Optical Indistinguishability via Twinning Fields

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Here we introduce the concept of the twinning field—a driving electromagnetic pulse that induces an identical optical response from two distinct materials. We show that for a large class of pairs of generic many-body systems, a twinning field which renders the systems optically indistinguishable exists. The conditions under which this field exists are derived, and this analysis is supplemented by numerical calculations of twinning fields for both the 1D Fermi-Hubbard model, and tight-binding models of graphene and hexagonal boron nitride. The existence of twinning fields may lead to new research directions in nonlinear optics, materials science, and quantum technologies.

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Introduction.—As our understanding of the physical world has progressed to mastery over it, it has become apparent that the qualities which define a material at equilibrium may be modified under driving. This phenomenon underpins both quantum simulation [1] and Floquet engineering [2–4]. One of the principal goals of quantum control theory [5] is the specification of the driving fields necessary either to steer a system to some desired state [6–11], or fulfill a prespecified condition on its expectations [12,13].

Using tracking quantum control [14–21], recent work has demonstrated that almost arbitrary control over the optical response of a large class of solid-state systems can be achieved [22,23]. One consequence of this is that two specially tailored driving fields will induce an identical response from two distinct systems. Given the essential malleability of quantum systems under driving, one might ask whether it is possible to fulfil the stronger condition of obtaining identical responses using the *same* driving field on each system. Put differently, do there exist fields for which a pair of systems' response are indistinguishable?

Consider two distinct systems $|\psi_1\rangle$ and $|\psi_2\rangle$, with an identical control field impinging on each of them (see Fig. 1). Each system will generate an optical response $J^{(k)}(t)$, and if $J^{(1)}(t) = J^{(2)}(t)$ for all times t, then the driving field is what we term a "twinning field," and the systems are *optically indistinguishable*.

In the regime of linear response, this may initially appear trivial, as many systems possess extremely similar absorption and emission spectra over a broad range of frequencies (e.g., large organic molecules [24]). Indeed, such is the closeness of these systems' response that quantum control [25–31] (including tracking control [32]) must be exploited to accurately detect these systems. Of course, similar is not *identical*, and purely linear response would require identical susceptibilities for identical responses. In general, however, materials also have a nonlinear component to differentiate them (see, e.g., [33]). In fact many important phenomena—e.g., high harmonic generation [34–36], the workhorse of attosecond physics [37–39]—explicitly rely on optical nonlinearities [40].

For this reason, true optical indistinguishability must be considered in the context of the nonlinear response that arises in a fully quantum treatment of materials. Some preliminary hints of this indistinguishability have been observed experimentally, specifically in the nonuniqueness of the parametrization of second-order nonlinear spectra [41]. To date, however, a theoretical justification for such results has been lacking. In this Letter we address this issue and present a framework for achieving driven indistinguishability. The main result is a demonstration that for any pair of generic many-electron systems on a lattice, there always exists a twinning field which will elicit an identical response from each system. Furthermore, the conditions under which this field is unique are established. This framework is then extended by deriving a general twinning field which renders an arbitrary expectation identical between systems. Finally, we discuss the physical implications of twinning fields, and their potential utility.

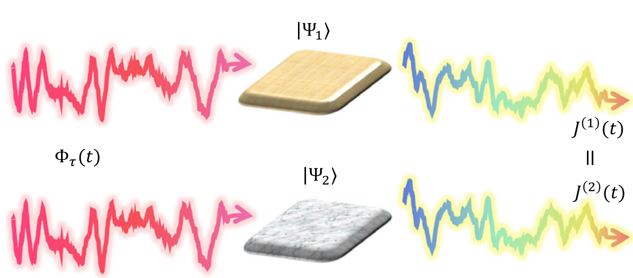

FIG. 1. Ordinarily, distinct systems will have different responses to the same driving field. A twinning field relates a pair of systems as the field under which the optical response $J^{(k)}(t)$ of each system is identical.

