Quantum Speed Limit Quantified by the Changing Rate of Phase

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The quantum speed limit is important in determining the minimum evolution time of a quantum system, and thus is essential for quantum community. In this Letter, we derive a novel unified quantum speed limit bound for Hermitian and non-Hermitian quantum systems. The bound is quantified by the changing rate of phase of the quantum system, which represents the transmission mode of the quantum states over their evolution. The bound leads to further insights beyond the previous bounds on concrete evolution modes of the quantum system, such as horizontal or parallel transition or horizontal joining of the two quantum states in Hilbert space. The bound is linked to the feasibility of the evolutions of the state vectors, and provides a tighter upper bound. In addition, the generalized Margolus-Levitin bound is discussed.

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Introduction.—As one of the fundamental questions, quantum speed limits (QSLs) play a vital role in the community of quantum physics. Recent advances in quantum engineering have spurred further investigation and understanding of how quickly a quantum physical system transitions between distinguishable states. This is also interesting from a practical point of view, as QSLs can be employed to estimate the speed of quantum simulations involving quantum information processing, quantum computation and quantum metrology [1–4]; identify decoherence time [5–9]; improve the rate of quantum information processing; make optimal control of the quantum system; and identify the precision bound in quantum metrology as well as the experimental measuring of the environment-assisted speed-up [10], and others [11–13]. New numerical methods such as machine learning can also be applied in the study of QSLs [14]. In the case of the unitary evolution, Mandelstam and Tamm derived the lower bound of the evolution time to be $\tau_{\mathrm{QSL}} = \pi\hbar/2\Delta E$, the Mandelstam-Tamm (MT) bound [15]. Margolus and Levitin presented a different bound for a closed system $\tau_{\mathrm{QSL}} = \pi\hbar/2\langle E\rangle$ [16], the Margolus-Levitin (ML) bound. Many researchers have attempted to develop QSL for the open quantum system [17–20], and have investigated the relations between QSL and the physical nature of the quantum system, including the role of entanglement in QSLs for open dynamics and a many-body system [21–26], nonequilibrium dynamics, relativistic and the distinct QSL bound [27–29], the relation between the maximum interaction speed and QSL in quantum spin systems [30], and the non-Markovianity effect of the environment on accelerating the speed of evolution [31,32]. QSLs have also been studied in the quantum representation of the Wigner function [33–37]. Also, in quantum thermodynamics the maximal rate of entropy production has been derived from the QSL [38]. It has been shown that such a speed limit could also exist in classical systems [33,39].

The previous theoretical framework for the generalization of QSL bounds employs the distance between the states in the manifold space as well as certain inequalities [5,7,18,27,40,41]. The relevant derivations provide the intuitively understandable expressions of QSL bounds. Additionally, it is known that the family of QSLs could be constructed using the family of contractive Riemannian metrics [5]. This means that the different distances lead to different bounds [5,7]. Usually, it is difficult to obtain a tight and attainable bound through the suitable choice of the distances [5,7,20,25,42].

Among the generations and developments of QSLs, with few exceptions [18,21,43–46], these studies have successfully generalized the MT-type bounds [5,17–27,40,41]. Although there have been attempts to generalize ML-type bounds, the current studies of ML-type bounds suggest that this is a significantly harder task, and so their generation remains an open question. The challenges on the generation of the ML-type bound indicate that it is necessary to investigate and understand the physics behind the bounds.

The evolution of a quantum system is governed by the Schrödinger equation, however, what remains unnoticed is the paradoxical situation of the existence of the different bounds [47]. It is surprising that the different physical properties seem to exist for the same quantum state [7,27,47]. One typical scheme to overcome the paradoxical situation is to derive a unified bound by combining the MT bound and ML bound together, i.e., reads $\tau_{\mathrm{QSL}} = \max\{\pi\hbar/2\Delta E, \pi\hbar/2\langle E\rangle\}$ for a unitary system and orthogonal states [18,20,21,47]. These inspire us to think what the physics behind the different QSL bounds, and how the different QSL bounds are related to the evolution of the quantum system since the QSLs are the lower bound of the time required for a quantum system to evolve. Additionally, the usefulness of a bound should be considered since it is linked to the feasibility of its evaluation—to be computational or experimental.

