Statistical Interactions and Boson-Anyon Duality in Fractional Quantum Hall Fluids

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We present an exact scheme of bosonization for anyons (including fermions) in the two-dimensional manifold of the quantum Hall fluid. This gives every fractional quantum Hall phase of the electrons one or more dual bosonic descriptions. For interacting electrons, the statistical transmutation from anyons to bosons allows us to explicitly derive the microscopic statistical interaction between the anyons, in the form of the effective two-body and few-body interactions. This also leads to a number of unexpected topological phases of the single component bosonic fractional quantum Hall effect that may be experimentally accessible. Numerical analysis of the energy spectrum and ground state entanglement properties are carried out for simple examples.

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One of the most fascinating aspects of the two-dimensional systems is the possibility of anyonic statistics, that is both theoretically important and with promising practical applications [1–6]. The fractional quantum Hall (FQH) effect, realized by subjecting a two-dimensional electron gas to a strong perpendicular magnetic field, is an ideal platform for anyon fluids [7]. The possibility of anyons and non-Abelians hosted by gapped topological phases was proposed in [4,8–10], with tentative experimental signals in a number of recent works [11–13]. Even in simple FQH phases, there can be rich dynamics involving the interaction and transmutation between different types of anyons [14,15].

Theoretically, the statistics and dynamics of anyons can be understood in different ways. Haldane's generalized Pauli exclusion principle extends the notion of bosons and fermions by looking at the reduction of Hilbert space when a particle occupies a state [16]. In this perspective, a "hard-core" boson and a fermion are equivalent. It is however not apparent if all of the statistical aspects (e.g., the complex phases from adiabatic braiding) are captured within this formalism. Fundamentally, the statistical properties of anyons can be understood as complex interactions between particles. For example, statistical transmutation with flux attachment and various schemes of boson-fermion dualities have been proposed [17–24]. Such dualities can be established if there is an exact mapping of the energy spectrum or partition function from one system to another. It is, however, not easy to understand the "statistical interaction" beyond the mean-field level in the field theoretical description with flux attachment or singular gauge transformation.

In this Letter, we use anyons in fractional quantum Hall effect (FQHE) as the example and propose exact duality not only between bosons and fermions, but also between bosons and anyons. The statistical interaction between anyons can be microscopically derived order by order, shown to be equivalent to the few-body interactions between bosons in the dual description. The ability to bosonize anyons in two dimensions could be understood as a consequence of bulk-edge correspondence [25–27] and the chiral Luttinger liquid description [28,29] of the quantum Hall edge. It is bosonization of the *entire* Hilbert space with conformal symmetry and without non-Abelian parafermions [17,30–32], in contrast to the approach with Jordan-Wigner transformation on lattice systems [33]. A direct consequence of this bosonization scheme is a large family of bosonic FQH phases, with explicit model Hamiltonians that in principle can be constructed exactly from the corresponding fermionic FQH phases. These new topological phases are dual descriptions of the familiar FQH phases of the electrons, though they occur at different filling factors and topological shifts.

A bosonic description for fermions.—The concepts of this work are most conveniently illustrated on the spherical geometry [34,35] with rotational symmetry. For gapped topological phases, rotational symmetry can also be relaxed, since FQH topological orders do not require any symmetry protection [41,42]. Without loss of generality we focus on spinless electrons in the lowest Landau level (LLL). With a monopole of total magnetic flux $2S$ at the center of the sphere, the number of single particle states (or orbitals) in the LLL is $N_o = 2S + 1$.

It is useful to define the vacuum in this Hilbert space as the highest density state $|\psi_{\mathcal{H}}\rangle$ of a particular sub-Hilbert space $\mathcal{H}$ within the LLL. Basis states of interest are thus created from the vacuum by inserting the magnetic fluxes. For example, if $\mathcal{H}$ is the full Hilbert space of the LLL, then the vacuum is the fully filled LL. All other basis in $\mathcal{H}$ can be obtained by flux insertion, or the creation of holes, from the vacuum. The vacuum and one such basis are shown as follows:

$$11111\cdots 111, \qquad 011111\ldots 111 \tag{1}$$

where each digit represents an orbital arranged sequentially from the north pole to the south pole on the sphere, with "1" indicating the orbital is occupied by an electron, and "0" otherwise. The basis on the right has one hole at the north pole. In both cases, electrons and holes are fermions, with single particle orbitals that are eigenstates of the angular momentum operator $\hat{L}_z$ and $\hat{L}^2$. They thus behave like spinors, in the sense that each electron or hole can be represented as a spinor with total angular momentum, or spin $\boldsymbol{S}$, and the quantum state can be labeled as $|\boldsymbol{s}, \boldsymbol{S}\rangle$, where $\boldsymbol{s} = -\boldsymbol{S}, -\boldsymbol{S}+1, \ldots, \boldsymbol{S}-1, \boldsymbol{S}$ is the index of the single particle orbitals.

