Strange Metals as Ersatz Fermi Liquids

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A long-standing mystery of fundamental importance in correlated electron physics is to understand strange non-Fermi liquid metals that are seen in diverse quantum materials. A striking experimental feature of these metals is a resistivity that is linear in temperature (T). In this Letter we ask what it takes to obtain such non-Fermi liquid physics down to zero temperature in a translation invariant metal. If in addition the full frequency (ω) dependent conductivity satisfies ω/T scaling, we argue that the T-linear resistivity must come from the intrinsic physics of the low energy fixed point. Combining with earlier arguments that compressible translation invariant metals are "ersatz Fermi liquids" with an infinite number of emergent conserved quantities, we obtain powerful and practical conclusions. We show that there is necessarily a diverging susceptibility for an operator that is odd under inversion and time reversal symmetries, and has zero crystal momentum. We discuss a few other experimental consequences of our arguments, as well as potential loopholes, which necessarily imply other exotic phenomena.

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A diverse variety of "strange" metals are seen [1–5] to not fit the basic predictions of Fermi liquid theory. Examples include cuprate high temperature superconductors, some heavy electron materials tuned to a quantum critical points, and a growing number of other correlated metals (see, e.g., Refs. [6–9]) The non-Fermi liquid physics manifests itself through unconventional power laws that go down to energy scales much lower than any microscopic scale. A striking example is a resistivity that increases linearly with temperature over a wide range that extends to very low temperature. There is currently very little understanding of this linear resistivity and other properties in most experimental systems. Understanding these strange metals is widely regarded as one of the biggest challenges in modern physics.

Here we present a number of general theoretical observations that provide strong restrictions on the dynamics of a class of *clean* strange metals. We expect that this class includes both the cuprate strange metal as well as non-Fermi liquid heavy fermion quantum critical metals. Remarkably we show, under some very general conditions discussed below, that obtaining a linear resistivity down to $T = 0$ in a clean metal requires the divergent susceptibility of an observable that is odd under inversion and time reversal, transforms as a vector under lattice rotations, and has zero crystal momentum. These are the same symmetries as those of the loop current order parameter [10] discussed in the cuprate materials. Thus our discussion of strange metal transport provides a very general reason for a diverging loop current susceptibility in the strange metal which may connect to the various reports and controversies (for a sampling of some representative papers, see Refs. [11–22]) surrounding such order in the proximate pseudogap metal.

We consider a putative non-Fermi liquid metal with the following assumed properties:

1. *Clean*. The system microscopically has U(1) charge conservation symmetry and lattice translation symmetry (no disorder).

2. *Conductivity scaling*. At low temperatures and frequencies, the conductivity approaches the universal scaling form

$$\sigma(\omega, T) = T^{-1}\Sigma(\omega/T) \tag{1}$$

for some function Σ, such that $\Sigma(0)$ is a nonzero finite number. In particular, the DC resistivity is proportional to T.

3. *Compressible*. The charge ν per unit cell can be continuously tuned as a function of some microscopic parameters without affecting the above properties, and is not pinned to any particular rational value.

These assumptions are strongly motivated by the observed non-Fermi liquid physics in the cuprates and at heavy electron quantum critical points. We could perhaps refer to these assumptions as our "Central Dogmas" [33,34]. So let us briefly discuss the experimental evidence for these assumptions.

We begin with the first assumption, whose nontrivial content is that the observed behavior is a property of a clean lattice system. Real materials of course have some level of disorder that breaks lattice translation invariance. Our assumption then is that to understand the essence of the strange metal physics including the linear-T resistivity, the disorder is unimportant. Support for this assumption comes from studies [35] on cuprates that are artificially damaged by electron irradiation, which provides a gentle way of tuning the disorder strength. It is seen that such irradiation increases the residual resistivity at zero temperature but does not change the slope of the linear resistivity. Moreover, for the cleanest samples the residual resistivity can be made very small, and generally is much smaller than the total resistivity in most of the temperature range where linear resistivity is observed. This suggests that the residual resistivity is related to disorder while the physics of linear resistivity is not affected by such disorder, and that there is a hypothetical perfectly clean limit in which the residual resistivity goes to zero while the linear resistivity remains. For heavy fermion quantum critical metals, some of them like $YbRh_2Si_2$ are stoichiometric compounds and it is perhaps not unreasonable that the basic non-Fermi liquid physics is not determined by disorder effects.