This Letter presents such a novel unified QSL bound that is quantified by the changing rate of phase of the quantum system. QSL bounds are therefore generalized within a single theoretical framework and is not limited to Mandelstam-Tamm's and Margolus-Levitin's formulate. Our QSL bound represents the "quantum speed" of the evolution of the quantum system since the phase can be thought as the nature of the quantum system evolving—the dynamical evolution of a quantum system leads to the accumulation of phase of the quantum system, and so it is natural for the transition speed of the quantum states to be quantified by the changing rate of phase. This implies that our novel unified bound is attainable, while also easier to compute and measure. This also indicates that it is a tight bound since the different QSL bounds represent the different evolution status of the quantum system. In contrast to previous studies, we show the changing rate of phase as a unified quantity, bounding the speed at which the quantum states evolve. This bound outstrips the previous generalization of QSL bounds, such as the zero ground-state energy requirement of the quantum system for ML-type bounds. In addition, this bound can provide tighter upper bounds than the previous results.

The concept of phase is of great practical importance in contemporary physics [48–51]. The QSL bound, quantified by using the changing rate of phase, helps us to understand the phase in great detail and leads to further insights in the quantum community, such as the maximum speed of quantum gate operations, quantum metrology, quantum control, adiabatic quantum computation, and indicators of quantum phase transitions, etc. [11,52–57].

A novel unified QSL bound theory.—Consider a quantum system that evolves according to the Schrödinger equation $H(t)|\psi(t)\rangle = i\hbar(\partial/\partial t)|\psi(t)\rangle$, where $H(t)$ is the Hamiltonian of the system. As usual, the states $|\psi(t)\rangle$ are assumed to be the normalized vectors in Hilbert space $\mathscr{H}$. In the language of differential geometry, the normalized state vectors $|\psi(t)\rangle$ in Hilbert space $\mathscr{H}$ form the Stiefel manifold S_N (here, we assume that the Hilbert space has dimension $N+1$ with N being a non-negative integer). The Stiefel manifold S_N is one of the typical Riemannian manifold. Roughly speaking, the Stiefel manifold S_N can be thought as the space of sets of N orthonormal vectors in an $N+1$-dimensional vector space. It is the complex homogeneous space $U(N+1)/U(1)$ [49,58,59]. Its corresponding projective space and base manifold are denoted as $\mathscr{P}$ and $\mathscr{B}$, respectively. As time marches on, we suppose the quantum system evolves into the state $|\psi(t+dt)\rangle$ from the state $|\phi(t)\rangle$ in the time interval dt.

In space $\mathscr{H}$, we define the time-parameter (from time $t=0$ to $t=\tau$) smooth curve, consisting of a family of vectors $|\psi(t)\rangle$, as $\mathscr{C} = \{|\psi(t)\rangle \in \mathscr{H}\}$.

From the Stiefel manifold S_N, the state $|\psi(t+dt)\rangle$ of the system, which evolves from the initial state $|\phi(t)\rangle$ by the distance ds along the curve $\mathscr{C}$, can be written as [58–62]

$$|\psi(t+dt)\rangle = e^{ids\mathbb{K}}|\phi(t)\rangle, \tag{1}$$

where $\mathbb{K}$ is the corresponding generator of ds as the parameter on the Stiefel manifold S_N.

The distance ds can be obtained by employing a horizontal geodesic in the base manifold $\mathscr{B}$ along the curve $\mathcal{C}$ using the Fubini-Study metric. Note that the curve $\mathcal{C}$ is the image of $\mathscr{C}$ in the projective space $\mathscr{P}$, that is, the curve $\mathscr{C}$ projects onto (with projection map Π) the image curve $\mathcal{C}$ in the projective space $\mathscr{P}$, namely, $\mathcal{C} = \Pi(\mathscr{C}) \subset \mathscr{P}$. By extending the state $|\phi\rangle$ to an orthonormal basis $\{|\phi_\alpha\rangle, \alpha = 1,2,\ldots,N+1\} \in \mathscr{B}$, and $|\phi_1\rangle \equiv |\phi\rangle$, the nonzero elements of $\mathbb{K}$ in this basis are $\mathsf{K} = \mathbb{K}_{12} = \mathbb{K}_{21} = \frac{1}{2}$ [59,60].

