Lieb-Robinson Bound and Almost-Linear Light Cone in Interacting Boson Systems

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In this work, we investigate how quickly local perturbations propagate in interacting boson systems with Bose-Hubbard–type Hamiltonians. In general, these systems have unbounded local energies, and arbitrarily fast information propagation may occur. We focus on a specific but experimentally natural situation in which the number of bosons at any one site in the unperturbed initial state is approximately limited. We rigorously prove the existence of an almost-linear information-propagation light cone, thus establishing a Lieb-Robinson bound: the wave front grows at most as $t\log^2(t)$. We prove the clustering theorem for gapped ground states and study the time complexity of classically simulating one-dimensional quench dynamics, a topic of great practical interest.

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Introduction.—In nonrelativistic quantum many-body systems, the speed limit of information propagation is characterized by the Lieb-Robinson bound [1–3], an effective light cone outside which the amount of transferred information rapidly decays with distance. In standard spin models such as the transverse Ising model, the light cone is linear over time and characterized by the Lieb-Robinson velocity, which depends only on the system details. As a fundamental restriction applied to generic many-body quantum systems, the Lieb-Robinson bound has been utilized to establish the clustering theorem on bipartite correlations in ground states [4–6] and efficient classical and quantum algorithms to simulate quantum many-body dynamics [7–9]. It has featured in many fields of quantum many-body physics including condensed matter theory [10–16], statistical mechanics [17–23], high-energy physics [24–30], and quantum information [31–35].

The Lieb-Robinson bound and the existence of a linear light cone are well understood under the following two conditions [3,5,6,36,37]: (i) the interaction is short-range, and (ii) the Hamiltonian is locally bounded. If either of these conditions is broken, as often happens in real-world quantum systems, the shape of the linear light cone becomes quite complicated. When there are long-range interactions, breaking the first condition, a comprehensive characterization of the shape of the light cone has been achieved [28,29,38–42]. However, it remains challenging to clarify the Lieb-Robinson bound when the second condition breaks down.

Quantum boson systems are representative examples of the breakdown of this second condition with locally unbounded Hamiltonians. The difficulty lies in the fact that the standard approach for the Lieb-Robinson bound necessarily results in a Lieb-Robinson velocity proportional to the norm of the local energy. When N bosons clump at a single location, the on-site energy can be as large as $\mathrm{poly}(N)$, leading to an infinite Lieb-Robinson velocity as $N\to\infty$. Even though it is quite unlikely that many bosons will clump together in realistic experiments, the theoretical possibility of such situations must be taken into account. If harmonic and anharmonic systems [43–47] and spin boson models [48–50] are considered, the Lieb-Robinson bound with the linear light cone has been established. However, we have no hope of unconditionally proving the existence of a Lieb-Robinson bound without restricting the form of Hamiltonians or initial states. (In Ref. [51], Eisert and Gross provided 1D quantum boson systems with nearest-neighbor interactions, inducing an exponential speed of information propagation.)

Recent experiments have focused on interacting bosonic systems of the Bose-Hubbard type [52–66], which typically appear in cold atom setups. Since the earliest experiments on the Lieb-Robinson bound [55,56], there have been many attempts to clarify information propagation in these models rigorously. However, with a few exceptions [67,68], establishing the Lieb-Robinson bound in Bose-Hubbard–type models remains an open problem. A previous rigorous study [67] showed that initially concentrated bosons in the vacuum spread at a finite speed. In Ref. [68], the Lieb-Robinson velocity was qualitatively improved from $\mathcal{O}(N)$ to $\mathcal{O}(\sqrt{N})$ (still infinitely large in the limit of $N\to\infty$), where N is the total number of bosons. On the other hand, numerical calculations and theoretical case studies indicate that a linear light cone should be observed in practical settings such as quench dynamics [69–76]. The most natural condition is to require a finite number of bosons at any one site in the initial state, for example, a Mott state. However, this condition can break down over time, and a large bias in the boson distribution may cause an unexpected acceleration of information propagation [67]. Until now, no theoretical tools have been developed to overcome this obstacle.