Results.—Here we outline the derivation of the twinning field for 1D systems, while the more general case is discussed in the Supplemental Material [42] which includes Refs. [43–45]. Here we consider two many-electron systems on a lattice, labeled by $k = 1, 2$. Each has a potential $\hat{U}^{(k)}$ due to electron-electron interactions. Both systems are excited by an identical laser pulse, described under the dipole approximation by the Peierls phase $\Phi(t)$ [46,47]. Such systems' evolution will then be determined by the Hamiltonian (in atomic units) [23,48]:

$$\hat{H}^{(k)}(t) = -t_0^{(k)} \sum_{j,\sigma} (e^{-i\Phi(t)} \hat{c}^\dagger_{j\sigma} \hat{c}_{j+1\sigma} + \text{H.c.}) + \hat{U}^{(k)}, \tag{1}$$

where $\hat{c}_{j\sigma}$ is the fermionic annhilation operator (acting on the appropriate system) for site j and spin σ, satisfying the anticommutation relation $\{\hat{c}^\dagger_{j\sigma}, \hat{c}_{j'\sigma'}\} = \delta_{\sigma\sigma'}\delta_{jj'}$, while $t_0^{(k)}$ is the hopping parameter describing the kinetic energy of the electrons.

Consider a typical example of an optically driven current. The current operator $\hat{J}^{(k)}$ is defined from a continuity equation for the electron density [22,23]. Provided all number operators $\hat{n}_{j\sigma} = \hat{c}^\dagger_{j\sigma}\hat{c}_{j\sigma}$ commute with $\hat{U}^{(k)}$, each system's current operator has the form [49]

$$\hat{J}^{(k)}(t) = -iat_0^{(k)} \sum_{j,\sigma} (e^{-i\Phi(t)} \hat{c}^\dagger_{j\sigma} \hat{c}_{j+1\sigma} - \text{H.c.}), \tag{2}$$

where $a^{(k)}$ is the lattice constant. It is important to note that the current expectation $J^{(k)}(t) = \langle \psi_k(t)|\hat{J}^{(k)}(t)|\psi_k(t)\rangle$ depends only implicitly on $\hat{U}^{(k)}$ through the evolution of $|\psi_k(t)\rangle$, significantly simplifying expressions.

Having dispensed with this preamble, we come to our main topic of investigation. For two systems $|\psi_1\rangle$ and $|\psi_2\rangle$ with potentials $\hat{U}^{(1)}$ and $\hat{U}^{(2)}$, does there exist a twinning field $\Phi_\tau(t)$ such that $J^{(1)}(t) = J^{(2)}(t)$, making the response of one system indistinguishable from the other?

To establish the existence of this field, we first express the nearest-neighbor expectation of each system in a polar form:

$$\hat{K} = \sum_{j,\sigma} \hat{c}^\dagger_{j\sigma} \hat{c}_{j+1\sigma}, \tag{3}$$

$$K(\psi_k) = \langle \psi_k(t)|\hat{K}|\psi_k(t)\rangle = R(\psi_k) e^{i\theta(\psi_k)}. \tag{4}$$

Note that in both this and later expressions, the argument ψ_k indicates that the expression is a functional of $|\psi_k\rangle \equiv |\psi_k(t)\rangle$. We emphasise that this functional will have a well-defined value for any state that has been obtained through evolution under the Hamiltonian given in Eq. (1). Using this, we may express the response expectation $J^{(k)}$ directly as

$$\begin{aligned} J^{(k)}(t) &= -iat_0 R(\psi_k)(e^{-i[\Phi(t)-\theta(\psi_k)]} - e^{i[\Phi(t)-\theta(\psi_k)]}) \\ &= -2at_0 R(\psi_k) \sin[\Phi(t) - \theta(\psi_k)]. \end{aligned} \tag{5}$$

It is straightforward to equate the currents $J^{(1)}(t)$ and $J^{(2)}(t)$ and obtain an expression for the twinning field Φ_τ in terms of the expectations of the two systems:

$$\Phi_\tau(t) = \arctan[\xi(\psi_1, \psi_2)] \tag{6}$$

$$\xi(\psi_1, \psi_2) = \frac{\lambda R(\psi_1)\sin[\theta(\psi_1)] - R(\psi_2)\sin[\theta(\psi_2)]}{\lambda R(\psi_1)\cos[\theta(\psi_1)] - R(\psi_2)\cos[\theta(\psi_2)]}, \tag{7}$$

where $\lambda = \{[a^{(1)}t_0^{(1)}]/[a^{(2)}t_0^{(2)}]\}$. Critically, in this 1D case one is able to obtain a closed form for $\Phi_\tau(t)$, such that the existence of this field can be assessed purely by considering its right-hand side. Given both the range and domain of arctan extends over the reals and $\xi(\psi_1, \psi_2)$ is real by definition (and has a definite value for any pair of states), we can immediately conclude that a twinning field between any two systems described by Eq. (1) always exists.

