Macroscopically Nonlocal Quantum Correlations

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(Received 19 April 2021; accepted 14 July 2021; published 16 September 2021)

It is usually believed that coarse graining of quantum correlations leads to classical correlations in the macroscopic limit. Such a principle, known as macroscopic locality, has been proved for correlations arising from independent and identically distributed (IID) entangled pairs. In this Letter, we consider the generic (non-IID) scenario. We find that the Hilbert space structure of quantum theory can be preserved in the macroscopic limit. This leads directly to a Bell violation for coarse-grained collective measurements, thus breaking the principle of macroscopic locality.

DOI: 10.1103/PhysRevLett.127.120401

Introduction.—Quantum mechanics does not impose any limit on the size of the system it describes, which can, in principle, be as large as a cat [1]. However, quantum behavior is not observed at the macroscopic scale, where the world appears to be classical. The idea that quantum mechanics must reproduce classical physics in the limit of large quantum numbers is known as the correspondence principle [2]. Yet, this principle, in all its generality, has not been rigorously stated and proved, mostly because the concept of "macroscopic" remains somewhat vague. While different interpretations of the macroscopic limit may lead, in general, to different conclusions, there is still confidence that quantum behavior must somehow disappear in the limit.

A possible explanation for the emergence of classicality from quantum theory is via the coarse graining of the measurements [3–10]. In this respect, one important consequence of the correspondence principle is the concept of *macroscopic locality* (ML) [6]: Coarse-grained quantum correlations become local (in the sense of Bell [11]) in the macroscopic limit. ML has been challenged in different circumstances, both theoretically and experimentally [12–19] (see Ref. [20] for a review). However, as far as we know, nonlocality fades away under coarse graining when the number of particles N in the system goes to infinity. In this sense, ML was proposed by Navascués and Wunderlich (NW) [6] as an axiom for discerning physical postquantum theories. In particular, they considered a bipartite Bell-type experiment where the parties measure intensities with a resolution of the order of $\sqrt{N}$ or, equivalently, $O(\sqrt{N})$ coarse graining. Then, under the independent and identically distributed (IID) assumption, i.e., under the premise that particles are entangled only by *independent and identically distributed* pairs, they prove ML for quantum theory.

In this Letter, we generalize the concept of ML to any level of coarse graining $\alpha \in [0,1]$, meaning that the intensities are measured with a resolution of the order of N^α. We drop the IID assumption, and we investigate the existence of a boundary between quantum (nonlocal) and classical (local) physics, identified by the minimum level of coarse graining α required to restore locality. To do this, we introduce the concept of *macroscopic quantum behavior* (MQB), demanding that the Hilbert space structure, such as the *superposition principle*, is preserved in the thermodynamic limit. Then, we provide a concrete example of MQB at $\alpha = 1/2$ which violates ML. This is the opposite of what happens in the IID case, where ML is known to hold, as shown by NW. Finally, we analyze the effects of noise and particle losses, showing robustness of the macroscopic statistics. Altogether, our findings shed new light on the problem of the transition (if any) between quantum and classical physics.

Experimental setup and macroscopic locality.—We consider a simple Bell-type setting as illustrated in Fig. 1. A state $\rho^{[2N]}$ of $2N$ particles is produced, out of

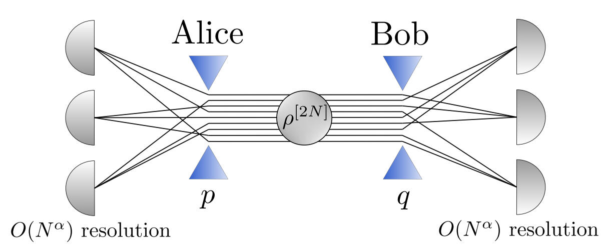

FIG. 1. Macroscopic Bell-type experiment. A source produces a $2N$-particle state and sends half of the particles to Alice and half to Bob. The parties perform collective measurements on their beams, specified by the settings (p, q). For each outcome, local detectors count the number of particles with resolution of the order of N^α.

which N are sent to Alice and N to Bob. Alice performs a collective measurement described by the (single-particle) positive operator-valued measure (POVM) elements $E^A_{a|p}$, where $a \in \Omega_A$ is her outcome and $p \in \Sigma_A$ is her measurement setting (and similarly does Bob). Alice (Bob) has the following limitations. (i) *Intensity measurement*: No access to individual outcomes but only to their sum or *intensity* $I_A = \sum_{i=1}^N a_i$. (ii) $O(N^\alpha)$ *coarse graining*: The measuring scale for I_A has a limited resolution of the order of N^α, where $\alpha \in [0,1]$ is the *order* or *level of coarse graining*.