In this work, we establish the Lieb-Robinson bound with an almost linear light cone when a local perturbation is added to quantum states that are initially time independent and have low boson density [see the condition (6)]. Our Lieb-Robinson bound characterizes a wave front that propagates as $t\log^2(t)$ with time. As a practical application, we derive the clustering theorem for noncritical ground states by extending the technique in [4–6]. In addition, we extend our theory to analyze the time complexity of computing quantum dynamics by quenching the Hamiltonian parameter, a topic of major research interest [69–93]. We rigorously establish the time complexity of $e^{t\log^3(t)}$ to simulate local quench dynamics for one-dimensional Bose-Hubbard–type Hamiltonians.

Setup and main result.—We consider a quantum system on a finite-dimensional lattice (graph), where bosons interact with each other. An unbounded number of bosons can sit on each of the sites, and the local Hilbert dimension is thus infinitely large. We denote by Λ the set of all sites on the lattice. For an arbitrary partial set $X \subseteq \Lambda$, we denote the cardinality (the number of sites contained in X) by $|X|$. For arbitrary subsets $X, Y \subseteq \Lambda$, we define $d_{X,Y}$ as the shortest path length on the graph that connects X and Y. For a subset $X \subseteq \Lambda$, we define the extended subset $X[r]$ by length r as

$$X[r] \coloneqq \{i \in \Lambda | d_{X,i} \le r\}, \tag{1}$$

where $X[0] = X$ and r is an arbitrary positive number (i.e., $r \in \mathbb{R}^+$).

We define b_i and $b_i^\dagger$ as the annihilation and creation operators of the boson, respectively. We also define $\hat{n}_i \coloneqq b_i^\dagger b_i$ as the number operator of bosons on site i. We consider a Hamiltonian of the form

$$H \coloneqq \sum_{\langle i,j \rangle} J_{i,j}(b_i b_j^\dagger + \text{H.c.}) + \sum_{Z\subset\Lambda:|Z|\le k} v_Z, \tag{2}$$

where $|J_{i,j}| \le \bar{J}$ and $\sum_{\langle i,j \rangle}$ denotes summation over all pairs of adjacent sites $\{i,j\}$ on the lattice. Here, v_Z consists of finite-range boson-boson interactions on subset Z. We now assume that v_Z is given as a function of the number operators $\{\hat{n}_i\}_{i\in Z}$. The simplest example is the Bose-Hubbard model:

$$H = \sum_{\langle i,j \rangle} J(b_i b_j^\dagger + \text{H.c.}) + \frac{U}{2}\sum_{i\in\Lambda}\hat{n}_i(\hat{n}_i - 1) - \mu\sum_{i\in\Lambda}\hat{n}_i,$$

where U and μ are $\mathcal{O}(1)$ constants. For an arbitrary operator O, the time evolution due to another operator A is

$$O(A,t) \coloneqq e^{iAt} O e^{-iAt}. \tag{3}$$

[We abbreviate $O(H,t)$ as $O(t)$ for simplicity.]

Let ρ_0 be a time-independent quantum state, i.e., $[\rho_0, H] = 0$. We consider propagation of a local perturbation to ρ_0 such as $\rho \to O_{i_0}\rho_0 O_{i_0}^\dagger$, where $i_0 \in \Lambda$ and O_{i_0} can take the form of a projection onto site i_0. We are interested in how fast this perturbation propagates. Mathematically, after the time evolution, $\rho(t)$ is given by $O_{i_0}(t)\rho_0 O_{i_0}(t)^\dagger$. Thus, we must estimate the approximation error of

$$O_{i_0}(t)\rho_0 \approx O_{i_0[R]}^{(t)}\rho_0, \tag{4}$$

where $O_{i_0[R]}^{(t)}$ is an appropriate operator supported on subset $i_0[R]$ [see the notation (1)]. Our main result concerns the approximation error for finite R (see Sec. S.II. in the Supplemental Material [94] for the formal expression).