The fermionic nature of the electrons or holes manifests from the $\hat{L}^2$ eigenstates for two particles. With more than one particle, we define $\hat{L}_\alpha = \sum_i \hat{L}_{\alpha,i}$, $\hat{L}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2$, where $\alpha = x, y, z$ and i is the index of *electrons*. The total angular momentum $\boldsymbol{S}_{\text{tot}}$ of the two particles can be any integer between 0 and $2\boldsymbol{S}$. For fermions, however, only $\boldsymbol{S}_{\text{tot}} = 2\boldsymbol{S} - k$ with odd integer k is allowed. In contrast for bosons, k can only be even. The counting, or the number of $\hat{L}^2$ eigenstates for each $\boldsymbol{S}_{\text{tot}}$ with more than two particles [36], are also different between fermions and bosons. Such counting is the *signature* of the particle statistics.

We now show that the fermionic holes in a single LL can be "bosonized." Starting with a fully filled LL with N_e electrons and as the vacuum for the holes, an insertion of one magnetic flux creates a single hole with total spin $\boldsymbol{S}_{1,h} = N_e/2$. If two magnetic fluxes are inserted, we create two holes each with spin $\boldsymbol{S}'_{1,h} = (N_e+1)/2$. The total spin of the two holes is thus $\boldsymbol{S}_{2,h} = N_e + 1 - k'$ with k' odd. The key observation here is that if instead of treating each hole as a fermion with spin $\boldsymbol{S}'_{1,h}$, we can also treat it as a particle with spin $\boldsymbol{S}_{1,h}$, so that the total spin is $\boldsymbol{S}_{2,h} = N_e - k$ with k even: the holes behave like bosons. This applies for the insertion of multiple fluxes with a fixed number of N_e: as fermions, each hole has $\boldsymbol{S}'_{1,h}$ that depends on the number of fluxes inserted, but they are also bosons with a fixed $\boldsymbol{S}_{1,h}$ that only depends on N_e. In fact, when a fully filled LL with fixed N_e is defined as the vacuum, the inserted fluxes should be understood as "particles" of spin $\boldsymbol{S}_{1,h}$, since in this way all holes have the same spin, independent of the number of fluxes inserted.

This seemingly trivial reinterpretation has important consequences. If we insert N_h fluxes to the fully filled LL, with the number of orbitals $N_o = N_e + N_h$, the Hilbert space is spanned by fermionic product states $|\boldsymbol{s}'_1, \boldsymbol{S}'_{1,h}; \boldsymbol{s}'_2, \boldsymbol{S}'_{1,h}; \ldots; \boldsymbol{s}'_{N_h}, \boldsymbol{S}'_{1,h}\rangle$ with $\boldsymbol{S}'_{1,h} = (N_o - 1)/2$ and $\boldsymbol{s}'_i$ running from $-\boldsymbol{S}'_{1,h}$ to $\boldsymbol{S}'_{1,h}$. The same space is also spanned by the bosonic product states $|\boldsymbol{s}_1, \boldsymbol{S}_{1,h}; \boldsymbol{s}_2, \boldsymbol{S}_{1,h}; \ldots; \boldsymbol{s}_{N_h}, \boldsymbol{S}_{1,h}\rangle$ with $\boldsymbol{S}_{1,h} = N_e/2$ and $\boldsymbol{s}_i$ running from $-\boldsymbol{S}_{1,h}$ to $\boldsymbol{S}_{1,h}$, where each bosonic product state is a linear combination of the fermionic counterpart (so an entangled many-body state of fermionic holes, see [36]). Thus all physics in a single LL can either be understood from quantum states of fermions (with N_e electrons and N_o orbitals), or quantum states of bosons (with $N_o - N_e$ bosons, and $N_e + 1$ orbitals). The two Hilbert spaces have the same dimension.