These assumptions naturally lead to the following macroscopic variable:

$$X_\alpha^{[N]} = \frac{1}{N^\alpha}\sum_{i=1}^N (a_i - \langle a_i \rangle). \tag{1}$$

This quantity, with the mean set to zero, has well-studied limit properties in the classical domain [21]. The special case $\alpha = 1/2$ (also discussed here) is closely related to the central limit theorem [22], for which $X_{\alpha=1/2}^{[N]}$ properly captures the quantum fluctuations of the intensity about its mean [23]. The above macroscopic variable, in turn, defines the POVM associated to it:

$$E(X_\alpha^{[N]}) = \sum_{\sum_{i=1}^N (a_i - \langle a_i \rangle)/N^\alpha = X_\alpha^{[N]}} E^A_{a_1} \otimes \cdots \otimes E^A_{a_N}. \tag{2}$$

Altogether, $X_\alpha^{[N]}$ and $E(X_\alpha^{[N]})$ specify Alice's measurement and likewise holds for Bob. Finally, Alice and Bob repeat their experiment many times in order to extract the bipartite distribution. The central quantity of interest is the limit thereof, namely,

$$P(x,y) = \lim_{N\to\infty} \mathrm{tr}\rho^{[2N]} E(X_\alpha^{[N]}) \otimes E(Y_\alpha^{[N]}), \tag{3}$$

where $X_\alpha^{[N]} \to x$ and $Y_\alpha^{[N]} \to y$ denote convergence in distribution. A necessary condition for convergence is that the variance scaling of the measured intensity (as determined by the state $\rho^{[2N]}$) matches the order of coarse graining, i.e., that if the variance of the intensity $\mathrm{Var}(I)$ scales as $N^{2\beta}$, then $\alpha = \beta$. Otherwise, if $\alpha < \beta$, the distribution (3) will simply not converge, and if $\alpha > \beta$, the distribution will converge to a Dirac delta function, thus giving trivial statistics.

The question of ML refers to the locality properties of the limit distribution (3). We will say a theory possesses ML at the order of α if the limit distributions $P(x,y)$ for any choice of measurements can be described by a local model

$$P(x,y) = \int d\lambda \mu(\lambda) P_A(x|\lambda) P_B(y|\lambda).$$

On the other hand, if the above factorization does not hold, we say that the theory exhibits *macroscopically nonlocal correlations* (at the order of α). It seems natural to conjecture the existence of a quantum-to-classical transition point, i.e., a critical value α_c such that quantum theory violates ML at any $\alpha < \alpha_c$, while locality is restored for $\alpha > \alpha_c$. Note that, at $\alpha = \alpha_c$, both ML and violation of ML are possible. Intuitively, we expect quantum theory to violate ML at $\alpha = 0$ (no coarse graining) and to satisfy it at $\alpha = 1$ (full coarse graining). Indeed, there are strong evidences that this is the case, as presented in Refs. [17,24], respectively. A more interesting result is that of NW. In their paper, the authors consider the case of $O(\sqrt{N})$ coarse graining ($\alpha = 1/2$) with IID states, described by a density matrix of the form $\rho^{[2N]} = (\rho_{AB})^{\otimes N}$. Then, according to the central limit theorem, the distributions of the macroscopic variables (1) are Gaussian. Using this, they show that the corresponding bipartite Gaussian distributions (3) are local, implying that $\alpha_c^{\mathrm{IID}} \leq 1/2$. However, it is far from clear whether actually $\alpha_c^{\mathrm{IID}} = 1/2$, and the possibility that even $\alpha_c^{\mathrm{IID}} = 0$ is not discarded [16]. The IID assumption might then be too restrictive. Our goal here is to drop it and consider the most general scenario. In the next section, we show how non-IID states at $\alpha = 1/2$ can give rise to non-Gaussian distributions in the limit. This result will lead us to full quantum behavior at $\alpha = 1/2$, for which we also show violation of ML. Thus, we prove that $\alpha_c \geq 1/2$ in the general case (non-IID).

