Resummation of Quantum Radiation Reaction in Plane Waves

Greger Torgrimsson*
Helmholtz-Zentrum Dresden-Rossendorf, Bautzner Landstraße 400, 01328 Dresden, Germany

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We propose a new approach to obtain the momentum expectation value of an electron in a high-intensity laser, including multiple photon emissions and loops. We find a recursive formula that allows us to obtain the $\mathcal{O}(\alpha^n)$ term from $\mathcal{O}(\alpha^{n-1})$, which can also be expressed as an integro-differential equation. In the classical limit we obtain the solution to the Landau-Lifshitz equation to all orders. We show how spin-dependent quantum radiation reaction can be obtained by resumming both the energy expansion as well as the α expansion.

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An electron in an electromagnetic field emits photons and the recoil it experiences is called radiation reaction (RR) [1,2]. Classically, the standard equation is the Abraham-Lorentz-Dirac (LAD) equation, but since it leads to unphysical solutions (with either preacceleration, where the particle starts to accelerate before the field turns on, or runaways, where the particle's momentum diverges even in the absence of a field) it is common to replace it with the Landau-Lifshitz (LL) equation, which is free from such unphysical solutions. Since preacceleration occurs on timescales that are actually within the quantum regime, and since the LAD and LL equations agree quite well within the classical regime (see, e.g., [3]), it is common to view the LL equation as giving a correct description for practical purposes. A more practical problem is how to describe quantum RR. If RR is important one would in general expect the particle to emit many photons, making higher orders in $\alpha = e^2/(4\pi)$ important. The first RR experiments with high-intensity lasers were performed recently [4,5] (see also [6]), and in upcoming experiments one may expect significant quantum effects [7–11]. Clearly, one needs some approximation to obtain higher orders in α. Standard approaches assume [12] $a_0 \coloneqq E/\omega \gg 1$ and then approximate higher orders as incoherent products [13] of $\mathcal{O}(\alpha)$ photon emissions calculated with a locally constant field (see, e.g., [7–11,24]). A question is how to take spin into account. See, e.g., [25–27] for spin-dependent RR. In this Letter we propose a new method for obtaining quantum RR in high-intensity lasers. It too is based on incoherent products of $\mathcal{O}(\alpha)$ terms. The main differences compared to previous approaches are: (1) We treat spin transitions using Mueller matrices, which allows us to consider arbitrary spin, field polarization and field shape, and, importantly, we are not limited to large a_0. (2) We consider the contribution from loop diagrams explicitly. (3) While we can resum the α expansion right from the start into a new integro-differential equation, we also show that resummation methods based, e.g., on Padé approximants can be faster.

Let p_μ and P_μ be the initial and final momentum of the electron, k_μ the wave vector of the plane wave, which is described by a gauge potential $a_\perp(\sigma)$ where [12] $\sigma = kx = \omega(t+z)$ is the (rescaled) light-front time and $a_\pm = 0$. Light-front coordinates are defined by $v^\pm = 2v_\mp = v^0 \pm v^3$ and $v_\perp = \{v_1, v_2\}$. We will focus on $\langle kP\rangle = \sum_{n=0}^\infty \langle kP\rangle^{(n)}$, where $\langle kP\rangle^{(n)} = \mathcal{O}(\alpha^n)$. The reason for focusing on the light-front longitudinal momentum ($kP = 2k_+P_-$) is because in a plane wave the probabilities of the whole process and each intermediate step only depend nontrivially on this momentum component. The dependence on the initial and final spin can be expressed as $\langle kP\rangle^{(n)} = (1/2)\mathbf{N}_0 \cdot \mathbf{M}^{(n)} \cdot \mathbf{N}_f$, where $\mathbf{N} = \{1, \mathbf{n}\}$ is the Stokes vector for spin along the unit vector $\mathbf{n}$ and $\mathbf{M}^{(n)}$ are "strong-field-QED Mueller matrices" [21,28,29]. Averaging and summing over the initial and final spin gives $\{1, \mathbf{0}\} \cdot \mathbf{M} \cdot \{1, \mathbf{0}\}$. The Mueller-matrix approach has been developed recently [21,28,29]. It allows us to obtain $\mathcal{O}(\alpha^n)$ probabilities to leading order for long pulses or large $a_0 = E/\omega$ by multiplying $\mathcal{O}(\alpha)$ Mueller matrices. In this case we need $\mathbf{M}^C$ for (nonlinear) Compton scattering and $\mathbf{M}^L$ for the cross term between the $\mathcal{O}(\alpha^0)$ and $\mathcal{O}(\alpha)$ parts of the amplitude for $e^- \to e^-$. In [21,28,29] we described how to write down expressions for individual higher-order diagrams. To use these methods to obtain RR we need a way to evaluate all the relevant higher-order diagrams and to resum them. Both photon emissions and loops [30] contribute and there are 2^n diagrams at $\mathcal{O}(\alpha^n)$. More details of this derivation can be found in the Supplemental