Following the suggestion made by Aharonov and Anandan [59], the states of $|\psi(t+dt)\rangle$ and $|\phi(t)\rangle$ are the points in space $\mathscr{P}$ that lie on the geodesic separated by the distance ds. After including the action of the generator, the moment of the Aharonov and Anandan distance $ds\mathsf{K}$ is equal to the gauge distance [27,58,59,61,63], namely

$$d\ell = ds\mathsf{K}. \tag{2}$$

In the evolution of the quantum system from state $|\phi(t)\rangle$ to state $|\psi(t+dt)\rangle$, we have [48,49,64,65]

$$|\psi(t+dt)\rangle = e^{id\varphi_p}|\phi(t)\rangle, \tag{3}$$

and $d\varphi_p$ is the accumulated phase of the quantum system in the evolution from state $|\phi(t)\rangle$ to state $|\psi(t+dt)\rangle$.

Now, the maximal quantum speed v_{QSL}, after considering Eqs. (1)–(3), turns to be [5,7,17,27]

$$v_{\mathrm{QSL}} = \left|\frac{d\ell}{dt}\right| \equiv \left|\frac{d\varphi_p}{dt}\right|. \tag{4}$$

For the quantum evolution, however, as usually, QSL bound involves estimation of the minimal evolution time τ_{QSL} [5,7]. The QSL time τ_{QSL} is defined as the ratio between some distance between states and the average speed induced by the quantum evolution, namely

$$\tau_{\mathrm{QSL}} = \frac{\mathcal{L}[\psi(0),\psi(\tau)]}{\frac{1}{\tau}\int_0^\tau |\frac{d\varphi_p}{dt}| dt}, \tag{5}$$

where $\mathcal{L}[\psi(0),\psi(\tau)]$ is the length of the geodesic line connecting the rays $|\psi(0)\rangle$ and $|\psi(\tau)\rangle$ on the base manifold $\mathscr{B}$.

The expression v_{QSL} of Eq. (4) shows that the changing rate of phase provides the upper bound to the evolution speed of the quantum system. Equation (5) establishes the expression of QSL time. This novel expression of the QSL bound shows us the geometric and kinematic characters of the quantum system during its evolution since the total phase consists of the geometric phase and dynamical phase. As the dynamical evolution of the quantum system results in the accumulation of phase [48–50,66], hence it is naturally the changing rate of phase that shows us the "quantum speed" of the quantum system evolution. This enriches our understanding of the phase of quantum system. Obviously, the QSL time in Eq. (5) is superior to the previous expression in terms of its feasibility and attainability, because the natures of phase in quantum system have been widely investigated and well understood [49,64]. In addition, the bound of Eq. (5) naturally provides a tight bound since it is obtained from the evolution of the quantum system.

The phase difference between $|\psi(0)\rangle$ and $|\psi(t)\rangle$ can be obtained by employing the following definition [67]:

$$e^{i\varphi_P} = \frac{\langle\psi(0)|\psi(t)\rangle}{|\langle\psi(0)|\psi(t)\rangle|}. \tag{6}$$

Correspondingly, the changing rate of phase of the quantum system can be calculated using Eq. (6).

To illustrate the novel bound given by Eq. (5), we apply it to some nontrivial cases.

Mandelstam-Tamm bound.—We first consider case of the MT bound derived from the bound of Eq. (5). In this case, the evolution speed of the quantum system from state $|\phi(t)\rangle$ to state $|\psi(d+dt)\rangle$ can be described using the geodesic joining their projection of a horizontal geodesic in the base manifold $\mathscr{B}$ [59]. The horizontal geodesic vector $|\bar{\psi}(t)\rangle$ in $\mathscr{P}$ space satisfies the following geodesic equation [59,68]

$$\frac{d^2}{dt^2}|\bar{\psi}(t)\rangle + \left[\frac{\Delta E(t)}{\hbar}\right]^2|\bar{\psi}(t)\rangle = 0. \tag{7}$$