Non-IID states and non-Gaussian limit distributions.— In the following, we will consider the case $\alpha = 1/2$ for a single party, say, Alice, and a system of N spin-1/2 particles (qubits). For simplicity, we will consider projective (von Neumann) measurements, with $E_a^2 = E_a$, for which we choose binary outcomes $a = \pm 1$. It is useful to define the (single-particle) observable $A = \sum_a a E_a$, such that the macroscopic variable (1) is naturally promoted to the macroscopic observable

$$\hat{X}_{\alpha=1/2}^{[N]} = \frac{1}{\sqrt{N}}\sum_{i=1}^N (A_i - \langle A_i \rangle). \tag{4}$$

We consider an example of non-IID state, the W state

$$|W\rangle = \frac{1}{\sqrt{N}}(|100\ldots0\rangle + |010\ldots0\rangle + \cdots + |000\ldots1\rangle).$$

The characteristic function of $\hat{X}_{\alpha=1/2}^{[N]}$ for this state can be written in terms of $\mathcal{A} = e^{it(A-\langle A\rangle)/\sqrt{N}}$ as $\chi(t) = \langle W|\mathcal{A}^{\otimes N}|W\rangle$. Direct computation gives

$$\chi(t) = \mathcal{A}_{00}^{N-2}[\mathcal{A}_{11}\mathcal{A}_{00} + (N-1)\mathcal{A}_{10}\mathcal{A}_{01}],$$

where $\mathcal{A}_{ij} = \langle i|\mathcal{A}|j\rangle$. Now, expanding $\mathcal{A} = 1 + it(A - \langle A\rangle)/\sqrt{N} - t^2(A - \langle A\rangle)^2/N + O(N^{-3/2})$ and using that $\langle A\rangle = (1 - 1/N)A_{00} + 1/NA_{11}$, the thermodynamic limit reads $\chi(t) = e^{-\sigma^2 t^2/2}(1 - \sigma^2 t^2)$, where we have defined the variance $\sigma^2 = \langle A^2\rangle - \langle A\rangle^2$. The Fourier transform of $\chi(t)$ gives the non-Gaussian limit distribution

$$P(x) = \frac{1}{\sqrt{2\pi\sigma^2}}\frac{x^2}{\sigma^2}e^{-x^2/(2\sigma^2)}.$$

Similarly, for any N-particle Dicke state [25]

$$|N,k\rangle = \frac{1}{\sqrt{\binom{N}{k}}}(|\underbrace{1\ldots1}_{k}0\ldots0\rangle + \text{permutations}),$$

it has been shown in Ref. [26] that the limit distribution of macroscopic observables of the form of Eq. (4) can be written in terms of Hermite polynomials as $P_k(x)\sim e^{-x^2/2}H_k^2(x)$. Thus, for such a family of states, we find non-Gaussian limiting behavior, which is conventionally associated to nonclassical phenomena. Moreover, such a distribution coincides with the Born rule for the distribution in position of the kth excited state of the harmonic oscillator, $P(x) = |\langle x|k\rangle|^2$, where the wave function is

$$\langle x|k\rangle = \frac{1}{(2\pi)^{1/4}}\frac{1}{\sqrt{k!}}e^{-x^2/4}H_k(x).$$

Here, $H_k(x)$ are the Hermite polynomials [27]

$$H_k(x) = (-1)^k e^{x^2/2}\frac{d^k}{dx^k}e^{-x^2/2}.$$

Macroscopic quantum behavior.—The analysis provided above suggests the following limit mapping:

$$|N,k\rangle \underset{N\to\infty}{\longrightarrow} |k\rangle, \qquad \hat{X}^{[N]}_{\alpha=1/2} \underset{N\to\infty}{\longrightarrow} \hat{x}, \tag{5}$$

where $|k\rangle$ is the number basis of the harmonic oscillator and $\hat{x}$ is the position operator. The identification between Dicke states and eigenstates of the harmonic oscillator is natural: It is known that the SU(2) algebra of angular momentum contracts to the Heisenberg algebra of creation and annihilation operators in the limit of large total angular momentum [28]. This is also related to the so-called photon-spin mapping [29]. However, Eq. (5) is not just a map of states but a joint map of states and observables. We would like to formalize this idea and generalize it to general POVM measurements and to any level of coarse graining. We shall define a mapping that preserves the (separable) Hilbert space structure of quantum theory, i.e., the *superposition principle* as well as the *Born rule*. To achieve this, let us introduce the concept of MQB in the following way.