Following Ref. [41], we define the shape of the light cone in the following sense. We say that the Hamiltonian dynamics e^{-iHt} have an effective light cone with velocity $v_{t,\delta}$ if the following inequality holds for an arbitrary error $\delta \in \mathbb{R}$ and t:

$$\|[O_{i_0}(t) - O_{i_0[R]}^{(t)}]\| \le \delta \|O_{i_0}\| \quad \text{for } R \ge v_{t,\delta}|t|. \tag{5}$$

When $v_{t,\delta}$ converges to a finite value for $t \to \infty$ (i.e., $v_{\infty,\delta} = \text{const.}$), we say that the effective light cone is linear. From the definition, the amount of information propagation is smaller than δ outside the region separated by the distance $v_{t,\delta}|t|$.

Main theorem.—Let us assume that the number of boson creations by O_{i_0} is finitely bounded. Then, for an arbitrary time-independent quantum state ρ_0 satisfying the low-boson-density condition

$$\max_{i\in\Lambda} \text{tr}(e^{c_0(\hat{n}_i - \bar{q})}\rho_0) \le 1 \qquad c_0 \le 1, \tag{6}$$

we can approximate $O_{i_0}(t)\rho_0$ by another operator $O_{i_0[R]}^{(t)}$ supported on $i_0[R]$ with the following approximation error:

$$\|(O_{i_0}(t) - O_{i_0[R]}^{(t)})\rho_0\| \le \|O_{i_0}\| \exp\left(c_0\bar{q} - C_1 \frac{R}{t\log(R)} + C_2 \log(R)\right), \tag{7}$$

where $t \ge 1$, and C_1 and C_2 are constants of $\mathcal{O}(1)$ that are independent of $\bar{q}$ and only depend on the details of the system. For a general operator O_{X_0}, we can obtain a similar inequality by slightly changing (7).

Condition (6) ensures that the probability for many bosons to be concentrated on one site is exponentially small in the initial state ρ_0. We notice that the condition can break down as time increases. By applying the inequality

(7) to (5), we obtain $v_{t,\delta} \propto \log^2(t)[\log(1/\delta) + c_0\bar{q}]$. Hence, information propagation is restricted in the region that is separated from i_0 by at most $\mathcal{O}(\bar{q})t\log^2(t)$. Therefore, we can ensure that the acceleration of information propagation observed in Ref. [51] cannot occur in our model, because the speed of information becomes at most polylogarithmically large with time, i.e., $\leq \log^2(t)$.

Clustering theorem.—As an immediate application of the main theorem, we consider the exponential decay of bipartite correlations in gapped ground states, i.e., the clustering theorem. Here, we denote the nondegenerate ground state by $|E_0\rangle$ and the spectral gap by ΔE. We prove an upper bound on the correlation function $\mathrm{Cor}(O_X, O_Y) \coloneqq \langle E_0|O_X O_Y|E_0\rangle - \langle E_0|O_X|E_0\rangle\langle E_0|O_Y|E_0\rangle$, where O_X and O_Y are operators supported on X and Y. For simplicity, we let $\bar{q} = \mathcal{O}(1)$. Then, the following inequality holds if $|E_0\rangle$ satisfies condition (6) (see Sec. S.III. in the Supplemental Material [94]):

$$\mathrm{Cor}(O_X, O_Y) \leq C_3\|O_X\| \cdot \|O_Y\| \exp\left(-\sqrt{\frac{C_3'\Delta E}{\log(R)}}R\right), \quad (8)$$

where C_3, C_3', and C_3'' are $\mathcal{O}(1)$ constants. From the inequality, the bipartite correlations decay beyond $R \approx \tilde{\mathcal{O}}(1/\Delta E)$. This subexponential decay, which is weaker than the exponential decay described in Refs. [4–6], is a consequence of the asymptotic form of $e^{-\mathcal{O}(R/(t\log R))}$ in our Lieb-Robinson bound (7).