Material (SM) [35]. One of the main steps is to note that $\mathbf{M}^{(n)}$ can be obtained by prepending a Compton scattering or a loop step at the initial position of $\mathbf{M}^{(n-1)}$, which gives the following recursive formula:

$$\mathbf{M}^{(n)}(b_0,\sigma) = \int_\sigma^\infty \frac{d\sigma'}{b_0}\int_0^1 dq\{\mathbf{M}^L(b_0,\sigma',q)\cdot\mathbf{M}^{(n-1)}(b_0,\sigma') + \mathbf{M}^C(b_0,\sigma',q)\cdot\mathbf{M}^{(n-1)}(b_0[1-q],\sigma')\}, \quad (1)$$

where $b_0 = kp$, $q = kl/kp$ and l is the photon momentum, and $\mathbf{M}^{(0)} = b_0\mathbf{1}$ [$\mathbf{M}^{(0)} = \mathbf{1}$ for the probability]. The lower integration limit has been introduced so that the products of Mueller matrices are light-front–time ordered. The final result is obtained by setting $\sigma = -\infty$. The shift $b_0 \to b_0(1-q_1)$ in the Compton term takes RR into account.

With $\mathbf{M} = \sum_{n=0}^\infty \mathbf{M}^{(n)}$, (1) implies

$$\frac{\partial\mathbf{M}}{\partial\sigma} = -\int_0^1 \frac{dq}{b_0}\{\mathbf{M}^L\cdot\mathbf{M}(b_0) + \mathbf{M}^C\cdot\mathbf{M}(b_0[1-q])\}. \quad (2)$$

This integro-differential matrix equation gives quantum RR to all orders in α with spin taken into account. Note that even if we are only interested in unpolarized initial and final states, we still need to solve a matrix equation before we can project with $\{1,\mathbf{0}\}\cdot\mathbf{M}\cdot\{1,\mathbf{0}\}$. Note also that (2) holds even if a_0 is not large, provided the field is sufficiently long and the full version of $\mathbf{M}^{C,L}$ is used. Equation (2) looks similar to kinetic RR equations [10,36] (see also [37]), so one may expect that it can be solved with similar methods. However, in this Letter we will instead use (1) to obtain the first $\mathbf{M}^{(n)}$ terms before performing a resummation. It turns out that this resummation approach can be much faster.

We consider first the classical limit. In the SM we show that (1) leads to a geometric series in the classical limit. In fact, we also show that the $\langle P_{\perp,+}\rangle$ components can be calculated in a similar way. We find

$$\lim_{b_0\ll 1}\langle P_{-,\perp}\rangle = \pi_{-,\perp} + \frac{\Delta}{1+\Delta\int d\sigma\mathbf{a}'^2}\left[\pi' - \int d\sigma\mathbf{a}'^2\pi\right]_{-,\perp}, \quad (3)$$

where $\pi_\mu = p_\mu - a_\mu + (2ap - a^2)k_\mu/(2kp)$ is the Lorentz momentum, and $\langle P_+\rangle$ satisfies the mass-shell condition $P_+ = (P_\perp^2+1)/(4P_-)$. This resummation [38] agrees exactly with the exact solution to LL [40] (see also [41] for an exact solution to LL; [10,36,42] for comparisons between LL and kinetic equations; and [43–47] for comparisons between classical equations and the classical limit of quantum RR at first order). Note that the loops are needed even in the classical limit, cf. [46–48].

