exact Profiles of Charges and Currents, Phys. Rev. Lett. **117**, 207201 (2016).
[15] V. B. Bulchandani, R. Vasseur, C. Karrasch, and J. E. Moore, Bethe-Boltzmann hydrodynamics and spin transport in the XXZ chain, Phys. Rev. B **97**, 045407 (2018).
[16] J.-S. Caux, B. Doyon, J. Dubail, R. Konik, and T. Yoshimura, Hydrodynamics of the interacting Bose gas in the quantum Newton cradle setup, SciPost Phys. **6**, 070 (2019).
[17] M. Schemmer, I. Bouchoule, B. Doyon, and J. Dubail, Generalized Hydrodynamics on an Atom Chip, Phys. Rev. Lett. **122**, 090601 (2019).
[18] J. Eisert, M. Friesdorf, and C. Gogolin, Quantum many-body systems out of equilibrium, Nat. Phys. **11**, 124 (2015).
[19] F. H. L. Essler and M. Fagotti, Quench dynamics and relaxation in isolated integrable quantum spin chains, J. Stat. Mech. (**2016**) 064002.
[20] R. Vasseur and J. E. Moore, Nonequilibrium quantum dynamics and transport: From integrability to many-body localization, J. Stat. Mech. (2016) 064010.
[21] E. Ilievski, M. Medenjak, T. Prosen, and L. Zadnik, Quasilocal charges in integrable lattice systems, J. Stat. Mech. (2016) 064008.
[22] D. Bernard and B. Doyon, Conformal field theory out of equilibrium: A review, J. Stat. Mech. (2016) 064005.
[23] L. D'Alessio, Y. Kafri, A. Polkovnikov, and M Rigol, From quantum chaos and eigenstate thermalization to statistical mechanics and thermodynamics, Adv. Phys. **65**, 239 (2016).
[24] C. Gogolin and J. Eisert, Equilibration, thermalisation, and the emergence of statistical mechanics in closed quantum systems, Rep. Prog. Phys. **79**, 056001 (2016).
[25] B. Doyon, Lecture notes on generalised hydrodynamics, SciPost Phys. Lect. Notes 18 (2020).
[26] V. F. Lazutkin, *KAM Theory and Semiclassical Approximations to Eigenfunctions*, 1st ed., A Series of Modern Surveys in Mathematics Vol. 24 (Springer-Verlag, Berlin, 1993).
[27] I. E. Mazets, T. Schumm, and J. Schmiedmayer, Breakdown of Integrability in a Quasi-1D Ultracold Bosonic Gas, Phys. Rev. Lett. **100**, 210403 (2008).
[28] M. Moeckel and S. Kehrein, Interaction Quench in the Hubbard Model, Phys. Rev. Lett. **100**, 175702 (2008).
[29] M. Marcuzzi, J. Marino, A. Gambassi, and A. Silva, Prethermalization in a Nonintegrable Quantum Spin Chain after a Quench, Phys. Rev. Lett. **111**, 197203 (2013).
[30] F. H. L. Essler, S. Kehrein, S. R. Manmana, and N. J. Robinson, Quench dynamics in a model with tuneable integrability breaking, Phys. Rev. B **89**, 165104 (2014).
[31] J. Lux, J. Müller, A. Mitra, and A. Rosch, Hydrodynamic long-time tails after a quantum quench, Phys. Rev. A **89**, 053608 (2014).
[32] M. Mierzejewski, T. Prosen, and P. Prelovšek, Approximate conservation laws in perturbed integrable lattice models, Phys. Rev. B **92**, 195121 (2015).
[33] B. Bertini, F. H. L. Essler, S. Groha, and N. J. Robinson, Prethermalization and Thermalization in Models with Weak Integrability Breaking, Phys. Rev. Lett. **115**, 180601 (2015).
[34] N. Nessi and A. Iucci, Glass-like behavior in a system of one dimensional fermions after a quantum quench, arXiv:1503.02507.
[35] B. Bertini, F. H. L. Essler, S. Groha, and N. J. Robinson, Thermalization and light cones in a model with weak integrability breaking, Phys. Rev. B **94**, 245117 (2016).
[36] F. R. A. Biebl and S. Kehrein, Thermalization rates in the one-dimensional Hubbard model with next-to-nearest neighbor hopping, Phys. Rev. B **95**, 104304 (2017).
[37] V. Alba and M. Fagotti, Prethermalization at Low Temperature: The Scent of Long-Range Order, Phys. Rev. Lett. **119**, 010601 (2017).
[38] F. Lange, Z. Lenarčič, and A. Rosch, Time-dependent generalized Gibbs ensembles in open quantum systems, Phys. Rev. B **97**, 165138 (2018).
[39] M. Van Regemortel, H. Kurkjian, I. Carusotto, and M. Wouters, Prethermalization to thermalization crossover in a