Application to quench dynamics.—We next consider the application of our results to quench dynamics, the most popular setup in the study of nonequilibrium quantum systems. Here, a system is initially prepared in a steady state ρ_0 (e.g., the ground state), and then evolves unitarily in time under the sudden change of the Hamiltonian $H \to H'$. We consider the case where the Hamiltonian H' is given by $H' = H + h_{X_0}$, where we assume H' still has the form of Eq. (2). In addition, the interaction h_{X_0} includes only polynomials of finite degree in $\{\hat{n}_i\}_{i\in X_0}$, such as $\hat{n}_i^2$ and $\hat{n}_i^2\hat{n}_j^3$, etc.

Our purpose is to find an appropriate unitary operator $U_{i_0[R]}$ supported on $i_0[R]$ that gives $\rho_0(H', t) \approx U_{i_0[R]}\rho_0 U_{i_0[R]}^\dagger$. We can prove the following theorem (see Sec. S.IX. in the Supplemental Material [94] for details):

Quench theorem.—For initial state ρ_0 with the conditions $[\rho_0, H] = 0$ and (6), we have

$$\begin{aligned}&\|\rho_0(H', t) - U_{i_0[R]}\rho_0 U_{i_0[R]}^\dagger\|_1\\ &\quad \leq \exp\left(c_0\bar{q} - C_1'\frac{(R - r_0)}{t\log(R)} + C_2'\log(R)\right),\end{aligned} \quad (9)$$

where we define r_0 such that $X_0 \in i_0[r_0]$ for an appropriate $i_0 \in \Lambda$, and C_1' and C_2' are constants of $\mathcal{O}(1)$ that are independent of $\bar{q}$ and only depend on the details of the system. Moreover, the computational cost of constructing the unitary operator $U_{i_0[R]}$ is at most $\exp[\mathcal{O}(R^D\log(R))]$.

This theorem immediately gives the following corollary on the time complexity of preparing $U_{i_0[R]}$:

Corollary.—The computational cost of calculating the quench dynamics on 1D chains up to an error ϵ is at most

$$\exp[t\log^3(t) + t\log(1/\epsilon)\log\log^2(1/\epsilon)], \quad (10)$$

where we assume $r_0 = \mathcal{O}(1)$ and $\bar{q} = \mathcal{O}(1)$. When the error ϵ is fixed, we have a time complexity of $e^{t\log^3(t)}$. This is the first rigorous result on the efficiency of the classical simulation of interacting boson systems.

Proof of the main theorem.—For the proof, we connect the Lieb-Robinson bounds for small time evolutions step by step, based on previous analyses of the Lieb-Robinson bound in long-range interacting systems [29,97]. The great merit of this approach is that we have to derive the Lieb-Robinson bound *only for short-time evolution.* We decompose the total time t into m_t pieces and define $\Delta t \coloneqq t/m_t$ with $m_t = \mathcal{O}(t)$. Note that we can make Δt arbitrarily small by making m_t sufficiently large. For a fixed R, we define the subset X_m as follows:

$$X_m \coloneqq i_0[m\Delta r], \qquad \Delta r = \lfloor R/m_t \rfloor,$$

where $X_m = X_0[m\Delta r]$ and $X_{m_t} \subseteq i_0[R]$.