This does not mean that LAD does not agree with the classical limit of QED, because our approximation neglects terms that have subdominant scaling with respect to the pulse length and/or a_0. From Eq. (4.39) in [47] we see that the difference between the LL and LAD equations at $\mathcal{O}(\alpha^2)$ is a term proportional to $\mathbf{a}'^2$, i.e., a term without an integral. This term vanishes at asymptotic σ, but is also subdominant at finite times because it has no pulse-length scaling. We can therefore not rule out that the classical limit of all QED contributions may agree with the LAD equation rather than the LL equation.

In the limit of a very long pulse we have (cf. [3,49–51])

$$\langle P_-\rangle \approx \frac{p_-}{\Delta\int d\sigma'\mathbf{a}'^2} \quad \langle P_\perp\rangle \approx \frac{\int d\sigma'\mathbf{a}'^2[a(\sigma')-a(\sigma)]_\perp}{\int d\sigma'\mathbf{a}'^2}, \quad (4)$$

so $\langle P_-\rangle$ becomes small, $\langle P_\perp\rangle$ stays $\mathcal{O}(1)$ and $\langle P_+\rangle$ becomes large, and (since $\Delta \propto p_-$) all components of $\langle P_\mu\rangle$ become independent of the initial momentum [3,49–51]. This observation will be important for resumming the α expansion in the quantum regime.

Having shown that (1) gives the expected classical limit, we now turn to quantum RR. For simplicity we focus on $\langle P_-\rangle$, and we sum over the final spin, i.e., $\mathbf{N}_f = \{1,\mathbf{0}\}$ is fixed and we have an overall factor of 2, so we use $\mathbf{N}^{(n)} \coloneqq \mathbf{M}^{(n)}\cdot\{1,\mathbf{0}\}$. In this first application we consider a constant field, so the σ integrals give $\int d\sigma_n\ldots d\sigma_1 \to (\Delta\phi)^n/n!$, and it is convenient to rescale $\mathbf{N}^{(n)}$ so that

$$a_0\langle kP\rangle^{(n)} = T^n\mathbf{N}_0\cdot\mathbf{N}^{(n)}, \quad (5)$$

where $T = \alpha a_0\Delta\phi$ is our effective expansion parameter, which can be $T > 1$ for large $a_0\Delta\phi$. In general $\mathbf{N}$ has four elements, but here we consider only initial and final states that are either unpolarized or polarized (anti-)parallel to the magnetic field, and then only two elements contribute. Omitting the irrelevant elements we have $\mathbf{N}_0 = \{1,0\}$ and $\mathbf{N}_0 = \{1,\pm 1\}$ for unpolarized and polarized states. We have

$$\mathbf{N}^{(n)} = \int_0^1 \frac{dq}{n\chi}\{\mathbf{M}^C\cdot\mathbf{N}^{(n-1)}(\chi[1-q]) + \mathbf{M}^L\cdot\mathbf{N}^{(n-1)}(\chi)\}, \quad (6)$$

where $\mathbf{N}^{(0)}(\chi) = \chi\{1,0\}$, $1/n$ comes from $(\Delta\phi)^n/n!$, $\chi = a_0b_0$ is the quantum nonlinearity parameter (gives the electric field in the rest frame of the electron), from [28,29]

$$\mathbf{M}^C = \begin{pmatrix} -\mathrm{Ai}_1(\xi) - \kappa\frac{\mathrm{Ai}'(\xi)}{\xi} & \frac{q}{s_1}\frac{\mathrm{Ai}(\xi)}{\sqrt{\xi}} \\ q\frac{\mathrm{Ai}(\xi)}{\sqrt{\xi}} & -\mathrm{Ai}_1(\xi) - 2\frac{\mathrm{Ai}'(\xi)}{\xi} \end{pmatrix} \quad (7)$$

and

$$\mathbf{M}^L = \begin{pmatrix} \mathrm{Ai}_1(\xi) + \kappa\frac{\mathrm{Ai}'(\xi)}{\xi} & -q\frac{\mathrm{Ai}(\xi)}{\sqrt{\xi}} \\ -q\frac{\mathrm{Ai}(\xi)}{\sqrt{\xi}} & \mathrm{Ai}_1(\xi) + \kappa\frac{\mathrm{Ai}'(\xi)}{\xi} \end{pmatrix}, \quad (8)$$