Bosonization of anyons.—This bosonic description of the Hilbert space can be easily generalized to anyons. An insertion of the magnetic flux creates a fermion in the full Hilbert space, but it will create an anyon in the truncated Hilbert space $\tilde{\mathcal{H}} \in \mathcal{H}$. The Hilbert space truncation can either be implemented via model Hamiltonians [43], or more generally using local exclusion conditions (LEC) [44]. Each truncated Hilbert space corresponds to a topological FQHE phase, which we can index with the filling factor ν and the topological shift S [45,46]. Note we have the relationship $N_o = \nu^{-1} N_e - S$ for the ground state, which serves as the vacuum. The previously discussed full Hilbert space and its fully filled LL as the vacuum can thus be denoted as $\mathcal{H}_{[\nu,S]} = \mathcal{H}_{[1,0]}$ and $|\psi\rangle_{[\nu,S]} = |\psi\rangle_{[1,0]}$.

Let us illustrate the duality between anyons and bosons with the simple example of the Laughlin phase at filling factor $\nu = 1/3$. The universal topological properties of this phase are defined by the null space of the Haldane pseudopotential interaction $\hat{V}_1^{2\text{bdy}}$, denoted as $\mathcal{H}_{[\frac{1}{3},-2]}$, spanned by the exact zero energy states [7], or Laughlin ground states and quasihole states. Thus the vacuum, or the highest density state for a given N_e, is the Laughlin ground state denoted as $|\psi\rangle_{[\frac{1}{3},-2]}$. Insertion of the magnetic fluxes creates anyons of charge $e/3$, instead of the fermionic holes in the full Hilbert space.

It is natural to organize the Laughlin quasiholes into eigenstates of $\hat{L}^2$ and $\hat{L}_z$, and the counting of these eigenstates clearly indicates the quasihole subspace is spanned by the bosonic degrees of freedom. For a single Laughlin quasihole, it has total angular momentum $\boldsymbol{S}_{1,\text{qh}} = N_e/2$, same as the fermionic holes. They are many-body wave functions in the electron basis denoted as $|\boldsymbol{s}, \boldsymbol{S}_{1,\text{qh}}\rangle$ with $\boldsymbol{s} = -N_e/2, -N_e/2+1, \ldots, N_e/2$. We also know they are Jack polynomials, and using $N_e = 4$ as an example, the root configurations of the five single-quasihole states are [47]:

$$01001001001, \qquad 10001001001, \qquad 10010001001$$
$$10010010001, \qquad 10010010010. \tag{2}$$

Inserting a second magnetic flux to those root configurations creates the two-quasihole states. By diagonalizing the quasihole subspace with $\hat{L}^2$ we see it is more natural to treat these quasiholes as bosons, each with $\boldsymbol{S}_{1,\text{qh}} = N_e/2$, independent of the total number of orbitals. Same as the fermionic holes, for the Hilbert space with N_e electrons and N_{qh} Laughlin quasiholes, we can construct an orthonormal basis denoted as $|\boldsymbol{s}_1, \boldsymbol{S}_{1,\text{qh}}; \boldsymbol{s}_2, \boldsymbol{S}_{1,\text{qh}}; \ldots; \boldsymbol{s}_{N_{\text{qh}}}, \boldsymbol{S}_{1,\text{qh}}\rangle$ with $\boldsymbol{s}_i$

running from $-\boldsymbol{S}_{1,\mathrm{qh}}$ to $\boldsymbol{S}_{1,\mathrm{qh}}$. Each state is a strongly entangled state in the electron basis, but can be interpreted as a "product state" in the bosonic quasihole basis [36]. Again the two descriptions are equivalent in $\mathcal{H}_{[\frac{1}{3},-2]}$. One should note that for two localized Laughlin quasiholes far apart from each other, braiding one quasihole around the other leads to anyonic Berry phase. What we have shown here is that for the many-quasihole states, there are always proper linear combinations of them that give states containing particles behaving like *bosons*, just like the case with the fermionic holes in $\mathcal{H}_{[1,0]}$.

The bosonization of anyons can be applied to the null spaces of model Hamiltonians of other Abelian FQH phases, as well as Hilbert spaces defined by LEC, as long as the counting of quasiholes is Abelian. It can also be applied to the Hilbert space spanned by the ground state and quasihole states of the Abelian composite fermion (CF) states, as long as $\mathcal{H}_{[\nu,S]}$ and $|\psi\rangle_{[\nu,S]}$ can be properly constructed from the CF theory by mapping the FQH states to the IQH states of CFs (with the built-in assumption of bulk-edge correspondence) [48–51], even in the absence of an exact model Hamiltonian. Fundamentally, this possibility of bosonization in two dimensions is due to the conformal mapping and bulk-edge correspondence of the FQH fluids. All quasihole states on the spherical geometry can be conformally mapped to the disk geometry, where the insertion of the magnetic fluxes in the bulk is equivalent to edge excitations at the boundary [52]. Bosonic representation of the anyons in the bulk (even for geometries without boundaries) is thus the dual description of the density modes of the edge of the quantum Hall fluids [see Fig. (1)]. However, if $\mathcal{H}_{[\nu,S]}$ is non-Abelian, one cannot fully bosonize the quasiholes within $\mathcal{H}_{[\nu,S]}$ due to the presence of parafermions [17,30–32].

