Inverse Scattering of the Zakharov-Shabat System Solves the Weak Noise Theory of the Kardar-Parisi-Zhang Equation

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We solve the large deviations of the Kardar-Parisi-Zhang (KPZ) equation in one dimension at short time by introducing an approach which combines field theoretical, probabilistic, and integrable techniques. We expand the program of the weak noise theory, which maps the large deviations onto a nonlinear hydrodynamic problem, and unveil its complete solvability through a connection to the integrability of the Zakharov-Shabat system. Exact solutions, depending on the initial condition of the KPZ equation, are obtained using the inverse scattering method and a Fredholm determinant framework recently developed. These results, explicit in the case of the droplet geometry, open the path to obtain the complete large deviations for general initial conditions.

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Large deviation rate functions characterize rare events and play a key role in nonequilibrium statistical physics as generalizations of the thermodynamic potentials [1–3]. They have been much studied for interacting particle models in one dimension. For diffusive systems, the macroscopic fluctuation theory (MFT) [4] provides a powerful framework to calculate the large deviation of the density, of the current, and of other fluctuating quantities, in agreement with the available exact solutions [5]. For driven diffusive systems, however, such as the asymmetric exclusion process (ASEP) [6], there is not yet a general approach to calculate the large deviations for all geometries. Exact results from the matrix product ansatz [7] and the Bethe ansatz are available in special cases, for instance, in a stationary regime, either on a finite ring, where rate functions are found to exhibit a universal shape, for the totally asymmetric simple exclusion process (TASEP) [8,9], the ASEP [10,11], and the Kardar-Parisi-Zhang (KPZ) equation [12,13], or in open geometries [14–16].

The KPZ equation is a prominent example of the driven diffusive class. It allows for a few exact solutions valid for all times [17–31], which exhibit at large times the universal typical fluctuations common to systems in the KPZ class [32–34]. Much recent attention has shifted to its large deviation properties, at late times [35–46] and also at short times [37,38,47–64] where two main approaches were developed. The first one uses the aforementioned exact solutions for all times, obtained from a mapping of KPZ observables to the integrable (replica) delta Bose gas. This allowed researchers to obtain the short time large deviations in a few cases [37,38,47–50]. A more versatile approach, closer in spirit to the MFT, is the weak noise theory (WNT) [53–63,65]. It is a saddle point method on the dynamical field theory, which is exact at short time. It leads to a system of two coupled nonlinear partial differential equations, which determine the "optimal" KPZ height field and noise producing the rare fluctuation. Until now, however, these equations have been solved only numerically, except in some limits where useful but approximate solutions were found. Although the existence of multisoliton solutions was noted [55], no exact solution allowing for the full calculation of the large deviations was obtained.

In this Letter, we construct the exact solution to the weak noise theory of the KPZ equation. Through the integrability of the Zakharov-Shabat (ZS) system, originally introduced to solve the nonlinear Schrödinger equation (NLS) [66], we show that the full space time dependence of the optimal height and noise fields admit representations in terms of Fredholm determinants. We provide an explicit formula for the KPZ droplet initial condition and give the general form for a large class of initial conditions.

The KPZ equation [67] describes the stochastic growth in time τ of the height field $h(y,\tau)$ of an interface, here in one space dimension $y \in \mathbb{R}$:

$$\partial_\tau h(y,\tau) = \nu \partial_y^2 h(y,\tau) + \frac{\lambda_0}{2}[\partial_y h(y,\tau)]^2 + \sqrt{D}\eta(y,\tau), \quad (1)$$

where $\eta(y,\tau)$ is a standard space time white noise, i.e., $\overline{\eta(y,\tau)\eta(y',\tau')} = \delta(\tau-\tau')\delta(y-y')$, where $\overline{\cdots}$ denotes averages over the noise. We choose units such that $D = \lambda_0 = 2$ and $\nu = 1$ [68]. We consider the probability