By employing Eqs. (4), (6), and (7), we can obtain the speed of evolution of the state vector $v_{\mathrm{QSL}} = [\Delta E(t)/\hbar]$ [59,68]. The QSL time of Eq. (5) reduces to the MT-type bound $\tau_{\mathrm{QSL}} = [\hbar/\overline{\Delta E(\tau)}]\mathcal{L}[\psi(0),\psi(\tau)]$, with $\overline{\Delta E(\tau)} = \frac{1}{\tau}\int_0^\tau \Delta E(t)dt$. In particular, when the quantum system evolves from a initial state to its orthogonal final state for the case of time-independent Hamiltonian: $\overline{\Delta E(\tau)} = \Delta E$, $\mathcal{L}[\psi(0),\psi(\tau)] = \pi/2$. Then, we have the MT bound

$$\tau_{\mathrm{QSL}} = \frac{\pi\hbar}{2\Delta E}. \tag{8}$$

From the above derivation, it is clear that the MT bound is obtained from the evolution of the quantum system on the base manifold $\mathscr{B}$, namely, the MT bound describes the evolution speed bound of the quantum system by employing the state vector in the projection space $\mathscr{P}$. This limit can be quantified using the horizontal geodesic joining the initial and final state vectors.

Margolus-Levitin bound and its generalization.—After including the influence of the geometric phase [49,66] in Margolus and Levitin's technological route [16], we derive the generalized ML bound [69]

$$\tau_{\mathrm{QSL}} = \frac{\pi\hbar}{2\langle E\rangle} + \frac{\hbar}{\langle E\rangle}[\varphi_g(t) - \varphi_g(0)]. \tag{9}$$

Equation (9) shows that contributions of geometric phase exist in the ML bound. Clearly, if the geometric phase remains constant $[\varphi_g(t) - \varphi_g(0) = 0]$ during the evolution of the quantum system, the generalized ML bound of Eq. (9) reduces to the ML bound $\tau_{\mathrm{QSL}} = \pi\hbar/2\langle E\rangle$. This means that the ML bound is obtained under the condition that the geometric phase remains constant [namely, $(d/dt)\varphi_g(t) = 0$] while the quantum system evolves. A typical case is where the state vector evolves along the geodesic that meets $\varphi_g[\mathcal{C}] = 0$ [65].

So, based on this setting, we find that

$$\left|\frac{d}{dt}\varphi_P\right| = \frac{1}{\hbar}\langle\psi(t)|H(t)|\psi(t)\rangle \equiv \frac{\langle E(t)\rangle}{\hbar}, \tag{10}$$

because the total phase is equal to the sum of the dynamical phase and the geometric phase [48,49,65].

Substituting Eq. (10) into Eq. (5) for the time-independent Hamiltonian H, and recalling that $|\psi(0)\rangle$ and $|\psi(\tau)\rangle$ are orthogonal, we obtain

$$\tau_{\mathrm{QSL}} = \frac{\pi\hbar}{2\langle E\rangle}, \tag{11}$$

which is the ML bound.

The above derivation of the generalized ML bound illustrates the difficulty of generalizing the ML bound using only the requirement of inequalities. We can derive the generalized ML bound after including the evolution condition under a constant geometric phase of the quantum system.

The other type bound.—When the quantum system evolves parallel along the curve $\mathscr{C}$ with the projection $\mathcal{C}\in\mathscr{B}$, we have the dynamical phase $\varphi_{\mathrm{dyn}}[\mathscr{C}] = 0$ [65] along the lift curve $\mathscr{C}$ of $\mathcal{C}$, and we have $d\varphi_P(t)/dt = d\varphi_g(t)/dt$ [49,65].