Definition 1: Let $\mathcal{H}^N$ be the Hilbert space of N particles. For every $N \geq N_0$, let $\mathcal{M}_d^N \subset \mathcal{H}^N$ be a subspace of fixed dimension $d \geq 2$, and let $\mathcal{M}_d^\infty$ be an auxiliary d-dimensional Hilbert space. Since these (sub)spaces are all d dimensional, they are all isomorphic as vector spaces:

$$\mathcal{M}_d^{N_0} \cong \mathcal{M}_d^{N_0+1} \cong \mathcal{M}_d^{N_0+2} \cong \cdots \cong \mathcal{M}_d^\infty.$$

Let us fix the sequence of isomorphisms by a choice of basis in every space:

$$|k\rangle_{N_0} \mapsto |k\rangle_{N_0+1} \mapsto |k\rangle_{N_0+2} \mapsto \cdots \mapsto |k\rangle_\infty,$$

for every $k = 1, \ldots, d$. This gives a unique identification among states $|\Psi_N\rangle = \sum_{k=1}^d c_k|k\rangle_N \in \mathcal{M}_d^N$ for all $N \geq N_0$, including $|\psi\rangle = \sum_{k=1}^d c_k|k\rangle_\infty \in \mathcal{M}_d^\infty$. Now, we say that the sequence of spaces $\mathcal{M}_d^N$, together with the corresponding choice of bases (isomorphisms), possesses MQB at the order of α if, for any $|\Psi_N\rangle \in \mathcal{M}_d^N$, we have

$$\lim_{N\to\infty}\langle\Psi_N|E(X_\alpha^{[N]})|\Psi_N\rangle = \langle\psi|e(x)|\psi\rangle \tag{6}$$

for all measurements specified by Eqs. (1) and (2). Here, $X_\alpha^{[N]} \to x \in \Omega$, and $e(x)$ is a POVM element acting in $\mathcal{M}_d^\infty$, satisfying $\sum_{x\in\Omega} e(x) = 1$.

This definition clearly ensures that both the Born rule and the superposition principle remain valid in the macroscopic limit. In this case we write

$$|k\rangle_N \overset{M}{\longmapsto} |k\rangle, \qquad E(X_\alpha^{[N]}) \overset{M}{\longmapsto} e(x).$$

In order to fulfill the above definition, the macroscopic variable as defined in Eq. (1) needs to be modified, as it exhibits a nonlinear dependence on the input state via the mean value $\langle a_i\rangle$. Such a behavior is inconsistent with the Born rule, in general, and the simplest way to fix this is to substitute this mean value by some constant μ, thus redefining

$$X_\alpha^{[N]} = \frac{1}{\tau N^\alpha}\sum_{i=1}^N (a_i - \mu), \tag{7}$$

where the parameter τ is introduced for future convenience. Such a modification of $X_\alpha^{[N]}$ consequently induces a modification of the POVM element $E(X_\alpha^{[N]})$ given in Eq. (2). With this, we are ready to provide examples.

MQB at $\alpha = 1/2$.—We consider again Dicke states, i.e., the sequence of spaces $\mathcal{M}^N = \mathrm{Span}\{|N,k\rangle\}_k$ together with the identification $|N,k\rangle \mapsto |N+1,k\rangle$, where $k = 0, 1, \ldots, d-1$ for some finite d which we leave unspecified for now. We shall directly evaluate the limit (6). Given a single-particle POVM with elements E_a, let us define the matrix $A = \sum_a aE_a$. The explicit calculation of the distribution of $X_{\alpha=1/2}^{[N]}$ on a state $|\Psi_N\rangle = \sum_k c_k|N,k\rangle$ is provided in Supplemental Material, Sec. A [30]. Choosing $\mu = A_{00}$ and $\tau = |A_{01}|$ for the macroscopic variable (7), the limit distribution can be written as

$$P(x)=\sum_{k,l}e^{-ik\varphi}c_k^*c_le^{il\varphi}\int dx'\frac{e^{-(x-x')^2/2s^2}}{\sqrt{2\pi s^2}}\langle k|x'\rangle\langle x'|l\rangle,\quad (8)$$