$P(H,T)$ to observe the value $h(0,T) = H - H_0$ at time $\tau = T$, where H_0 is a constant chosen below. At short time, although the typical height fluctuations are Gaussian with Edwards-Wilkinson scaling $\delta H \sim T^{1/4}$, the KPZ nonlinearity leads to nontrivial and nonperturbative tails for $P(H,T)$, describing rare events. For $T \ll 1$, it takes the large deviation form

$$P(H,T) \sim \exp[-\Phi(H)/\sqrt{T}], \tag{2}$$

where the exact rate function $\Phi(H)$ was obtained for droplet, Brownian, and flat initial height profiles, from the exact solutions [47,48,50,58].

We now explain how to obtain such a rate function from the WNT: We first derive the WNT equations in a way leading directly to the so-called $\{P,Q\}$ system, which we then analyze. To that aim, it is useful to define the rescaled time and space variables as $t = \tau/T$ and $x = y/\sqrt{T}$, where T, the observation time, is fixed. Through the Cole-Hopf map, the KPZ field is equivalently described introducing $Z(x,t) = e^{h(y,\tau)+H_0}$, which satisfies the (rescaled) stochastic heat equation (SHE) in the Ito sense

$$\partial_t Z(x,t) = \partial_x^2 Z(x,t) + \sqrt{2}T^{1/4}\tilde{\eta}(x,t)Z(x,t), \tag{3}$$

where $\tilde{\eta}(x,t)$ is another standard space time white noise. This equation is now studied for $t \in [0,1]$. The noise amplitude is now of the order of $T^{1/4}$; hence, a short observation time $T \ll 1$ corresponds to a weak noise. Our convenient choice is $H_0 = \frac{1}{2}\log T$ [69]. It is convenient to study the following generating function which admits a large deviation principle at short time $T \ll 1$, with $z \geq 0$:

$$\overline{\exp(-ze^H/\sqrt{T})} \sim \exp[-\Psi(z)/\sqrt{T}]. \tag{4}$$

Inserting Eq. (2) into the lhs, we see that, for $T \ll 1$, $\Psi(z)$ and $\Phi(H)$ are related through a Legendre transform:

$$\Psi(z) = \min_H [ze^H + \Phi(H)]. \tag{5}$$

Here we aim to calculate $\Psi(z)$ and $\Phi(H)$ using the WNT, for an initial condition of the form $e^{h(y,0)} = (1/\sqrt{T}) \times Z_0(y/\sqrt{T})$, where $Z_0(x)$ is given, an example being the droplet initial condition $Z_0(x) = \delta(x)$.

Any average of the form (4) can be represented using the dynamical field theory associated to the rescaled SHE (3) as $\overline{e^{(1/\sqrt{T})\iint dt dx j(x,t)Z(x,t)}} = \int \mathcal{D}Z\mathcal{D}\tilde{Z} e^{-(1/\sqrt{T})S[\tilde{Z},Z,j]}$ with the dynamical action

$$S[\tilde{Z},Z,j] = \int_0^{+\infty} dt \int_{\mathbb{R}} dx [\tilde{Z}(\partial_t - \partial_x^2)Z - \tilde{Z}^2 Z^2 - jZ], \tag{6}$$

where $\tilde{Z}/\sqrt{T}$ is the response field. In Eq. (4), the source field is $j(x,t) = -z\delta(x)\delta(t-1)$. For $T \ll 1$, the action is evaluated by a saddle point method. Defining $\tilde{Z} = -zP$, $Q = Z$, and $g = -z$, the saddle point equations of the WNT, $(\delta S/\delta\tilde{Z}) = 0$ and $(\delta S/\delta Z) = 0$, take the form of the $\{P,Q\}$ system

$$\begin{aligned} \partial_t Q &= \partial_x^2 Q + 2gPQ^2, \\ -\partial_t P &= \partial_x^2 P + 2gP^2 Q, \end{aligned} \tag{7}$$