In this case, Eq. (5) reduces to the previous result [27] as follows:

$$\tau_{\mathrm{QSL}} = \frac{1}{\frac{1}{\tau}\int_0^\tau |\dot{\varphi}_g|dt}\mathcal{L}[\psi(0),\psi(\tau)]. \tag{12}$$

In Fig. 1, we give the sketch of the above three evolution modes of the quantum system in Hilbert space $\mathscr{H}$ with its projection in base manifold $\mathscr{B}$. The green curve $\mathscr{C}_1$ denotes

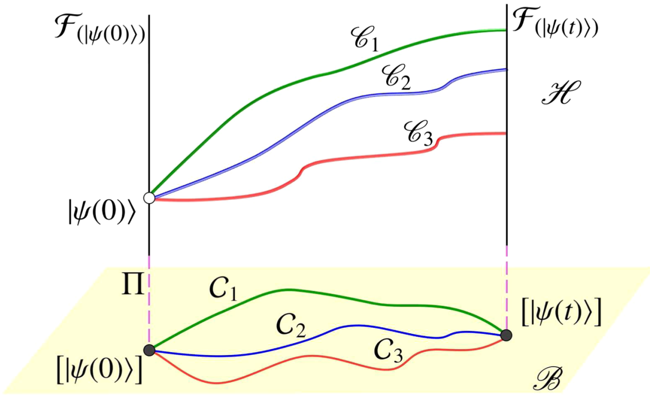

FIG. 1. Sketch of quantum system evolution in the Hilbert space $\mathscr{H}$ with the projection of its evolution paths in base manifold $\mathscr{B}$.

the evolution of the quantum system governed by the Schrödinger equation, and it accumulates the total phase φ_P. The blue curve $\mathscr{C}_2$ denotes the geodesic evolution of quantum system which accumulates the dynamical phase $\varphi_{\rm dyn}$. The red curve $\mathscr{C}_3$ denotes the parallel evolution of quantum system which accumulates the goemetric phase φ_g. Their QSL bounds correspond to Eqs. (8), (11), and (12), respectively.

Non-Hermitian Hamiltonians.—The novel unified bound of QSL given by Eq. (5) is not limited to the Hermitian quantum mechanics, and can be applied to non-Hermitian quantum systems. For non-Hermitian quantum mechanics, we can introduce the complete biorthonormal set of basis vectors, in which the pair of topological vector spaces over complex number field are in duality: we denote the topological vector space as $\mathscr{V}$ and its dual space as $\mathscr{V}^*$ [63,70]. By employing the dual topological vector spaces, the phases of the non-Hermitian quantum system have been investigated [63,71,72]. Based on investigations of the phase of a non-Hermitian quantum system, the QSL of Eq. (5) can be applied in a non-Hermitian quantum system.

Example 1.—We first consider a typical case of the ratio $\alpha = (\Delta E/\langle E\rangle) = 1$. It is shown [69] that our novel unified bound of Eq. (5) presents the same conclusion of Theorem 1 in Ref. [47]: the equivalent MT and ML bounds can be attained through a specific setting of the initial states.

Example 2.—Other interesting cases have been considered for $\alpha = (\Delta E/\langle E\rangle) \neq 1$. In Ref. [47], Levitin and Toffoli investigated the upper bounds of the QSL time for a three-level system with a time-independent Hamiltonian H, where the equation of eigenenergy is $H|E_n\rangle = E_n|E_n\rangle$.

(1) Case of $\alpha < 1$. The following initial state $|\psi(0)\rangle = \sum_n c_n|E_n\rangle$ was considered in Ref. [47], with $|c_n|^2 = p_n$ $(n = 0, 1, 2)$. Under the condition that $0 < p_0 \ll 1$, p_n was assumed as [47] $p_0 = (\delta/2) + \mathcal{O}(\delta^2)$, $p_1 = \frac{1}{2} - (\delta/4)[1 + \cos(\omega_1 t)] + \mathcal{O}(\delta^2)$, and $p_2 = \frac{1}{2} - (\delta/4)[1 - \cos(\omega_1 t)] + \mathcal{O}(\delta^2)$, with $\delta \ll 1$.

The time-evolving vectors $|\psi(t)\rangle$ can, after including the influence of the geometric phase, be written as follows (under the assumption that the ground energy of the system is zero)

$$|\psi(t)\rangle = \sum_n c_n e^{-i\omega_n t} e^{i\varphi_g^{(n)}(t)}|E_n\rangle, \tag{13}$$

where $\omega_n = E_n/\hbar$.