Bosonic topological phases.—For noninteracting electrons in a single LL with no disorder, the Hamiltonian is just an identity. In the dual description, even though the statistical properties of the particles have changed, the bosons are also noninteracting. Similarly, if we bosonize within $\mathcal{H}_{[\frac{1}{3},-2]}$ when electrons interact with $\hat{V}_1^{\mathrm{2bdy}}$ (so that Laughlin quasiholes are noninteracting), the resulting bosons will be noninteracting as well. Thus in a system with no dynamics at all, statistical interaction is also absent. If we introduce interaction between electrons or anyons, nontrivial interaction between bosons in the dual picture will develop. The latter can be determined by imposing exact mapping of the energy spectra in the two pictures, in addition to the mapping of the many-body states we have established. The microscopic interaction between bosons then also captures the statistical interaction between fermions or anyons in the respective (truncated) Hilbert space.

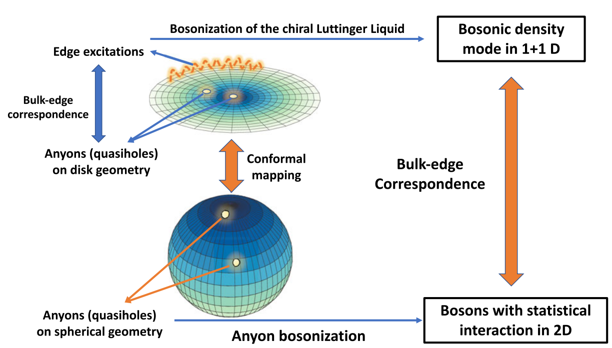

FIG. 1. A schematic illustration of the bosonization of anyons in 2D.

We illustrate this with the full Hilbert space $\mathcal{H}_{[1,0]}$, and introduce $\hat{V}_1^{\mathrm{2bdy}}$ between electrons. For N_e electrons and N_h holes, we can label the eigenstates of $\hat{V}_1^{\mathrm{2bdy}}$ with $|\boldsymbol{S},\alpha,N_h\rangle_h$ and the respective energy $E_{\boldsymbol{S},\alpha,N_h}$. Only the highest weight states are needed so that $\hat{L}_z|\boldsymbol{S},\alpha,N_h\rangle_h = \boldsymbol{S}|\boldsymbol{S},\alpha,N_h\rangle_h$ and $\hat{L}^2|\boldsymbol{S},\alpha,N_h\rangle_h = \boldsymbol{S}(\boldsymbol{S}+1)|\boldsymbol{S},\alpha,N_h\rangle_h$, with α the index labeling the degeneracy of the highest weight states in each total angular momentum sector. We now know that for a bosonic Hilbert space with N_h bosons and N_e+1 orbitals, each $|\boldsymbol{S},\alpha,N_h\rangle_h$ have a one-to-one mapping to a bosonic state $|\boldsymbol{S},\alpha,N_h\rangle_b$ with the same quantum numbers. The effective Hamiltonian between the bosons is given as follows:

$$\hat{H}_b = \sum_{n=2}^{\infty}\sum_{k,\alpha_k}\lambda_{\boldsymbol{S}_k,\alpha_k,n}\hat{V}_{k,\alpha_k}^{n\text{-bdy}}, \tag{3}$$

where $\boldsymbol{S}_k = \boldsymbol{S}-k$, $\hat{V}_{k,\alpha_k}^{n\text{-bdy}}$ are the n-body pseudopotentials [43] with total relative angular momentum k, and α_k labels the degeneracy of the pseudopotentials. We thus require

$${}_b\langle \boldsymbol{S}_k,\alpha_k,N_h|\hat{H}_b|\boldsymbol{S}_k,\alpha_k,N_h\rangle_b = E_{\boldsymbol{S}_k,\alpha_k,N_h}. \tag{4}$$