Next we consider the second assumption, of conductivity scaling. Linear dc resistivity down to ultra-low temperatures is of course seen in many non-Fermi liquid metals. We note, however, that our assumption implies the absence of a residual zero temperature resistivity; as mentioned above, we expect that in the clean limit, the residual resistivity would indeed go to zero. The ω/T scaling of the frequency dependent conductivity has been directly demonstrated recently [36] in $YbRh_2Si_2$. Evidence for such scaling in the cuprate strange metal regime has long been reported [37], at least up to $\hbar\omega$ slightly bigger [38] than $k_B T$.

Finally the third assumption—that the metal is compressible—is widely made in the literature though it has not been scrutinized in detail experimentally. In the cuprates the hypothesized quantum critical doping associated with the strange metal occurs at slightly different values in different materials. This is consistent with assuming that the critical doping can be continuously tuned by varying microscopic parameters. It may be possible to demonstrate this directly by studying the change of critical doping with pressure in a single cuprate material.

We will obtain some striking theoretical constraints on metals with these assumed properties, without actually constructing a specific model of such a metal. Assumptions 1 and 3 are also shared by conventional Fermi liquids in clean systems; however, these manifestly do not satisfy Assumption 2. For example, the dc conductivity of a clean Fermi liquid scales like $\sigma(0,T) \propto T^{-2}$ (or faster if umklapp is not effective). Our discussion builds on the results of Ref. [39] which focused on the kinematics of compressible translation invariant quantum phases and phase transitions. A key result is that any such metallic phase has a very large emergent symmetry and associated conservation laws. Non-Fermi liquids with such an emergent symmetry were dubbed "ersatz Fermi liquids." Here we focus on the dynamics of such ersatz Fermi liquids.

Strange metal transport is "intrinsic".—An important distinction to make is between "intrinsic" and "extrinsic" resistivity. It is helpful to use the language of the renormalization group (RG). Quite generally the low-energy physics of the system is described by some RG fixed point. The resistivity is intrinsic if this RG fixed point theory itself has nonzero dc resistivity at nonzero temperature. By contrast, the resistivity is extrinsic if the dc resistivity of the RG fixed point theory is zero (even at nonzero temperature); then nonzero resistivity must arise entirely from RG-irrelevant couplings.

In a conventional clean Fermi liquid, the resistivity is extrinsic. There the only source of resistivity is umklapp scattering which is an irrelevant perturbation to the Fermi liquid fixed point [40]. By contrast, for systems satisfying Assumption 2, the resistivity must be intrinsic. To see this, note that the conductivity of the system as a function of frequency, $\sigma(\omega, T)$, in general depends on the values of irrelevant couplings. However, the only way for the asymptotic behavior at small ω and T to be described by Eq. (1), is if the right-hand side of Eq. (1) represents the conductivity *of the fixed point theory* with irrelevant terms set to zero; we give a careful proof of this statement in the Supplemental Material [23]. Since Assumption 2 then states that $\Sigma(0) < \infty$, it follows that the dc conductivity of the fixed-point theory is not infinite. We remark that at first glance Eq. (1) as a result for an RG fixed-point theory might seem surprising from the point of view of dimensional analysis. However, recall that the Fermi liquid fixed point also satisfies Eq. (1), albeit with the scaling function Σ being a delta function. The point is that in the RG for the Fermi liquid fixed point, there is a length scale k_F (the Fermi wave vector) that does not scale in the RG flow.

The emergent symmetries of a strange metal.—Now we turn to examining the consequences of the assumptions that the metal is clean, and is compressible (Assumptions 1 and 3). We first recall the result of Ref. [39]. For any system satisfying Assumptions 1 and 3, the group of emergent internal symmetries in the IR fixed point theory *cannot* be a compact finite-dimensional Lie group. What then could the emergent internal symmetry group of a strange metal be? A hint is provided by ordinary Fermi liquids which, since they satisfy Assumptions 1 and 3 must indeed obey the constraints of Ref. [39].