The corresponding horizontal state $|\bar{\psi}(t)\rangle$ of $|\psi(t)\rangle$ in Eq. (13) is

$$|\bar{\psi}(t)\rangle = e^{\frac{i}{\hbar}\int_0^t \langle H(t')\rangle dt'}|\psi(t)\rangle = \sum_n c_n e^{i[\varphi_g^{(n)}(t) - \omega_n t + \frac{1}{\hbar}\int_0^t \langle H(t')\rangle dt']}|E_n\rangle, \tag{14}$$

where $\langle H(t)\rangle \equiv \langle E\rangle = \langle\psi(t)|H(t)|\psi(t)\rangle = p_1E_1 + p_2E_2$. Then, using Eqs. (14) and (6), the changing rate of phase $|d\varphi_p/dt|$ is obtained as [69] $|\dot{\varphi}_P| = (\Delta E/\hbar)(1 - \delta\Upsilon_1) + \mathcal{O}(\delta^2)$, where $\Upsilon_1 = \{[\omega_1 + \omega_2 + (\omega_1 - \omega_2)\cos(\omega_1 t)]/[2(\omega_2 - \omega_1)]\} + [(\omega_1\omega_2)/(\omega_2 - \omega_1)^2]$.

Applying the condition that the finial state is orthogonal to the initial state $\mathcal{L} = \pi/2$, and using Eq. (5), we have $\tau_{\rm QSL} = [\pi\hbar/2\Delta E][1 + (1/\tau)\int_0^\tau p_0\Upsilon_1 dt]$, with $\tau = \pi/4(\omega_2 - \omega_1)$.

Thus, choosing

$$p_0 < \varepsilon\tau\left(\int_0^\tau \Upsilon_1 dt\right)^{-1}, \tag{15}$$

we obtain, after considering Eq. (8),

$$\frac{\pi\hbar}{2\Delta E} < \tau_{\rm QSL} \leq \frac{\pi\hbar}{2\Delta E}(1 + \varepsilon). \tag{16}$$

This is the same expression as in Ref. [47].

(2) Case of $\alpha > 1$. In this case, the initial state is assumed to be as follows [47] $|\psi(0)\rangle = c_0|0\rangle + c_1|E_1\rangle + c_{2k+1}|E_{2k+1}\rangle$, where $k = 1, 2, \ldots$; $E_{2k+1} = (2k+1)E_1$. Let $p_0 = \frac{1}{2}$, $p_1 = \frac{1}{2}[1 - (\beta/k^2)]$, $p_{2k+1} = (\beta/2k^2)$, where $p_{2k+1} = (\beta/2k^2) \ll 1$, $p_\nu = |c_\nu|^2$ $(\nu = 0, 1, 2k+1)$. Correspondingly, the time-dependent states can be written as $|\psi(t)\rangle = c_0|0\rangle + e^{-i\omega_1 t}c_1|E_1\rangle + e^{-i\omega_{2k+1}t}c_{2k+1}|E_{2k+1}\rangle$, where $\omega_1 = (E_1/\hbar)$ and $\omega_{2k+1} = (E_{2k+1}/\hbar)$.

Then, using Eq. (6) and $|\psi(t)\rangle$, we can obtain [69] $|\dot{\varphi}_P| = (\langle E\rangle / \hbar)\{1 - (2\beta / k)[1 - \cos(2k\omega_1 t) + [(\sin(2k\omega_1 t) \times \sin(\omega_1 t))/(1 + \cos(\omega_1 t))]]\}$.

Applying the condition that the finial state is orthogonal to the initial state $\mathcal{L} = \pi/2$, and using Eq. (5), we have

$$\tau_{\rm QSL} = \frac{\pi\hbar}{2\langle E\rangle}\left(1 + \frac{1}{\tau}\int_0^\tau \frac{\beta}{k}\Upsilon_2 dt\right), \tag{17}$$