where $\varphi=\arg(-A_{01})$, $s^2=\sigma^2/\tau^2-1$ in terms of the limit variance $\sigma^2=\langle 0|\sum_a a^2E_a|0\rangle-\langle 0|A|0\rangle^2$, $|x\rangle$ is the position basis, and $|k\rangle$ is the number basis of the harmonic oscillator. As we see, for the limit variable we get $x\in\mathbb{R}$. Therefore, the MQB at $\alpha=1/2$ is given by

$$|N,k\rangle\overset{M}{\longmapsto}|k\rangle,\qquad E(X^{[N]}_{\alpha=1/2})\overset{M}{\longmapsto}U_\varphi^\dagger e_s(x)U_\varphi,\quad (9)$$

where $U_\varphi=e^{i\varphi\hat{k}}$, in terms of the number operator $\hat{k}$ for the harmonic oscillator, and $e_s(x)$ is the Gaussian POVM element

$$e_s(x)=\frac{1}{\sqrt{2\pi s^2}}\int dx'e^{-(x-x')^2/2s^2}|x'\rangle\langle x'|.\quad (10)$$

We see that the limit (auxiliary) space $\mathcal{M}_d^\infty$ is naturally embedded in the infinite-dimensional space of the harmonic oscillator. Since the dimension d is arbitrarily large, we can freely set $\mathcal{M}_d^\infty$ to be the whole space of the harmonic oscillator.

Let us now analyze in more detail the simple case of projective measurements. For these, the values of σ and τ coincide, so that $s=0$. Then $e_s(x)$ becomes the projector on position $|x\rangle\langle x|$, and the observable $\hat{X}^{[N]}=\sum_{X^{[N]}}X^{[N]}E(X^{[N]})$ becomes in the limit the phase-space observable

$$U_\varphi^\dagger\hat{x}U_\varphi=\hat{x}\cos\varphi+\hat{p}\sin\varphi,\quad (11)$$

depending on A only through the off-diagonal phase $\varphi=\arg(-A_{01})$. With this, one can see that the operators $\hat{X}^{[N]}_{\alpha=1/2}$ form a noncommutative bosonic algebra in the thermodynamic limit. This has been known in the context of fluctuation observables [23,31]. From the perspective of MQB, we obtain incompatibility of measurements along with the superposition principle in the macroscopic limit. This is a strong hint for violation of ML.

Violation of ML at $\alpha=1/2$.—Now we consider again the bipartite Bell scenario depicted in Fig. 1 and assume for simplicity projective measurements. Let the source produce a bipartite $2N$-particle state of the form

$$|\Psi_{2N}\rangle=\sum_k c_k|N,k\rangle_A\otimes|N,k\rangle_B.\quad (12)$$

Let Alice measure the macroscopic observable $\hat{X}^{[N]}_{\alpha=1/2}$ with settings $p\in\Sigma_A$, and let Bob measure $\hat{Y}^{[N]}_{\alpha=1/2}$ with settings $q\in\Sigma_B$. Using the MQB (9) for both, these measurements become phase-space observables (11) for different angles φ_A and φ_B on a state

$$|\psi\rangle=\sum_k c_k|k\rangle_A\otimes|k\rangle_B.$$

Such a system exhibits Bell nonlocality for a suitable choice of the constants c_k and phase-space measurements [32] and can be easily generalized to the multipartite case. In Supplemental Material, Sec. B [30], we show explicit violation of the Clauser-Horne-Shimony-Holt inequality.

Robustness.—In this part, we will study robustness of our MQB. We will analyze this robustness in two ways: losses and noise at the microscopic level and global noise of the order of $\sqrt{N}$. First, we consider the case of losses, where individual particles reach the detectors only with some probability $p\in[0,1]$, and they are lost with probability $1-p$. If the parties are able to measure the number of received particles (with a precision of the order of $\sqrt{N}$), then, as shown in Supplemental Material, Sec. C [30], this loss simply translates into a rescaled variable $x\to x/p$ together with $\sigma^2\to\sigma^2/p$. Rescaling back to the old variable, we get an effective broadening of the limit Gaussian POVM (10):