dilute Bose gas following an interaction ramp, Phys. Rev. A **98**, 053612 (2018).
[40] K. Mallayya, M. Rigol, and W. De Roeck, Prethermalization and Thermalization in Isolated Quantum Systems, Phys. Rev. X **9**, 021027 (2019).
[41] Y. Tang, W. Kao, K.-Y. Li, S. Seo, K. Mallayya, M. Rigol, S. Gopalakrishnan, and B. L. Lev, Thermalization near Integrability in a Dipolar Quantum Newton's Cradle, Phys. Rev. X **8**, 021030 (2018).
[42] A. Friedman, S. Gopalakrishnan, and R. Vasseur, Diffusive hydrodynamics from integrability breaking, Phys. Rev. B **101**, 180302 (2020).
[43] T. Mori, T. N. Ikeda, E. Kaminishi, and M. Ueda, Thermalization and prethermalization in isolated quantum systems: A theoretical overview, J. Phys. B **51**, 112001 (2018).
[44] A. Mitra, Quantum quench dynamics, Annu. Rev. Condens. Matter Phys. **9**, 245 (2018).
[45] A. Riera, C. Gogolin, and J. Eisert, Thermalization in Nature and on a Quantum Computer, Phys. Rev. Lett. **108**, 080402 (2012).
[46] J. Sirker, N. P. Konstantinidis, F. Andraschko, and N. Sedlmayr, Locality and thermalization in closed quantum systems, Phys. Rev. A **89**, 042104 (2014).
[47] M. P. Mueller, E. Adlam, L. Masanes, and N. Wiebe, Thermalization and canonical typicality in translation-invariant quantum lattice systems, Commun. Math. Phys. **340**, 499 (2015).
[48] B. Doyon, Thermalization and Pseudolocality in extended quantum systems, Commun. Math. Phys. **351**, 155 (2017).
[49] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.127.130601 for technical derivations of the main results and references to [50–52].
[50] J. De Nardis and M. Panfil, Density form factors of the 1D Bose gas for finite entropy states, J. Stat. Mech. (2015) P02019.
[51] E. Granet and F. H. L. Essler, A systematic $1/c$-expansion of form factor sums for dynamical correlations in the Lieb-Liniger model, SciPost Phys. **9**, 082 (2020).
[52] B. Doyon and T. Yoshimura, A note on generalized hydrodynamics: Inhomogeneous fields and other concepts, SciPost Phys. **2**, 014 (2017).
[53] J. Durnin, M. J. Bhaseen, and B. Doyon (to be published).
[54] C. N. Yang and C. P. Yang, Thermodynamics of a one-dimensional system of Bosons with repulsive delta-function interaction, J. Math. Phys. (N.Y.) **10**, 1115 (1969).
[55] M. Takahashi, *Thermodynamics of One-Dimensional Solvable Models* (Cambridge University Press, Cambridge, England, 1999).
[56] A. Zamolodchikov, Thermodynamic Bethe Ansatz in relativistic models: Scaling 3-state Potts and Lee-Yang models, Nucl. Phys. **B342**, 695 (1990).
[57] B. Doyon, Finite-temperature form factors in the Majorana theory, J. Stat. Mech. (2005) P11006.
[58] B. Doyon, Finite-temperature form factors: A review, SIGMA **3**, 011 (2007).
[59] Y. Chen and B. Doyon, Form factors in equilibrium and non-equilibrium mixed states of the ising model, J. Stat. Mech. (2014) P09021.
[60] J. De Nardis and M. Panfil, Particle-hole pairs and density-density correlations in the Lieb-Liniger model, J. Stat. Mech. (2018) 033102.
[61] A. Cortés Cubero and M. Panfil, Thermodynamic bootstrap program for integrable QFT's: Form factors and correlation functions at finite energy density, J. High Energy Phys. 01 (2019) 104.
[62] A. Cortés Cubero and M. Panfil, Generalized hydrodynamics regime from the thermodynamic bootstrap program, SciPost Phys. **8**, 004 (2020).
[63] J. De Nardis, D. Bernard, and B. Doyon, Hydrodynamic Diffusion in Integrable Systems, Phys. Rev. Lett. **121**, 160603 (2018).
[64] J. De Nardis, D. Bernard, and B. Doyon, Diffusion in generalized hydrodynamics and quasiparticle scattering, SciPost Phys. **6**, 049 (2019).
[65] J. Berges, S. Borsányi, and C. Wetterich, Prethermalization, Phys. Rev. Lett. **93**, 142002 (2004).
[66] D. Boyanovsky, I. D. Lawrie, and D. S. Lee, Relaxation and kinetics in scalar field theories, Phys. Rev. D **54**, 4013 (1996).
[67] J. Rau and B. Müller, From reversible quantum microdynamics to irreversible quantum transport, Phys. Rep. **272**, 1 (1996).
[68] S. Juchem, W. Cassing, and C. Greiner, Quantum dynamics and thermalization for out-of-equilibrium φ^4 theory, Phys. Rev. D **69**, 025006 (2004).
[69] D. Benedetto, F. Castella, R. Esposito, and M. Pulvirenti, From the n-body Schrödinger equation to the quantum Boltzmann equation: A term-by-term convergence result in the weak coupling regime, Commun. Math. Phys. **277**, 1 (2008).
[70] S. Hollands and G. Leiler, On the derivation of the Boltzmann equation in quantum field theory: Flat spacetime, arXiv:1003.1621.
[71] M. L. R. Fürst, J. Lukkarinen, P. Mei, and H. Spohn, Derivation of a matrix-valued Boltzmann equation for the Hubbard model, J. Phys. A **46**, 485002 (2013).
[72] M. Stark and M. Kollar, Kinetic description of thermalization dynamics in weakly interacting quantum systems, arXiv:1308.1610.
[73] M. Drewes, S. Mendizabal, and C. Weniger, The Boltzmann equation from quantum field theory, Phys. Lett. B **718**, 1119 (2013).
[74] A. LeClair and G. Mussardo, Finite temperature correlation functions in integrable QFT, Nucl. Phys. **B552**, 624 (1999).
[75] J. Berges and J. Cox, Thermalization of quantum fields from time-reversal invariant evolution equations, Phys. Lett. B **517**, 369 (2001).
[76] J. Berges, Quantum fields far from equilibrium and thermalization, in *Proceedings of the Sixth Workshop, Non-Perturbative QCD*, edited by H. M. Fried, Y. Gabellini, and B. Müller (World Scientific, Singapore, 2002), p. 85, https://doi.org/10.1142/4809.
[77] B. Doyon, Diffusion and superdiffusion from hydrodynamic projections, arXiv:1912.01551.
[78] A. B. Zamolodchikov, Integrable field theory from conformal field theory, Adv. Stud. Pure Math. **19**, 641 (1989).
[79] F. A. Smirnov and A. B. Zamolodchikov, On space of integrable quantum field theories, Nucl. Phys. **B915**, 363 (2017).
[80] A. Cavaglià, S. Negro, I. M. Szécsényi, and R. Tateo, $T\bar{T}$ deformed 2D quantum field theories, J. High Energy Phys. 10 (2016) 112.