Using different values of N_h, the coefficients of $\lambda_{\boldsymbol{S}_k,\alpha_k,n}$ can be computed iteratively [36]. For example, it is easy to check that $\lambda_{N_e,1,2}=1$, and $\lambda_{N_e-2p,1,2}=0$ for integer $p>0$. In the thermodynamic limit, we can thus show analytically the following [36]:

$$\lambda_{\boldsymbol{S}_n,1,n} = E_{\boldsymbol{S}_n,1,n} - \sum_{k'=2}^{n-1}\lambda_{\boldsymbol{S}_{k'},1,k'}\frac{n!}{(n-k')!k'!}, \tag{5}$$

$$E_{\boldsymbol{S}_n,1,n} = \sum_{m',n'=0}^{n-1}\frac{(m'+n'-1)!}{2^{m'+n'-1}m'!n'!}(m'-n')^2, \tag{6}$$

where $\boldsymbol{S}_n = (nN_e/2)$. Here $\lambda_{\boldsymbol{S}_n,1,n}$ is the coefficient of the leading n-body bosonic pseudopotential with *zero* total relative angular momentum. In general, $\lambda_{\boldsymbol{S}_n-2p,1,n}$ *decreases rapidly* with increasing p, but rather slowly with increasing n (see [36]). Thus while the interaction between fermionic holes is just a short-range two-body interaction, the interaction between the bosons in the dual picture is longer-ranged, with few-body interactions involving clusters of bosons. The complexity of the bosonic interaction reflects the statistical interaction between the underlying fermions, which we can now quantify order by order.

If we choose $N_h = 2(N_e - 1)$, we are at the filling factor $\nu = 1/3$ with $S = -2$, and $\hat{V}_1^{2\mathrm{bdy}}$ gives the familiar Laughlin phase. In the bosonic picture, the corresponding filling factor is $\nu = 2$ with $S = 2$. We will thus obtain a previously unreported gapped topological phase with the effective interaction $\hat{H}_b$, that has the exact same spectrum as the fermionic Laughlin phase. It can potentially be realized in experiments, because while $\hat{H}_b$ seems very artificial and hard to engineer experimentally, for gapped phases one can have realistic interaction adiabatically connected to the model interactions (an example is the MR state in the second LL [53]). In Fig. (2) we show that with a much simpler truncated $\hat{H}_b$, the bosonic spectrum highly resembles the fermionic topological phase, and low-lying states in the ground state entanglement spectrum captures the Laughlin edge modes as expected.

We can also define more than one vacuum for the same topological phase. For example, if we are interested in the Laughlin phase at $\nu = 1/5$, we can treat the states as quantum fluids of holes created from $|\psi\rangle_{[1,0]}$. This is always possible for any quantum fluids in the LLL. In the Hilbert space of $N_o = 5N_e - 4$, bosonization of the fermionic holes maps it to the bosonic system with $N_e + 1$ orbitals

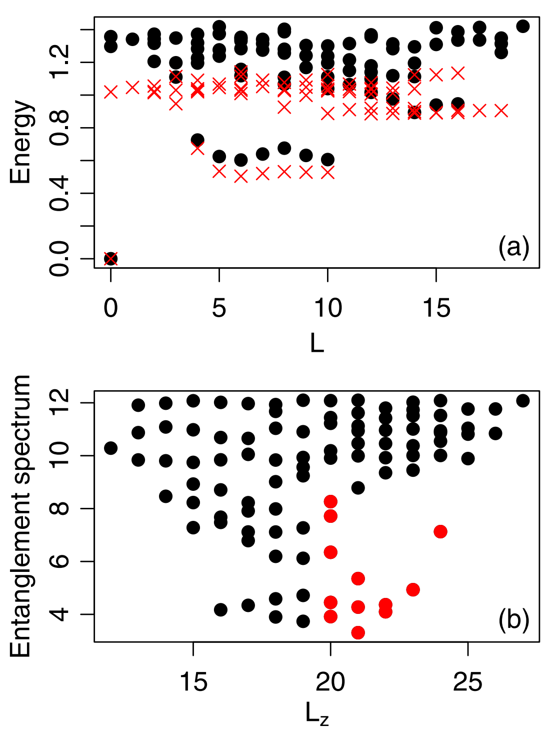

FIG. 2. (a) The black dots give the energy spectrum with 18 bosons, 11 orbitals, and a truncated Hamiltonian dual to $\hat{V}_1^{2\mathrm{bdy}}$ as detailed in [36], and the fermionic spectrum of $\hat{V}_1^{2\mathrm{bdy}}$ is shown here with red crosses as comparison. (b) The ground state entanglement spectrum, with low-lying states (shown in red) having the correct Virasoro counting.