A Fermi liquid manages to have a symmetry group that is not a compact finite-dimensional Lie group because that the charge at *each* point on the Fermi surface is separately conserved. Specifically (in two dimensions, say), any operator of the form

$$\int f(\theta)\hat{n}(\theta)d\theta \tag{2}$$

is conserved, for any smooth function $f(\theta)$, where θ is some coordinate parametrizing the Fermi surface, and where $\hat{n}(\theta)$ is the linear charge density operator with respect to θ; thus the total charge of the system is $\hat{Q} = \int \hat{n}(\theta) d\theta$, while the momentum can be expressed as

$$\hat{\mathbf{P}} = \int \mathbf{k}(\theta) \hat{n}(\theta) d\theta, \tag{3}$$

where $\mathbf{k}(\theta)$ is the momentum of point θ on the Fermi surface. Therefore, the emergent internal symmetry group of the Fermi liquid is an *infinite*-dimensional continuous group. As explained in Ref. [39] this emergent symmetry is anomalous. Some (though not all) universal properties of a Fermi liquid can be understood directly in terms of its emergent symmetry and the associated anomaly.

It is interesting to postulate that the emergent internal symmetry and anomaly of the strange metal fixed point is the same as a Fermi liquid, despite having no quasiparticles. Reference [39] introduced the term *ersatz Fermi liquid* to refer to such a system; now we will examine the very striking consequences of this postulate. In the Supplemental Material [23], we discuss other possibilities.

Charge transport in ersatz Fermi liquids.—There is a tension between the strange metal being an ersatz Fermi liquid and having nonzero resistivity. Any conserved quantity risks leading to dissipationless current flow if it has nonzero overlap with the electrical current, since the conservation law then prevents the current from fully relaxing.

For simplicity, let us first consider the case of an ersatz Fermi liquid in two spatial dimensions with continuous rotational symmetry. In that case, the only conserved quantities that can overlap with the current are n_1 and $n_{-1} = n_1^\dagger$, where we defined the Fourier components of the $\hat{n}(\theta)$'s: $\hat{n}_l = (1/2\pi) \int_0^{2\pi} e^{-il\theta} \hat{n}(\theta) d\theta$. These are closely related to the momentum, since we have $\mathbf{k}(\theta) = k_F(\cos\theta, \sin\theta)$ which implies from Eq. (3) that $P_x = \pi(\hat{n}_1 + \hat{n}_{-1})$ and $P_y = \pi(\hat{n}_1 - \hat{n}_{-1})/i$. Then [41–43] (see also the Supplemental Material [23] for an easy argument) the real part of the frequency-dependent conductivity is given by

$$\sigma(\omega) = \frac{\pi \mathcal{Q}^2}{\mathcal{M}} \delta(\omega) + (\text{nonsingular part}), \tag{4}$$

where $\mathcal{Q}$ is the charge density, and

$$\mathcal{M} \coloneqq \frac{1}{V} \chi_{P_x P_x} \coloneqq \frac{1}{V} \left(\frac{\partial}{\partial v} \right) \langle P_x \rangle_{H - vP_x} |_{v=0}, \tag{5}$$

can be interpreted as the "mass density," where $\langle \cdot \rangle_{H - vP_x}$ denotes a thermal expectation value with the Hamiltonian H replaced by $H - vP_x$. The delta function in Eq. (4) leads to infinite dc conductivity (unless its coefficient is zero). This is an example of the "momentum bottleneck" for current relaxation.

Is there any way to suppress the delta function in Eq. (4) in order to obtain finite dc conductivity? Strange metals are supposed to exist at finite charge density, so $\mathcal{Q} \neq 0$. Therefore, the only way to suppress the delta function is if $\mathcal{M}$ *is infinite*. Going beyond the assumption of continuous rotational symmetry, and for any spatial dimension, we show in the Supplemental Material [23] that it remains the case that the only way to suppress the delta function in the conductivity at zero frequency for an ersatz Fermi liquid, assuming generic charge density (Assumption 3), is for a certain susceptibility of the $\hat{n}(\theta)$'s to diverge. Therefore, we have reached one of the principal conclusions of our Letter: Assumptions 1–3, if satisfied by way of the system being an ersatz Fermi liquid, imply the divergence of a susceptibility of the emergent conserved quantities.