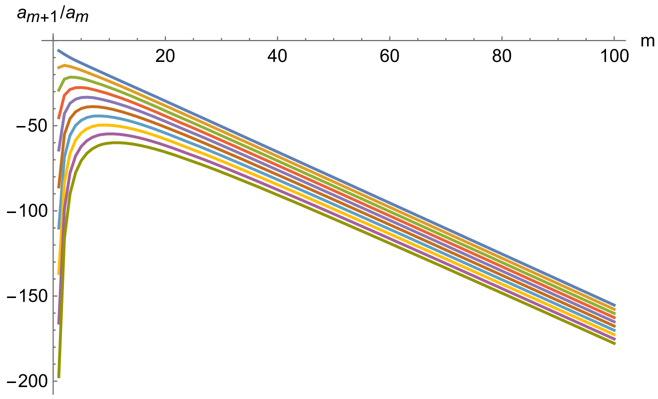

FIG. 1. The ratios of neighboring coefficients in the χ expansion for $\{1,0\}\cdot\mathbf{a}_m$. The different lines correspond to $\langle kP\rangle^{(1)}$ (top) to $\langle kP\rangle^{(10)}$ (bottom). The linear scaling at higher orders shows that the coefficients grow factorially and the χ expansion is hence asymptotic.

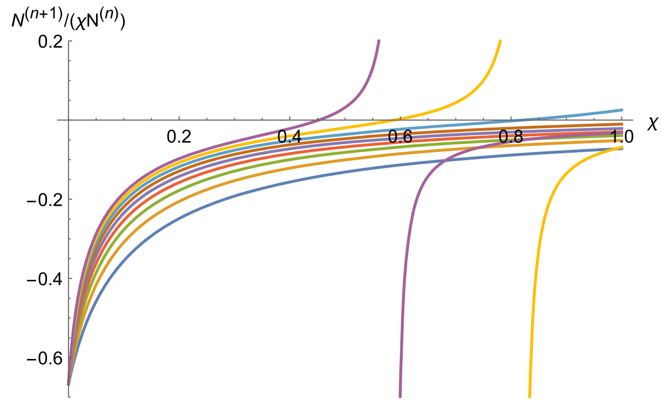

FIG. 2. The ratios $N^{(n+1)}/(\chi N^{(n)})$ of neighboring coefficients in the α expansion as a function of χ, for an unpolarized initial state, $N^{(n)}=\{1,0\}\cdot\mathbf{N}^{(n)}$. The ratios have been divided by χ in order to show the convergence to the classical limit $N^{(n+1)}/N^{(n)}\to -2\chi/3$ for $\chi\to 0$.

where $\xi=(r/\chi)^{2/3}$ with $r=(1/s_1)-1$, $\kappa=(1/s_1)+s_1$, $s_1=1-q$ and $\mathrm{Ai}_1(\xi)=\int_\xi^\infty dx\,\mathrm{Ai}(x)$.

In order to compute $\mathbf{N}^{(n)}$ we have used two completely different methods. In the first we compute $\mathbf{N}^{(1)}(\chi)$ between $\chi=0$ and some [52] $\chi_{\max}$ and make an interpolation function of it, which is then used in (6) to compute an interpolation function of $\mathbf{N}^{(2)}(\chi)$, and so on.