$$s_p^2=\frac{\sigma^2}{p^3\tau^2}-1.\quad (13)$$

Similarly, independent single-particle noise channels $\rho^{[N]}\mapsto\Gamma^{\otimes N}(\rho^{[N]})$ can be absorbed in the single-particle POVMs (see Supplemental Material, Sec. C [30]). Since any POVM is mapped to the limit POVM $U_\varphi^\dagger e_s(x)U_\varphi$ parametrized by s and φ, such noise channels can affect the macroscopic statistics only in two simple ways: coherently, by shifting the angle φ, or incoherently, by enlarging the width s. In Supplemental Material, Sec. C [30], we provide explicit calculations for the depolarizing and dephasing channels, showing broadening effects as in Eq. (13). Finally, we consider the measurement precision (both of the intensity and of the number of particles) to be captured by some classical independent noise bounded by $\tau\epsilon\sqrt{N}$ for some constant ϵ. This simply translates into an additive classical random variable r bounded by ϵ, so that the random variable in the MQB (9) becomes

$$X^{[N]}_{\alpha=1/2}\to x+r.\quad (14)$$

Altogether, this shows that our MQB is robust. In principle, the macroscopic Bell violation should still be observable for small enough values of the global noise ϵ and of the effective parameter s capturing noise and losses at the microscopic level.

MQB at $\alpha=1$.—We close with a final example of MQB for $\alpha=1$ (maximal coarse graining) with $2N$-particle Dicke states $|2N,N+k\rangle$. As before, we have $A=\sum_a aE_a$ and choose $\mu=\frac{1}{2}\mathrm{tr}A$ and $\tau=|A_{01}|$ for the macroscopic variable (7). In Supplemental Material, Sec. D [30], we compute the limit distribution $P(x)$ of the variable $X^{[2N]}_{\alpha=1}$ on a superposition state $|\Psi_{2N}\rangle = \sum_k c_k |2N, N+k\rangle$. This distribution has a finite support $x \in [-1, 1]$, as opposed to the case $\alpha = 1/2$, where we had $x \in \mathbb{R}$. It is therefore convenient to set $x = \cos\theta$, with $\theta \in [0, \pi]$, so that the distribution can be written as

$$P(\theta) = \sum_{k,l} e^{-ik\varphi} c_k^* c_l e^{il\varphi} \frac{e^{-i(k-l)\theta} + e^{i(k-l)\theta}}{2\pi}, \qquad (15)$$

where $\varphi = \arg(A_{01})$. Then, the sequence of spaces $\mathcal{M}^{2N} = \mathrm{Span}\{|2N, N+k\rangle\}_k$ together with the identification $|2N, N+k\rangle \mapsto |2(N+1), N+1+k\rangle$ has MQB at $\alpha = 1$ given by

$$|2N, N+k\rangle \overset{M}{\longmapsto} |k\rangle, \qquad E(X^{[2N]}_{\alpha=1}) \overset{M}{\longmapsto} U^\dagger_\varphi e(x) U_\varphi.$$

Here, $|k\rangle$ is the eigenbasis of the quantum rotor [33], with wave functions

$$\langle \pm\theta | k\rangle = \frac{1}{\sqrt{2\pi}} e^{\pm ik\theta}, \qquad \theta \in [0, \pi],$$

$U_\varphi = e^{i\varphi\hat{k}}$, and $e(\theta) = |\theta\rangle\langle\theta| + |-\theta\rangle\langle -\theta|$. Despite the MQB, the POVMs $U^\dagger_\varphi e(x) U_\varphi = |\theta - \varphi\rangle\langle\theta - \varphi| + |-\theta - \varphi\rangle\langle -\theta - \varphi|$ are compatible for all φ. Given this measurement compatibility, we can construct a joint distribution for all settings, thus restoring classicality and Bell locality. In Supplemental Material, Sec. D [30], we explicitly provide a local model for the bipartite distributions arising from this MQB.

Conclusions.—In this Letter, we have introduced a generalized concept of macroscopic locality at any level of coarse graining $\alpha \in [0, 1]$. We have investigated the existence of a critical value α_c that marks the quantum-to-classical transition. We have introduced the concept of MQB at level α of coarse graining, which implies that the Hilbert space structure of quantum mechanics is preserved in the thermodynamic limit. This facilitates the study of macroscopic quantum correlations. By means of a particular MQB at $\alpha = 1/2$, we show that $\alpha_c \geq 1/2$, as opposed to the IID case, for which $\alpha_c^{\mathrm{IID}} \leq 1/2$. An upper bound on α_c is, however, lacking in the general case. The possibility that no such transition exists remains open, and perhaps there exist systems for which ML is violated at $\alpha = 1$.

We thank Nicolás Medina-Sánchez and Joshua Morris for helpful comments. Both authors acknowledge support from the Austrian Science Fund (FWF) through BeyondC-F7112.

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