We connect the step-by-step approximations of the short-time evolution to reach the final approximation. Under the assumption of the time invariance of ρ_0 (i.e., $\rho_0(t) = \rho$), we can derive the following inequality [29]:

$$\begin{aligned}&\|[O_{i_0}(m_t\Delta t) - O_{X_{m_t}}^{(m_t)}]\rho_0\|_1\\ &\quad \leq \sum_{m=1}^{m_t}\|[O_{X_{m-1}}^{(m-1)}(\Delta t) - O_{X_m}^{(m)}]\rho_0\|_1,\end{aligned} \quad (11)$$

where $O_{X_0}^{(0)} = O_{X_0}$, and $O_{X_m}^{(m)}$ is recursively defined by approximating $O_{X_m}^{(m)}(\Delta t)$. When ρ_0 depends on the time, a severe modification is required in the inequality (11) (see Sec. IV. B in the Supplemental Material [94]). In order to reduce (11) to the main inequality (7), we need to obtain

$$O_{X_{m-1}}^{(m-1)}(\Delta t)\rho_0 \approx U_{X_m}^{(m)\dagger} O_{X_{m-1}}^{(m-1)} U_{X_m}^{(m)}\rho_0 = O_{X_m}^{(m)}\rho_0, \quad (12)$$

by using an appropriate unitary operator supported on X_m.

Therefore, our primary task is to estimate the approximation error of Eq. (12), which gives the Lieb-Robinson bound for the short time Δt. We can prove that, for a general operator O_X with $X \subseteq i[r]$ ($i \in \Lambda$), there exists a unitary operator $U_{X[\ell]}$ supported on $X[\ell]$ such that

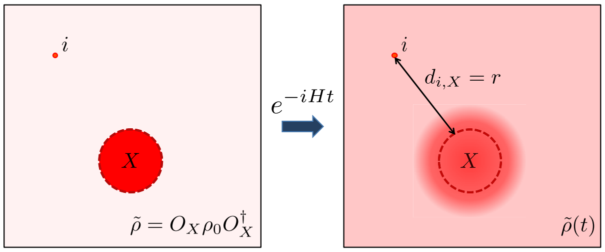

FIG. 1. Boson density after time evolution. In Ref. [67], all bosons were initially concentrated in a region X on the vacuum and were shown to spread beyond it with a finite velocity. However, if there is initially a finite number of bosons outside X, the upper bound of the boson number increases exponentially with t. This spoils the approach of Ref. [67] in general setups for long-time evolution. Our approach only considers the short time $t = \mathcal{O}(1)$, when the exponential increase $e^{\mathcal{O}(t)}$ is still $\mathcal{O}(1)$. We then ensure that the boson number distribution for $\hat{n}_i$ exponentially decays if the site i of interest is sufficiently separated from the region X.

$$\|(O_X(t) - U_{X[\ell]}^\dagger O_X U_{X[\ell]})\rho_0\|_1 \leq \|O_X\| e^{c_0 \bar{q} - \ell/\log(r) + C_0 \log(r)}, \tag{13}$$

for $t \leq \Delta t_0$ (see Subtheorem 1 in the Supplemental Material [94]), where C_0 and Δt_0 are $\mathcal{O}(1)$ constants. We here choose the time width Δt such that $\Delta t \leq \Delta t_0$. By using the inequality (13) with $\ell = \Delta r$ and $t = \Delta t$, we can reduce the inequality (11) to the desired form (7) by choosing C_1 and C_2 appropriately. This completes the proof of the main theorem. ■

Short-time Lieb-Robinson bound.—We have seen that the bosonic Lieb-Robinson bound can be immediately derived if we can prove the inequality (13), which includes all the difficulties in our proof. We will now provide a sketch of the proof; a fuller and more formal presentation can be found in the Supplemental Material [94] (Secs. S.V., S.VI., S.VII. and S.VIII.).

We first consider the boson density after short-time evolution (see Sec. S.VI. in the Supplemental Material [94]). For this purpose, we need to estimate

$$\mathrm{tr}[\hat{n}_i^s \tilde{\rho}(t)], \qquad \tilde{\rho}(t) = e^{-iHt} O_X \rho_0 O_X^\dagger e^{iHt}, \tag{14}$$

with $s \in \mathbb{N}$. This quantity characterizes the influence of the perturbation O_X on the boson density after time evolution. In the state $\tilde{\rho}(0)$, the boson number $\hat{n}_i$ ($i \notin X$) is exponentially suppressed because of condition (6), while the bosons may be highly concentrated in the region X. Time evolution will cause these concentrated bosons to spread outside X (see Fig. 1).