Note that, as defined by Eq. (5), $\mathcal{M}$ is the suceptibility of a quantity $\hat{P}_x$ that is odd under time-reversal and inversion symmetry. In fact, in the Supplemental Material [23] we show that even without continuous rotation symmetry, and in any spatial dimension, the operator for which the divergent susceptibility suppresses the delta function in Eq. (4) must share the same symmetry properties as the electrical current operator. Thus, it is odd under time-reversal symmetry and inversion symmetry, while under lattice rotation symmetry, it transforms as a vector. This suggests that the divergent susceptibility could potentially be a signature of a continuous phase transition into a phase that (among other features) spontaneously breaks inversion and time-reversal symmetry, a point we return to later.

Finally, note that at any $T > 0$ the susceptibility will probably be finite, while the emergent conservation laws will be violated by irrelevant operators. The role of these effects on charge transport at $T > 0$ is discussed in the Supplemental Material [23].

We next describe a number of experimental tests of the idea that the strange metal is an ersatz Fermi liquid.

Experimental test: Crossover to off-critical resistivity and scaling.—One signature is the scaling of resistivity in cases where the strange metal occurs at a quantum critical point proximate to a Landau Fermi liquid. Then, away from criticality, $\mathcal{M}$ in Eq. (4) will become finite, thereby reactivating the mechanism of conserved quantities preventing current decay. The conductivity will be dominated at low frequencies and temperatures by the delta function peak in Eq. (4) (which can get broadened with width $\propto T^2$ due to momentum relaxation from irrelevant couplings). But since the weight of this peak precisely goes to zero at the critical point, where the conductivity must instead have a different origin, we should *not* expect the conductivity at low temperatures and frequencies near the critical point to collapse onto a universal scaling curve. By contrast, if one of the loopholes discussed in the Supplemental Material [23] applies, and the strange metal is not an ersatz Fermi liquid, then it is conceivable that such a scaling collapse could occur. This conclusion will hold irrespective of the nature of the detailed crossover from the ersatz Fermi liquid quantum critical fixed point to the ordinary Fermi liquid.

It is sometimes observed that in the proximate Fermi liquid near a strange metal quantum critical point, the resistivity $\rho(T) - \rho(0) = AT^2$ with A diverging upon approaching the critical point [44] while the critical point itself shows linear resistivity. For an ersatz Fermi liquid, the diverging A coefficient *should not* be part of a scaling function with the linear resistivity. Such a scaling has been attempted in heavy electron systems (Ref. [47]); our discussion calls for careful scrutiny of this scaling plot. It would also be very interesting to determine how exactly the ac conductivity crosses over from the "broad" peak of width $\sim T$ in the strange metal [see Eq. (1)] to the much narrower Drude peak in the Fermi liquid.

Experimental test: Inversion or time reversal breaking order.—We argued—on general grounds—that if the strange metal is an ersatz Fermi liquid (or a variant) then it is necessary that a susceptibility of the emergent conserved $\hat{n}(\theta)$ must diverge in order to obtain the required resistivity. For this to work, the susceptibility must diverge in channels that have overlap with the current operator. Thus the low energy theory has observables O which are time reversal or inversion odd, live at zero crystal momentum, transforms as a vector under lattice rotations, and whose susceptibility diverges. This is a firm prediction of the ersatz Fermi liquid hypothesis for strange metals that could potentially be tested.