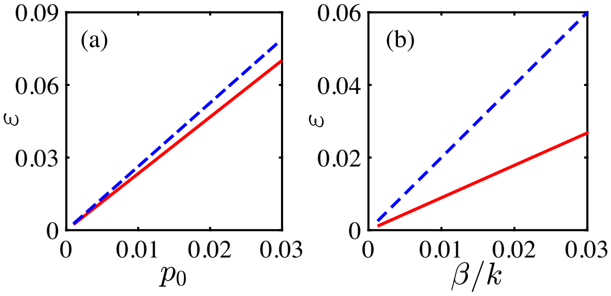

FIG. 2. ε as a function p_0 [panel (a)] and β/k [panel (b)] in the initial state. The solid red lines denote our results from Eq. (15) [panel (a)] and Eq. (18) [panel (b)]. The dashed blue lines denote the results from Eq. (25) and Eq. (31) of Ref. [47]. The other parameters for panel (a) are $E_1 = 18$, $E_2 = 40$ and those for panel (b) are $k = 40$, $E_1 = 18$, $\beta = (\alpha - 1)/4$.

where $\Upsilon_2 = 2\{1 - \cos(2k\omega_1 t) + [(\sin(2k\omega_1 t)\sin(\omega_1 t))/(1 + \cos(\omega_1 t))]\}$, and $\tau = \pi/\omega_1$.

Finally, choosing

$$\frac{\beta}{k} < \varepsilon\tau\left(\int_0^\tau \Upsilon_2 dt\right)^{-1}, \tag{18}$$

and combining Eqs. (11), (17), and (18), we obtain

$$\frac{\pi\hbar}{2\langle E\rangle} < \tau_{\mathrm{QSL}} \le \frac{\pi\hbar}{2\langle E\rangle}(1+\varepsilon). \tag{19}$$

This is the same expression as in Ref. [47].

In Fig. 2 we show the numerical results for ε as a function of p_0 [panel (a)] and β/k [panel (b)]. The solid red lines denote our results with $\varepsilon > (1/\tau)\int_0^\tau p_0\Upsilon_1 dt$ in Eq. (15), and with $\varepsilon > (1/\tau)\int_0^\tau(\beta/k)\Upsilon_2 dt$ in Eq. (18), which correspond to MT and ML-type bounds, respectively. The dashed blue lines denote the results of Ref. [47] with $\varepsilon > \frac{p_0}{2}\{(1/\alpha^2) - 1 - (4/\pi)\sin\frac{\pi}{2}[(1/\alpha) - 1]\}$ [panel (a)], and with $\varepsilon > 2\beta/k$ [panel (b)], respectively. As shown in the figure, the values of ε in Eqs. (15) and (18) are smaller than the values of ε given in Ref. [47]. This means that our results have tighter upper bounds than the results of Ref. [47] since ε sets the upper bound of τ_{QSL} in Eqs. (16) and (19).

Conclusions.—In this Letter, a novel unified expression of QSL has been derived for Hermitian and non-Hermitian quantum systems. The QSL bound is quantified by the changing rate of phase of the quantum system. The QSL bound is of the feasibility since the total phase of the quantum system consist of the geometric phase and the dynamical phase, which is explicitly linked to the initial state and the properties of the Hamiltonian. In addition, many computational methodologies and experimental measurements of phase for the quantum system are developed [49,64]. This shows that the QSL bound is theoretically and experimentally attainable.

As the bound can be obtained via the changing rate of phase, it provides us the potential possibility: how can we evolve faster? The evolution of a QSL along the curve $\mathscr{C}$ by $|\psi(t)\rangle$ could tell us if we can evolve faster along another curve that has the same speed, or confirm that we are already doing the best we can. In particular, in the community of quantum simulators and communications, the novel unified QSL bound can be applied to investigate the minimum time of operation of quantum gates that operate on qubits. Additionally, the bound could be employed as the cost function for optimizing the adiabatic quantum computations.

The tightness of QSL bounds plays an important role, as a tight QSL bound can be employed to estimate the fastest possible evolution speed. The QSL bound of Eq. (5) presents the bound of the quantum system evolving along the cure $\mathscr{C}$ by $|\psi(t)\rangle$. As shown above, it can be reduced to the celebrated MT and ML bounds and other previous bound based on the evolution modes of the quantum system. Our numerical results show that the QSL bound of Eq. (5) also gives us the tighter upper bounds when compared to previous results.

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