a close cousin of the NLS equation [70], which was also discussed in Ref. [55]. While the $\{P,Q\}$ system is interesting in its own right, we will apply its study to the following mixed boundary conditions, of interest for the WNT:

$$Q(x,0) = Q_0(x), \qquad P(x,1) = \delta(x). \tag{8}$$

The source j imposes this form for P at $t = 1$ [71,72], while Q is specified at $t = 0$ from the initial height of the KPZ equation, i.e., $Q_0(x) = Z_0(x)$. The function $\Psi(z)$ in Eq. (4) is obtained from the action S in Eq. (6) at the saddle point. Using the first equation in (7), it can be written in the form (5), allowing one to identify $\Phi(H) \equiv g^2 \int_0^1 dt \int_{\mathbb{R}} dx P^2 Q^2$, with $H = H_z^* \coloneqq \arg\min_H [ze^H + \Phi(H)]$, in agreement with Ref. [53] (see also [72]). The "optimal shape" $h_{\rm opt}(y,\tau)$ of the KPZ height field from the WNT, i.e., the most probable one realizing the value $h_{\rm opt}(0,T) = H - H_0$ at $\tau = T$ and $y = 0$, is obtained from the solution $Q(x,t)$ of (7) for $t \in [0,1]$ as $e^{h_{\rm opt}(y,\tau)} = (1/\sqrt{T})Q[(y/\sqrt{T}),(\tau/T)]$.

Let us first analyze the $\{P,Q\}$ system (7) for general initial conditions and return to the WNT later. Remarkably, (7) belongs to the Ablowitz-Kaup-Newell-Segur (AKNS) class of integrable nonlinear problems [78], for which there exists a Lax pair, i.e., a pair of linear differential equations whose compatibility conditions are equivalent to (7). Here the system reads $\partial_x \vec{v} = U_1 \vec{v}$, $\partial_t \vec{v} = U_2 \vec{v}$, where $\vec{v} = (v_1, v_2)^\intercal$ is a two-component vector (depending on x, t, and k) where

$$U_1 = \begin{pmatrix} -\mathbf{i}k/2 & -gP(x,t) \\ Q(x,t) & \mathbf{i}k/2 \end{pmatrix}, \quad U_2 = \begin{pmatrix} \mathsf{A} & \mathsf{B} \\ \mathsf{C} & -\mathsf{A} \end{pmatrix}, \tag{9}$$

where $\mathsf{A} = k^2/2 - gPQ$, $\mathsf{B} = g(\partial_x - \mathbf{i}k)P$, and $\mathsf{C} = (\partial_x + \mathbf{i}k)Q$. One can check that the compatibility condition $\partial_t U_1 - \partial_x U_2 + [U_1, U_2] = 0$ recovers the system (7) which we solve through the following scattering problem. Let $\vec{v} = e^{k^2 t/2}\phi$ with $\phi = (\phi_1, \phi_2)^\intercal$ and $\vec{v} = e^{-k^2 t/2}\bar{\phi}$ be two independent solutions of the linear problem such that at $x \to -\infty$, $\phi \simeq (e^{-\mathbf{i}kx/2}, 0)^\intercal$, and $\bar{\phi} \simeq (0, -e^{\mathbf{i}kx/2})^\intercal$. Assuming from now on that P and Q vanish at infinity, the $x \to +\infty$ behavior of these solutions defines scattering amplitudes

$$\phi \underset{x\to+\infty}{\simeq} \begin{pmatrix} a(k,t)e^{-\mathrm{i}kx/2} \\ b(k,t)e^{\mathrm{i}kx/2} \end{pmatrix}, \qquad \bar{\phi} \underset{x\to+\infty}{\simeq} \begin{pmatrix} \tilde{b}(k,t)e^{-\mathrm{i}kx/2} \\ -\tilde{a}(k,t)e^{\mathrm{i}kx/2} \end{pmatrix}. \tag{10}$$

Plugging this form into the ∂_t equation of the Lax pair at $x \to +\infty$, one finds a very simple time dependence: $a(k,t) = a(k)$ and $b(k,t) = b(k)e^{-k^2 t}$, $\tilde{a}(k,t) = \tilde{a}(k)$ and $\tilde{b}(k,t) = \tilde{b}(k)e^{k^2 t}$. Another relation is obtained from the Wronskian of the two solutions: $a(k)\tilde{a}(k) + b(k)\tilde{b}(k) = 1$ [79].