An important caveat to this statement is that the predicted twinning field may be identically zero depending on the initial states of the twinned systems. For example, if we attempt to twin two systems of noninteracting electrons ($\hat{U}^{(k)} = 0$), then $\hat{K}$ commutes with the Hamiltonian and $K(\psi_k)$ is constant. If the systems are evolved from their ground state, $\theta(\psi_k) = 0$, and by Eq. (6), $\Phi_\tau(t) = 0$. This scenario is consistent with the impossibility of twinning fields in linear optics, and can be avoided by having at least one of the system pair have a nonzero potential and hence a nonlinear response. Furthermore, while an equation for Φ_τ can still be obtained in higher dimensions, in general it will not be of a closed form, and therefore a twinning field is not guaranteed to exist. The additional requirements for a twinning field to exist in this scenario are detailed in the Supplemental Material [42].

Illustrations.—Here we provide examples of twinning fields for systems described by the Fermi-Hubbard model of strongly interacting electrons. In this case, each system has an on-site potential described by [48,49]:

$$\hat{U}^{(k)} = U^{(k)} \sum_j \hat{c}^\dagger_{j\uparrow}\hat{c}_{j\uparrow}\hat{c}^\dagger_{j\downarrow}\hat{c}_{j\downarrow} \tag{8}$$

where $U^{(k)}$ parametrizes the energy of the electron-electron repulsion, and all equilibrium properties are determined by the ratio $U^{(k)}/t_0^{(k)}$. Despite the simplicity of the potential, this model is rich in nontrivial behavior, including topological [50,51] and superconducting phases in 2D [52,53]. The Fermi-Hubbard model is computationally challenging and a complete understanding of its dynamics is believed to require a quantum computer [54]. It also exhibits a highly nonlinear optical response

[49,55,56], and therefore provides a suitable platform for numerical calculations of twinning fields.

Here we consider an $L = 10$ site chain with periodic boundary conditions and an average of one electron per site. For the sake of simplicity, in both systems we use the lattice constants $a^{(1)} = a^{(2)} = 4$ Å, with a hopping parameter of $t_0 \equiv t_0^{(1)} = t_0^{(2)} = 0.52$ eV. To avoid the trivial solution of $\Phi_\tau(t) = 0$, each system (initially in the ground state) is first pumped by a single cycle of a transform-limited field. Specifically, an enveloped sine-wave is used with an amplitude of $E_0 = 10$ MV/cm and frequency $\omega_0 = 32.9$ THz. All calculations were performed using exact diagonalization via the QuSpin PYTHON package [57,58].

Figure 2 shows examples of calculated twinning fields and the accompanying responses they generate for several pairs of systems. These pairs are parametrized by Δ, using $U^{(1)} = 0.5t_0$, while $U^{(2)} = U^{(1)} + \Delta$. Applying the twinning field calculated at each time, we find the current in each pair of systems is identical, as expected.

A more concretely physical example is to twin two commonly studied materials—graphene and hexagonal boron nitride (h-BN). Both of these structures have a bipartite lattice structure, as shown in Fig. 3. Critically, that lattice constant for both systems is almost identical, and can hence be modeled with the identical value $|\mathbf{a}_j^{(\mathrm{GR})}| = |\mathbf{a}_j^{(h\text{-BN})}| = 2.5$ Å [59,60]. This is critical for the existence of a twinning field, as in higher dimensions the ability to twin a pair of systems can only be guaranteed when they share the same lattice structure.

Both materials are well modeled by the tight-binding approximation [61], with an on-site potential of the same form as Eq. (8). $t_0^{(\mathrm{GR})} = t_0^{(h\text{-BN})} = 2.7$ eV, and for the graphene carbon atoms $U_c = 0$, while for h-BN $U_B = 3.3$ eV and $U_N = -1.4$ eV for boron and nitrogen atoms, respectively [62,63]. While it is possible to also include next-to-nearest hopping, the relative strength of this compared to nearest-neighbor hopping is only ∼5% [64,65], and we therefore neglect it for calculational simplicity. Further information on both the precise Hamiltonian describing these systems, and the derived twinning field equations may be found in the Supplemental Material [42].

Simulations are performed using $L = 12$ sites with periodic boundaries, and the systems are again prepared via the application of a pumping field. In this case, the polarization of this initial pump is of great consequence, and in order to generate a physically realizable twinning field, it must be aligned with one of the nearest neighbor vectors. Figure 4 shows an example of this, with the initial pump pulse and subsequent twinning field aligned along the $\mathbf{a}_1$ direction.