In the second method we expand $\mathbf{N}^{(1)}(\chi)$ in a power series in χ, which is then used to obtain a power series of $\mathbf{N}^{(2)}(\chi)$ and so on, see the SM. This gives $\mathbf{N}^{(n)}=\chi^{1+n}\sum_{m=0}^M\mathbf{a}_m^{(n)}\chi^m$. As illustrated in Fig. 1, the coefficients have alternating sign and grow factorially $\mathbf{a}_m\sim(-1)^m m!$. The χ expansion is therefore asymptotic with zero radius of convergence and needs to be resummed. There is no unique resummation method (unless, of course, one is able to find an exact expression). Different resummations are obtained by matching the series onto different (sums of) special functions. Recent examples involve, e.g., the Meijer-G or hypergeometric functions [23,53,54] (see [55] for further applications in strong-field QED). Such resummations require fewer terms, but usually some extra information, e.g., about the opposite limit (large χ in our case). However, in the present case, we can without problems obtain a large number of terms quickly. We can hence use the general Borel-Padé resummation method [56–65], which only requires the $\mathbf{a}_m^{(n)}$ coefficients. From the truncated series one first obtains a truncated Borel transform, $\mathbf{B}_M^{(n)}(t)=\sum_{m=0}^M(1/m!)\mathbf{a}_m^{(n)}t^m$. Next we project with the initial Stokes vector, $a_m^{(n)}\coloneqq\mathbf{N}_0\cdot\mathbf{a}_m^{(n)}$, and construct a Padé approximant $PB[I/J](t)=\sum_{i=0}^I c_it^i/(1+\sum_{j=1}^J d_jt^j)=B_N(t)+\mathcal{O}(t^{N+1})$. The result is then obtained from the following Laplace integral:

$$\mathbf{N}_0\cdot\mathbf{N}_{\rm re}^{(n)}(\chi)=\chi^{1+n}\int_0^\infty\frac{dt}{\chi}e^{-t/\chi}PB[I/J](t). \qquad (9)$$

Using either of these two approaches we obtain a set of functions, $\mathbf{N}^{(n)}(\chi)$, for $0<\chi<\chi_{\max}$. The result for an unpolarized initial state is shown in Fig. 2. In the low χ limit we find convergence towards the classical prediction.

In general one would also expect the α expansion to be asymptotic. However, an approximation does not have be asymptotic, see, e.g., [60,66–69] for the weak and strong field approximations of the QED effective action. (See [70,71] for other recent α resummations.) In classical RR, LAD leads to an asymptotic series [72], while LL has a finite radius of convergence. Since the coefficients we have calculated suggest a finite radius of convergence, we propose to resum the α expansion with a Padé approximant

$$a_0\langle kP\rangle\approx a_0\langle kP\rangle_{\rm re}=\chi+\frac{\sum_{i=1}^I A_i(\chi)T^i}{1+\sum_{j=1}^J B_j(\chi)T^j}, \qquad (10)$$

where A_i and B_i are obtained by matching $a_0\langle kP\rangle_{\rm re}=\chi+\sum_{n=0}^{I+J}T^n\mathbf{N}_0\cdot\mathbf{N}^{(n)}+\mathcal{O}(T^{I+J+1})$. Since we expect a finite limit for $T\gg 1$, we take $I=J$. This makes it possible for $\langle kP\rangle_{\rm re}\to 0$ as $T\to\infty$, which is what we expect [cf. (4)]. If we do not impose this limit, then we can take $|A_I/(\chi B_I)+1|$ as an upper-limit estimate on the relative error (the error goes to zero as $T\to 0$). Alternatively, we can demand $\langle kP\rangle_{\rm re}=\mathcal{O}(1/T)$ for $T\gg 1$, which fixes $A_I=-\chi B_I$. LL predicts that the leading order in $T\gg 1$ is independent of the initial momentum (4). If we assume that holds in general, then the $\mathcal{O}(1/T)$ term must be the same as the classical prediction, which implies

$$A_I=-\chi B_I\quad B_I=\frac{2}{3}(A_{I-1}+\chi B_{I-1}). \qquad (11)$$

Figure 3 shows that the resummation (10) converges quickly. On the scale of this plot, the $I=4$ and $I=5$ resummations are virtually indistinguishable, which are