In order to characterize the dynamics of the bosons, we utilize the method in Ref. [67]. We can prove that

$$\frac{\mathrm{tr}[\hat{n}_i^s \tilde{\rho}(t)]}{\|O_X\|^2} \leq c_1' e^{c_0 \bar{q}} |X|^3 (c_1 s |X|)^s e^{-d_{i,X}} + c_1'' e^{c_0 \bar{q}} (c_1 s)^s, \tag{15}$$

where c_1, c_1', and c_1'' depend on the time as $e^{\mathcal{O}(t)}$. The above upper bound induces an exponential increase of the boson density with time; hence we cannot use it for arbitrarily large t. However, the key point of our proof method is that we only need to treat the short-time evolution, where the coefficients c_1, c_1', and c_1'' are $\mathcal{O}(1)$ constants. By using Markov's inequality, we can ensure that the probability distribution of the boson number $\hat{n}_i$ obeys

$$P_{i,\geq z_0}^{(t)} \leq 2 c_1'' e^{c_0 \bar{q}} \|O_X\|^2 \left(\frac{\tilde{c}_1 d_{i,X}}{z_0} \right)^{\tilde{c}_1' d_{i,X}/\log(r)} \tag{16}$$

under the condition $d_{i,X} \gtrsim \log(r)$, where $P_{i,\geq z_0}^{(t)}$ is the probability that z_0 or more bosons are observed at the site i. (Recall that by definition $X \subseteq i[r]$.) Finally, we remark that it is essential to the proof that the Hamiltonian be the form (2); if the Hamiltonian includes interactions such as $\hat{n}_i \hat{n}_j b_i b_j^\dagger$, the inequality (15) may break down even for small t.

In the second technique, we construct an effective Hamiltonian that has bounded local energy in a specific region and approximates the exact dynamics (see Sec. S.VII. in the Supplemental Material [94]). The inequality (16) implies that the boson number $\hat{n}_i$ is strongly suppressed when the site i is sufficiently separated from the region X. Hence, we expect that, in the original Hamiltonian H, the maximum boson number at one site can be truncated during short-time evolution. We first define two regions $L_1 \coloneqq X[\ell_0]$ and $L_2 \coloneqq X[2\ell_0]$, where the length ℓ_0 is appropriately chosen. We then consider the boson truncation in the region $\tilde{L}$ which is defined as (see Fig. 2)

$$\tilde{L} \coloneqq L_2 \backslash L_1. \tag{17}$$

We now define $\bar{\Pi}_{\tilde{L},q}$ as the projection onto the eigenspace such that the boson number $\hat{n}_i$ ($\forall\, i \in \tilde{L}$) is truncated up to q, i.e., $\|\hat{n}_i \bar{\Pi}_{\tilde{L},q}\| \leq q$. We then approximate the time-evolution operator e^{-iHt} by using an effective Hamiltonian $\tilde{H}[\tilde{L}, q]$, defined by

$$\tilde{H}[\tilde{L}, q] \coloneqq \bar{\Pi}_{\tilde{L},q} H \bar{\Pi}_{\tilde{L},q}, \tag{18}$$

with a bounded local energy in the region $\tilde{L}$. In general, the time evolution $O_X(t)$ cannot be approximated by $O_X(\tilde{H}[\tilde{L}, q], t)$ at all, where we have used the notation (3). However, we are only interested in the norm difference between $O_X(t)\rho_0$ and $O_X(\tilde{H}[\tilde{L}, q], t)\rho_0$. We can prove