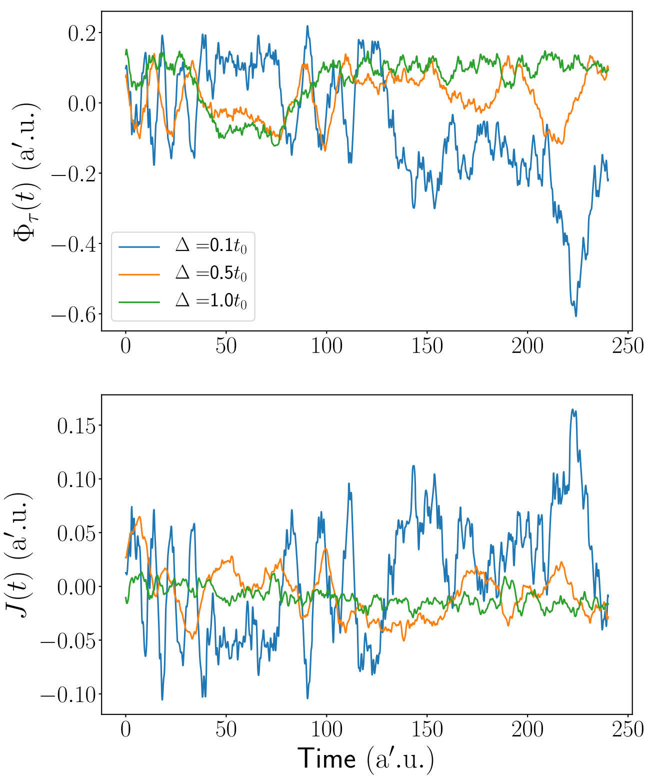

FIG. 2. Top panel: twinning fields and the accompanying optical response for three pairs of systems, following an initial pump pulse. Bottom panel: the current resulting from the application of the twinning field, which is identical in each pair of systems. In all cases $U^{(1)} = 0.5t_0$, while $U^{(2)} = U^{(1)} + \Delta$, where Δ is varied for each of the three pairs. a′.u. are atomic units with energy normalized to t_0.

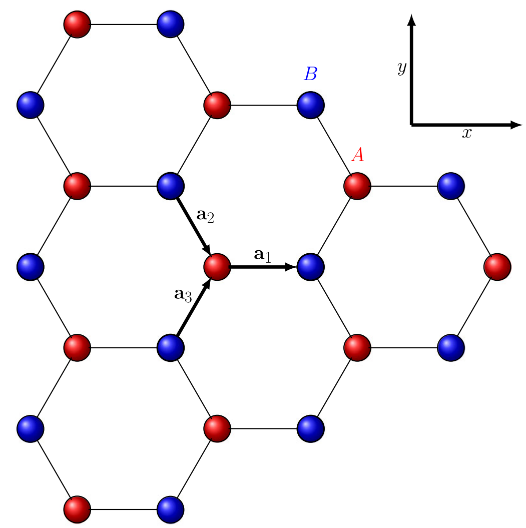

FIG. 3. Both graphene and h-BN can be modeled with the tight-binding approximation set on a hexagonal bipartite lattice, defined by the positions of the B sublattice atoms relative to those on the A sublattice. These are characterized by the three nearest neighbor vectors $\mathbf{a}_j$, with each having an angular separation of 120°, and length 2.5 Å.