Before providing explicit formulas for these scattering amplitudes, let us show how to obtain from them the solution for the $\{P, Q\}$ system, i.e., how to construct the inverse scattering transform. The spatial part of the Lax pair is a 1D Dirac equation called the ZS system, originally introduced to solve the NLS equation [66,80] and extended by AKNS [78]. It was shown very recently [81,82] that the inverse scattering problem can be solved by the means of Fredholm determinants (FDs). Introducing the two reflection coefficients $r(k) = b(k)/a(k)$ and $\tilde{r}(k) = \tilde{b}(k)/[g\tilde{a}(k)]$, one defines two functions [83]:

$$A_t(x) = \int_{\mathbb{R}} \frac{dk}{2\pi} r(k) e^{\mathrm{i}kx - k^2 t}, \qquad B_t(x) = \int_{\mathbb{R}} \frac{dk}{2\pi} \tilde{r}(k) e^{k^2 t - \mathrm{i}kx} \tag{11}$$

and two linear operators from $\mathbb{L}^2(\mathbb{R}^+)$ to $\mathbb{L}^2(\mathbb{R}^+)$ with respective kernels

$$\mathcal{A}_{xt}(v,v') = A_t(x+v+v'), \quad \mathcal{B}_{xt}(v,v') = B_t(x+v+v'). \tag{12}$$

Note that these functions and kernels obey the simple heat equation in space time, and we assume that $A_t(x)$ and $B_t(x)$ vanish fast enough for $x \to +\infty$. The solutions P and Q [81,82] are reconstructed as

$$\begin{aligned} Q(x,t) &= \langle \delta | \mathcal{A}_{xt}(I + g\mathcal{B}_{xt}\mathcal{A}_{xt})^{-1} | \delta \rangle, \\ P(x,t) &= \langle \delta | \mathcal{B}_{xt}(I + g\mathcal{A}_{xt}\mathcal{B}_{xt})^{-1} | \delta \rangle, \end{aligned} \tag{13}$$

where $|\delta\rangle$ is the vector with component $\delta(v)$ so that $\langle \delta | \mathcal{O} | \delta \rangle = \mathcal{O}(0,0)$ for any operator $\mathcal{O}$. The product PQ, which is a conserved charge, i.e., $\partial_t \int_{\mathbb{R}} dx PQ = 0$ as easily verified from (7), can be expressed from a FD as $gPQ = \partial_x^2 \log \mathrm{Det}(I + g\mathcal{B}_{xt}\mathcal{A}_{xt})$. The formula (13), thus, provides the general solution of the $\{P, Q\}$ system, parameterized by the two functions A_t and B_t, equivalently, by the scattering amplitudes. Although these are in one-to-one correspondence with the $\{P, Q\}$ boundary data, making it explicit is nontrivial and is our aim below. Particular cases are such that $\mathcal{A}_{xt}$ and $\mathcal{B}_{xt}$ are operators of finite ranks, leading to solitonic-type solutions [72]. The simplest one leads to $gPQ = \{[(\kappa+\mu)^2]/4\cosh^2 \frac{1}{2}(\kappa+\mu)[x - x_0(t)]\}$. In the context of WNT, this soliton has been used as an approximate solution for $H \to +\infty$, and another rank-one family was noticed in Ref. [55]. However, this is insufficient to obtain the full rate function $\Phi(H)$, which requires the (infinite-rank) general solution obtained in this work.