In the cuprates, there have been many reports (and controversies) of ordering that spontaneously breaks precisely the symmetries of such observables O (see, e.g., Refs. [11–17,19–22]). These have been usually interpreted microscopically in terms of loop current ordering. Remarkably, our considerations, which come from a completely different line of thought, demand the existence of critically diverging fluctuations of such order in the strange metal regime. This may be consistent with the emergence of static order in the pseudogap ground state. However we caution that the pseudogap ground state is not *just* an ordinary Fermi liquid metal in the presence of such order. Rather on top of whatever transformation underlies the evolution between the overdoped and underdoped metallic ground states (e.g., a Fermi surface jump), our considerations make it plausible that there is a breaking of time reversal and inversion symmetries. Furthermore the diverging susceptibility of an order parameter at a quantum critical points does not necessarily imply that one of the proximate phases has static order for the corresponding observable, as is known from a number of theoretical examples.

Experimental test: Quantum oscillations.—Another experimental test that one could consider in principle, although in practice it may be difficult to realize, is based on quantum oscillations. Consider any system in two spatial dimensions with lattice translation symmetry and U(1) charge conservation symmetry, and let ν be the average charge per unit cell. Then we say the system exhibits *universal quantum oscillations* if, upon applying a weak magnetic field B, the properties of the system (for example, resistivity) are periodic in $1/B$ with period

$$\Delta(1/B) = \frac{e}{2\pi\hbar}\frac{1}{\rho}, \tag{6}$$

where ρ is some number (the "effective charge density") such that $\rho V_{\mathrm{unit}} = \nu[\mathrm{mod}\, 1]$, where V_{unit} is the volume of the unit cell. (For spinful systems, there is an additional factor of 2 in this relation).

Quantum oscillations were originally derived for Fermi liquids, based on a semiclassical quantization argument for the orbits of quasiparticles. However, in fact one expects quantum oscillations for any system where the discrete microscopic translation symmetry gets extended to an emergent continuous symmetry [39,48], as happens, for example, for an ersatz Fermi liquid in which the Fermi surface does not wrap nontrivially around the Brillouin zone. We expect, moreover, that the converse also holds, so that universal quantum oscillations can be considered an experimental signature of the microscopic translation symmetry getting extended to an emergent continuous symmetry. When this occurs, then by a similar argument to the ersatz Fermi liquid case discussed above but, we emphasize, *without* needing to assume the system is an ersatz Fermi liquid, then the only way to get intrinsic resistivity would be to have a diverging "mass density," corresponding to a diverging susceptibility.

Unfortunately, for hole-doped cuprate materials that are clean enough that one can expect to observe quantum oscillations, the critical magnetic field required to suppress superconductivity down to zero temperature at critical doping is larger than is accessible with current technology; for example, it is estimated that the critical field is about 150 T in YBCO [49]. Meanwhile, at heavy fermion critical points, a magnetic field tunes the system out of criticality, again complicating a direct determination of the possibility of quantum oscillations associated with the quantum critical state. Therefore, it has not been possible to verify whether the zero-temperature quantum critical point that is believed to control a strange metal exhibits quantum oscillations. However, hopefully this might be possible in the future.

We remark that *electron*-doped cuprates exhibit T-linear resistivity in a range of dopings [50,51] (see Refs. [52,53] for reviews); quantum oscillations have been reported for some of these materials [54] but not in the same doping range as the T-linear resistivity. This seems worthy of further study.

Loopholes.—The results presented here have very important theoretical and experimental implications for strange metals, as we have described. Therefore, one should think carefully about whether there might be any potential loopholes. Of course, one possibility would be that our Assumptions 1–3 are not satisfied, although we have described the experimental basis to believe that they are. Beyond that, all the loopholes that we can think of would be extremely exotic. One possibility would be that the emergent internal symmetry group is not a compact finite-dimensional Lie group. Another would be a scenario that we call "ultralocal quantum criticality" (which is a stronger condition than what is often referred to as "local quantum criticality"). We discuss these loopholes further in the Supplemental Material [23].

Conclusion.—In this work, we have unveiled a powerful new approach to understanding strange metals. Rather than trying to find theoretical models that can reproduce the phenomenology, which has so far eluded the community, we are able to make considerable progress through general structural arguments based only on minimal assumptions. Through such an approach, we have made strong model-independent predictions about the nature of strange metals. We expect that our results will narrow down the search for a theoretical understanding of these mysterious and fascinating phases of matter.

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