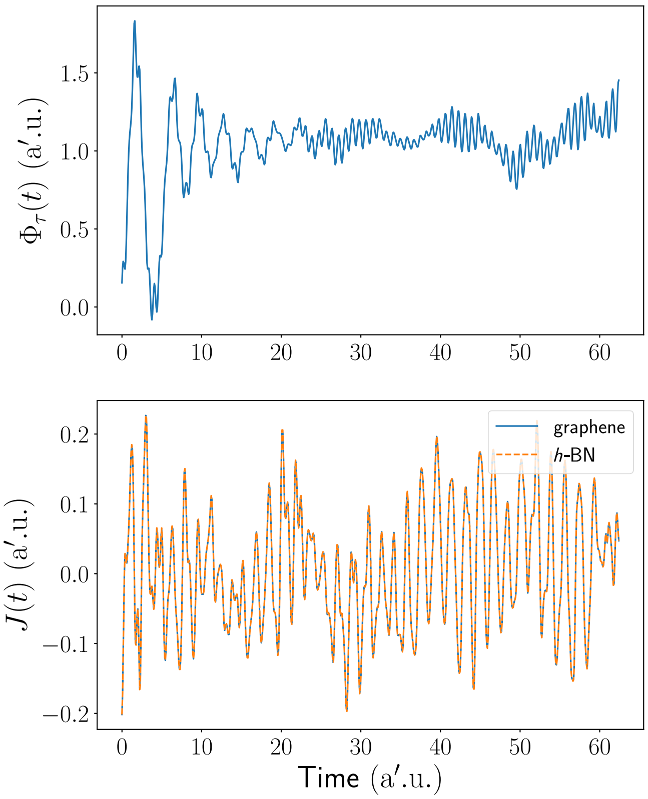

FIG. 4. Top panel: twinning field generated after a pump pulse in the $\mathbf{a}_1/x$ direction. The symmetry of the lattice guarantees that the resultant twinning field is *also* purely in the x direction. Bottom panel: the overall current $J(t)$ in the x direction due to application of the twinning field. As expected, this is identical in both systems.

Discussion.—We have introduced a nonlinear optical phenomenon where some pairs of quantum systems have a twinning field which generates an identical response in each individual system. In 1D the necessary conditions for a twinning field to exist are rather general, but in higher dimensions the two systems must possess a high degree of similarity in their lattice structures for the existence of a twinning field to be guaranteed. Conditions for the uniqueness of this field were derived, and numerical calculations provided examples of twinning fields in a Fermi-Hubbard system.

It is instructive to compare optical indistinguishability with the antihaecceitism [66,67] of quantum particles. The latter is responsible for both the Fermi-Dirac and Bose-Einstein distributions [68,69] (as well as the resolution of Gibbs' paradox [70,71]), and is an intrinsic and immutable property of said particles. For this reason it has been commonly assumed that systems governed by distinct Hamiltonians will be distinguishable from each other. Indeed, the effectiveness of spectroscopy is predicated on the notion that a material can be uniquely identified from its spectral response [72]. The existence of twinning fields gives the lie to this assumption, however, demonstrating that indistinguishability can arise as an emergent property under driving.

It may be tempting to think of twinning fields purely as an act of deception, where an unscrupulous salesman could use the technique to pass off a cheap and nasty material as something more costly. In fact, the analysis presented here demonstrates that it is a trick requiring highly specific conditions to be repeated, and such a fraud can be defeated by an arbitrary modification to the example driving field. Indeed, even if the twinning field is nonunique, the field up to the point that Lipschitz continuity (see [42]) is violated *will* be unique. Any field that is distinct from this initial trajectory will therefore be guaranteed to produce a response distinguishing the two systems. This has the important consequence of ensuring that techniques designed to discriminate between similar systems are well founded [27,28,32].

Of course, the existence of twinning fields forces one to consider both their feasibility and wider utility. While the twinning fields calculated here appear to be rather broadband pulses, the rapid improvement in both intensity and bandwidth of laboratory laser sources [73,74]—combined with the fact that similarly tailored tracking control fields can be well approximated by a few distinct frequencies [23]—suggests twinning fields may be experimentally realizable with current technology. In fact, deep learning networks have recently been employed to experimentally determine the driving field required to generate a desired response in a material, in a manner that is robust to noise [75]. It is likely that such techniques could be similarly applied for the practical calculation of twinning fields.

One potential application of these fields is to characterize the effect of interactions between systems. Applying the twinning field calculated for a noninteracting pair, it would be possible to identify the additional current generated by each system due to its interaction with the other. Twinning fields may also provide a method for creating alternative realizations of metamaterials [76] when a specific response is required. Given a field and a metamaterial's nearest-neighbor expectation, Eq. (6) can be used to calculate the properties of a different material producing the same effect. Consequently the search for cheaper alternative components in quantum technologies may be aided by twinning fields.

Naturally, there remain a number of unanswered questions. For instance, a given twinning field relates a pair of systems, but is this pair unique? Put differently, are there triplets, or n-tuplets of systems which exhibit optical indistinguishability? In the case of a nonunique twinning field, what are the physical consequences of choosing one solution over an another? Such questions may merit further investigation, as understanding these secondary properties provides both challenges and opportunities to illuminate the principles upon which driven systems operate.

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