[81] T. Bargheer, N. Beisert, and F. Loebbert, Long-range deformations for integrable spin chains, J. Phys. A **42**, 285205 (2009).

[82] B. Pozsgay, Y. Jiang, and G. Takács, $T\bar{T}$-deformation and long range spin chains, J. High Energy Phys. 03 (2020) 092.

[83] E. Marchetto, A. Sfondrini, and Z. Yang, $T\bar{T}$-Deformation and Integrable Spin Chains, Phys. Rev. Lett. **124**, 100601 (2020).

[84] B. Pozsgay, Current operators in integrable spin chains: Lessons from long range deformations, SciPost Phys. **8**, 016 (2020).

[85] A. B. Zamolodchikov and Al. B. Zamolodchikov, Factorized S-matrices in two-dimensions as the exact solutions of certain relativistic quantum field models, Ann. Phys. (N.Y.) **120**, 253 (1979).

[86] H. Spohn, *Large Scale Dynamics of Interacting Particles* (Springer-Verlag, Heidelberg, 1991).

[87] H. Spohn, The phonon Boltzmann equation, properties and link to weakly anharmonic lattice dynamics, J. Stat. Phys. **124**, 1041 (2006).

[88] B. Wennberg, Stability and exponential convergence for the Boltzmann equation, Arch. Ration. Mech. Anal. **130**, 103 (1995).

[89] F. Hérau, D. Tonon, and I. Tristani, Regularization estimates and Cauchy theory for inhomogeneous Boltzmann equation for hard potentials without cut-off, Commun. Math. Phys. **377**, 697 (2020).

[90] M. J. Bhaseen, J. P. Gauntlett, B. D. Simons, J. Sonner, and T. Wiseman, Holographic Superfluids and the Dynamics of Symmetry Breaking, Phys. Rev. Lett. **110**, 015301 (2013).

[91] B. Doyon, J. Dubail, R. Konik, and T. Yoshimura, Large-scale Description of Interacting One-Dimensional Bose Gases: Generalized Hydrodynamics Supersedes Conventional Hydrodynamics, Phys. Rev. Lett. **119**, 195301 (2017).

[92] G. A. El and A. Kamchatnov, Kinetic Equation for a Dense Soliton Gas, Phys. Rev. Lett. **95**, 204101 (2005).

[93] V. B. Bulchandani, On classical integrability of the hydrodynamics of quantum integrable systems, J. Phys. A **50**, 435203 (2017).

[94] https://doi.org/10.6084/m9.figshare.16553082.