and $4(N_e - 1)$ bosons, corresponding to a filling factor of $\nu = 4$ and $S = 2$, with the effective interaction Hamiltonian $\hat{H}_b$ defined in Eq. (3) and computed from Eq. (4). Note that E_{S_k,α_k,N_h} in Eq. (4) needs to be computed from the bare interaction from the electrons, i.e., the Haldane pseudopotentials $\hat{V}_1^{2\mathrm{bdy}} + \hat{V}_3^{2\mathrm{bdy}}$ for the case of the model Hamiltonian at $\nu = 1/5$.

Alternatively, we can treat the states as quantum fluids of the Laughlin $\nu = 1/3$ quasiholes created from $|\psi\rangle_{[\frac{1}{3},-2]}$. Bosonization of the same topological phase as quantum fluids of Laughlin $\nu = 1/3$ quasiholes leads again to the bosonic system with $N_e + 1$ orbitals and $2(N_e - 1)$ bosons, corresponding to a filling factor of $\nu = 2$ and $S = 2$. This is the same as the bosonic description of the Laughlin $\nu = 1/3$ phase in $\mathcal{H}_{[1,0]}$. These could be two competing bosonic phases at the same filling factor and topological shift, each with a well-defined model Hamiltonian. It would be interesting to see if these two Hamiltonians are topologically distinct, given that they are very different with the pseudopotential expansion [36], and are the dual description of the fermionic Laughlin phase at different filling factors. The topological entanglement entropy [54,55] could be the topological index that distinguishes between the two bosonic phases.

Summary and outlook.—We have established that *all* fractional quantum Hall fluids (including non-Abelian ones) can be understood as quantum fluids of bosons, when anyons (including fermionic holes) are bosonized either in the single LL Hilbert space $\mathcal{H}_{[1,0]}$, or other Abelian sub-Hilbert spaces. From the bare interaction between electrons, this bosonization scheme allows us to explicitly calculate the statistical interaction between anyons, and construct microscopic Hamiltonians for the dynamics of the anyon fluids. The duality hints that each quantum Hall fluid can be understood equivalently as particles of different statistical properties. This may explain various different effective schemes involving composite fermions or bosons in the literature as viewing the same physics from different perspectives [56–60].

The bosonization scheme allows us to identify topological FQH phases of interacting bosons with the topological orders inherent from their fermionic or anyonic counterparts. They are different from the two-component or lattice bosonic FQHE proposed in the literature [61–63], that require symmetry protection. A family of fermionic parton states was recently proposed in [14], and it is possible that their bosonic versions could be related to the topological phases proposed here [64]. We numerically analyzed the gapped energy spectrum and the ground state entanglement spectrum of a bosonic phase at $\nu = 2$ and $S = 2$, confirming the validity of the bosonization scheme. Note that one fractional quantum Hall phase for electrons can be interpreted either as a quantum fluid of fermionic holes, or different types of anyons. Each interpretation leads to a dual description of bosons with different microscopic interactions, at different filling factors. It will be interesting to see if we can experimentally realize such bosonic FQH phases in, for example, the cold atom systems; or to see how these bosonic excitations with fractional charges can be nucleated in the bulk of the fermionic FQH systems.

Fundamentally, the ability to bosonize anyons in two dimensions is related to the bulk-edge correspondence and the feasibility of bosonizing the chiral Luttinger liquid of the edge excitations. For FQH phases with ground states and quasiholes living in non-Abelian subspaces (e.g., $\mathcal{H}_{[\frac{1}{2},-2]}$ with vacuum $|\psi\rangle_{[\frac{1}{2},-2]}$, which is the null space of the Moore-Read model three-body Hamiltonian), we cannot fully bosonize the quantum Hall fluid, because of the presence of parafermions [65]. Nevertheless, we can still understand the ground state and quasiholes of such FQH phases as quantum fluids of bosons and parafermions, with explicit statistical interactions between them from a similar scheme. Further studies of such systems could enhance our understandings of the statistical nature of anyons and non-Abelians, and help to construct effective field theory descriptions of such exotic particles in a systematic manner.

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[65] Note that we can still bosonize the MR state within the Hilbert space of a single LL (which is Abelian). For example the MR ground state with N_e electrons is a quantum fluid with $N_e - 2$ quasiholes (when the vacuum is the fully filled LL), which can be bosonized in a completely analogous manner as the Laughlin states.