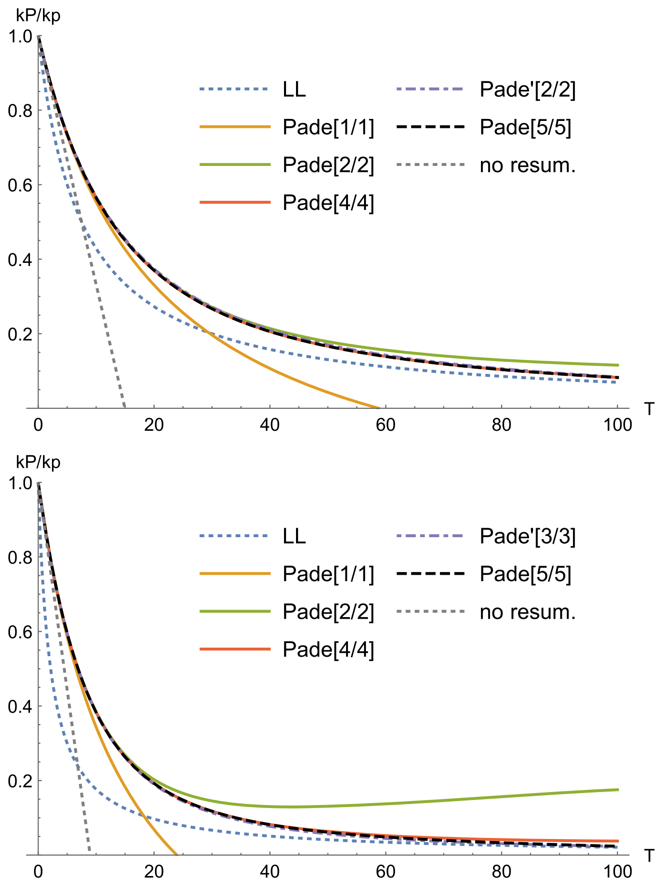

FIG. 3. Final result for $\chi = 0.2$ (upper) and $\chi = 0.7$ (lower) as a function of $T = \alpha a_0 \Delta\phi$. Padé$[I/I]$ corresponds to the resummation in (10) with A_i and B_i determined from the first $2I$ coefficients $\{1,0\}\cdot\mathbf{N}^{(n)}$, $1 \le n \le 2I$. Padé'$[I/I]$ is obtained from only $\{1,0\}\cdot\mathbf{N}^{(n)}$, $1 \le n \le 2I-2$, while the remaining two coefficients are determined from (11). The "no resum." line is just the sum of $\mathcal{O}(\alpha^0)$ and $\mathcal{O}(\alpha)$.

obtained from the first eight and ten $\mathbf{N}^{(n)}$ terms. For short to moderately large T, we see that the classical prediction overestimates the effect of RR, as expected [1]. However, for larger T the classical and quantum predictions seem to converge. This is what one would expect if the leading order at $T \gg 1$ is independent of the initial momentum. This motivates us to use the modified Padé approximant based on (11). With the two extra conditions in (11) we indeed find an even faster convergence, with a decent approximation already with $I = 2$, i.e., using only the $\mathcal{O}(\alpha)$ and $\mathcal{O}(\alpha^2)$ terms.

These resummations can be compared with the solution to the integro-differential equation corresponding to (6), i.e.,

$$\frac{\partial\mathbf{N}}{\partial T} = \int_0^1 \frac{dq}{\chi}\{\mathbf{M}^C\cdot\mathbf{N}(\chi[1-q]) + \mathbf{M}^L\cdot\mathbf{N}(\chi)\}, \qquad (12)$$

where $\mathbf{N} = \sum_{n=0}^{\infty} T^n \mathbf{N}^{(n)}$ and $\mathbf{N}(T=0) = \chi\{1,0\}$. We have solved (12) numerically with the Euler method and found good agreement with the resummations above. However, it takes much more time to solve (12) because we need $T_{\max}/dT$ steps, which is typically more than the ten (or fewer) steps we needed in the resummation approach.

At $T \gg 1$ we have the ansatz $\mathbf{N} \approx \{c/T + \mathcal{O}(1/T^2), \mathcal{O}(1/T^2)\}$, so $\partial\mathbf{N}/\partial T = \mathcal{O}(1/T^2)$, which gives a condition for c since the right-hand side of (12) is not automatically $\mathcal{O}(1/T^2)$. As mentioned, we expect c to be independent of χ, and now we can confirm that this is consistent with (12).