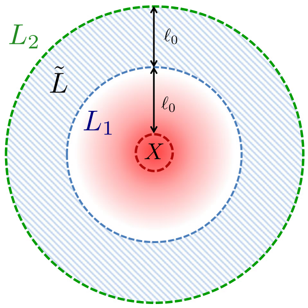

FIG. 2. Schematic picture of the region where the boson number is truncated. In the region L_1 $(= X[\ell_0])$, we cannot restrict the boson number distribution. On the other hand, as long as ℓ_0 is sufficiently large, the boson number outside L_1 can be truncated up to a finite value q. We perform the boson number truncation in the shaded region $\tilde{L} = L_2 \backslash L_1$ with $L_2 = X[2\ell_0]$. By using an effective Hamiltonian $\tilde{H}[\tilde{L}, q]$ as in Eq. (18), we can approximate the exact dynamics by choosing the appropriate q [i.e., $\eta\ell_0$; see the inequality (19)].

$$\|[O_X(t) - O_X(\tilde{H}[\tilde{L}, \eta\ell_0], t)]\rho_0\|_1 \le \frac{\|O_X\|}{2} e^{c_0\bar{q}} e^{-2\ell_0/\log(r)} \tag{19}$$

for $q = \eta\ell_0$ and $\ell_0 \ge C_0 \log^2(r)$, where η and C_0 are $\mathcal{O}(1)$ constants which are independent of $\bar{q}$, and r has been defined by $X \subseteq i[r]$. From this upper bound, we can see that the error exponentially decreases with the number of the boson truncation. Thus, by using the Hamiltonian $\tilde{H}[\tilde{L}, \eta\ell_0]$, the biggest obstacle, namely, the unboundedness of the interaction norms, has been removed, at least in the region $\tilde{L}$. However, outside this region, the norm is still unbounded. We thus need to consider how to derive the Lieb-Robinson bound for $e^{-i\tilde{H}[\tilde{L},\eta\ell_0]t}$ only from the finiteness of the Hamiltonian norm in the region $\tilde{L}$.

Our final task is to approximate the time evolution $O_X(\tilde{H}[\tilde{L}, \eta\ell_0], t)$ by $U_{L_2}^\dagger O_X U_{L_2}$, where U_{L_2} is an appropriate unitary operator supported on the subset L_2 (see Sec. S.VIII. in the Supplemental Material [94]). By a careful calculation based on the standard approach to deriving the Lieb-Robinson bound, we can show that the approximation error obeys

$$\|O_X(\tilde{H}[\tilde{L},\eta\ell_0],t) - U_{L_2}^\dagger O_X U_{L_2}\| \le \frac{\|O_X\|}{2} e^{-2\ell_0/\log(r)}, \tag{20}$$

assuming $t \le \Delta t_0$ with Δt_0 an $\mathcal{O}(1)$ constant. Therefore, under the conditions $\ell_0 \ge C_0 \log^2(r)$ and $t \le \Delta t_0$, we have the inequalities (19) and (20), which together yield the desired inequality (13) since $e^{c_0\bar{q}} \ge 1$.

Conclusion.—In this work, we have established the Lieb-Robinson bound (7) with an almost-linear light cone $R \propto t\log^2(t)$ for arbitrary initial steady states under the condition (6). Our bound leads to the clustering theorem (8) for gapped ground states and the efficient simulation of the quench dynamics as in (10). Our result gives the first rigorous characterization of the light cone of interacting boson systems under experimentally realistic conditions.

Nevertheless, this Lieb-Robinson bound might be further improved. First, the asymptotic form $e^{-R/(t\log R)}$ in (7) could be changed to e^{-R+vt}, which would induce a strictly linear light cone for information propagation. Second, there remains the challenge to clarify the class of quantum states that rigorously satisfies the assumption (6). Third, regarding the time independence of ρ_0, we conjecture that an information wave front of at least polynomial form (i.e., $R \propto t^\zeta$, $\zeta \ge 1$) can be derived when ρ_0 is time dependent. Although our current techniques cannot immediately accommodate these improvements, we hope to develop a better Lieb-Robinson bound for interacting bosons in the future.

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Note added.—For the reader's information, we would like to refer to a subsequent study by Yin and Lucas [98], which proves the linear light cone for interacting boson systems in another specific setup.

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