In order to obtain $\mathbf{N}^{(n)}$ from $\mathbf{N}^{(n-1)}$ we need to calculate both components of $\mathbf{N}^{(n-1)}$, even if we at the end are only interested in unpolarized particles. Hence, in calculating the results above we have obtained the necessary information to study a polarized initial state as a byproduct. We show in the SM for the difference in the final momentum due to the initial spin

$$\lim_{\chi\ll 1}\frac{a_0}{2}k\Delta\langle P\rangle = \sum_{n=1}^{\infty}\{0,1\}\cdot\mathbf{N}^{(n)}T^n = -\frac{3}{2}\chi^2\frac{\ln[1+\frac{2}{3}\chi T]}{(1+\frac{2}{3}\chi T)^2}. \qquad (13)$$

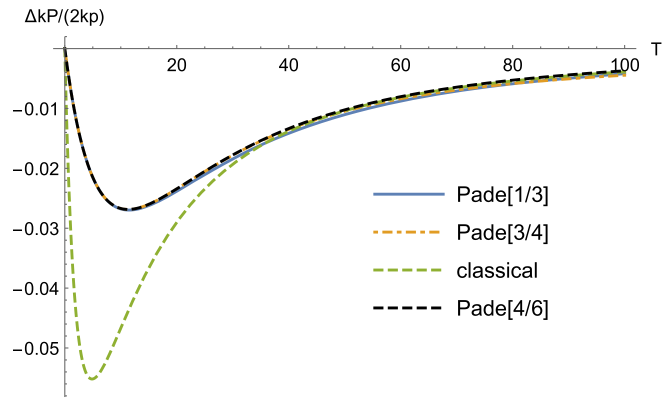

FIG. 4. Difference in the final momentum for initial state with spin up and down and $\chi = 0.2$.

We see that $\Delta\langle P\rangle$ too becomes independent of χ to leading order in $T \gg 1$, although this time the next-to-leading order is only logarithmically suppressed. Another difference is that (13) is nonmonotonic with a maximum $|\Delta\langle P\rangle|$ at $T \sim 1/\chi$.

The full quantum result can be obtained as described above. Although (13) might suggest using a resummation involving logarithms, we still find a fast convergence with Padé approximants as in (10), except that $P_- > 0$ implies that $|\Delta\langle P\rangle|$ must be smaller than $\langle P\rangle_\uparrow + \langle P\rangle_\downarrow$ and for that to hold at large T we need $J > I$. In practice, different choices of I and J can give good approximations, and a "wrong" choice just means that we need to include more terms. Figure 4 shows that the classical prediction overestimates the peak of $|\Delta\langle P\rangle|$ by about a factor of 2 for $\chi = 0.2$, but is close to the quantum result for large T.

In conclusion, we have developed a new approach for spin-dependent quantum RR using products of Mueller matrices for photon emissions and loops. We have found a recursive formula which gives $\mathcal{O}(\alpha^n)$ from $\mathcal{O}(\alpha^{n-1})$. In the classical limit we find the solution to LL to all orders. We obtain quantum RR either by resumming the recursive formula into an integro-differential matrix equation, or by resumming the α expansion with Padé approximants, which converge quickly. The $\mathcal{O}(\alpha^n)$ terms in the second approach can be obtained either numerically or by expanding each $\mathcal{O}(\alpha^n)$ in χ and resumming the resulting asymptotic series with, e.g., the Borel-Padé method.

The first laser-based RR experiments were performed just a couple of years ago [4,5], and further laser-electron experiments are planned for the near future, e.g., at LUXE [73,74] and FACET-II [75]. The experiment in [5] has already hinted that standard locally-constant-field (LCF) approaches might give significant discrepancies. One source of discrepancy within LCF could be spin and loop effects not taken into account in standard LCF approaches, but which can be included in a LCF version of our approach. If instead a_0 is not large enough for any LCF treatment, but if the field is long, then one can try our approach with a locally-monochromatic-field (LMF) approximation. If a_0 is not large enough for LCF and the pulse not long enough for LMF, but if a_0 is still moderately large and the pulse still moderately long, then one could try our approach with the exact $\mathcal{O}(\alpha)$ Mueller matrices [28,29], which would also be relevant for long pulses with some asymmetry that prevents a LMF treatment. In order to compare with experiments such as [5], it would be useful to generalize our approach to the momentum spectrum.

*g.torgrimsson@hzdr.de

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