Let us now apply this to the WNT, i.e., for the boundary data in Eq. (8), and characterize the scattering amplitudes. Integrating the ∂_x equation of the Lax pair at $t = 1$ for $\bar{\phi}$ and ϕ using Eq. (8) allows one to obtain [72] that $\tilde{b}(k) = ge^{-k^2}$. In addition, if the initial condition $Q(x,0)$ is even in x (which we assume from now on), then $\tilde{a}(k) = a(-k) = [a(k)]^*$ and $b(k)$ is real and even. From the Wronskian, this leads to the form

$$a(k) = e^{-\mathrm{i}\varphi(k)} \sqrt{1 - gb(k)e^{-k^2}}, \tag{14}$$

where we still have two unknown functions, a phase $\varphi(k)$, which is odd $\varphi(k) = -\varphi(-k)$, and $b(k)$.

To determine them, we have derived from Eqs. (12) and (13), and the boundary data (8) at $t = 1$, the following integral equation which the functions $A_{t=1}, B_{t=1}$ must obey (see [72]):

$$B_1(x) = \delta(x) + g\Theta(-x) \int_0^{+\infty} dv B_1(x+v) A_1(v), \tag{15}$$

where $A_1(v) = (p * A_0)(v) \coloneqq \int_{\mathbb{R}} dy p(v-y) A_0(y)$ denoting the heat kernel at unit time $p(z) \coloneqq (e^{-z^2/4}/\sqrt{4\pi})$.

Droplet initial condition.—Let us now specialize to $Q_0(x) = \delta(x)$. The solution of (7) then satisfies the symmetry $Q(x,t) = P(x, 1-t)$. This, in turn, implies that $A_t(x) = B_{1-t}(x)$ and $r(k) = e^{k^2}\tilde{r}(-k)$, also implying $b(k) = 1$, which we use below. Hence, in Eq. (15), we can replace $A_1(v)$ by $(p * B_1)(v)$ and we obtain a closed *nonlinear* integral equation for the function B_1 (which equals A_0). This equation still looks formidable; however, for readers familiar with random walks, it has a flavor of another famous integral equation, the Hopf-Ivanov (HI) equation [84,85], which, however, is *linear*, and reads

$$B_1(x) = \delta(x) + g\Theta(-x) \int_{-\infty}^0 dy p(x-y) B_1(y). \tag{16}$$

Amazingly, we found that these two equations, (16) and (15), are *equivalent*. This can be tested in perturbation in g and is shown to all orders in Ref. [72]. The HI equation arises in survival probabilities of random walks [86–89]. Indeed, writing $B_1(x)$ as a series, $B_1(x) = \delta(x) + \sum_{n=1}^{+\infty} g^n B_{1,n}(x)$, and inserting in (16) leads to the recursion $B_{1,n}(x) = \int_{y<0} p(x-y) B_{1,n-1}(y)$. The interpretation is then straightforward. Consider $X(j) \in \mathbb{R}$ a discrete time random walk, $X(j+1) = X(j) + z_j$, with z_j independent identically distributed with jump probability $p(z)$.

Then $B_{1,n}(x)$ is the probability that the walk starting at $X(0)=0$ arrives at $X(n)=x$ in $n\geq 1$ steps, while remaining negative, $\{X(j)\leq 0\}_{j=0,\ldots,n}$ [and $\int_{-\infty}^{0} dx B_{1,n}(x) = \binom{2n}{n}2^{-2n}$ is given by the universal Sparre Andersen theorem [90,91]].

To show that the solution of Eq. (16) also solves Eq. (15) then amounts to splitting the walk into two independent parts, upon crossing the level x for the last time [72]. Introducing the Laplace transform $\hat{B}_1(s)=\int_{-\infty}^{0} e^{sx}B_1(x)$, the solution of the HI equation is known to be [85]

$$\hat{B}_1(s)=\exp\left(-\int_{\mathbb{R}}\frac{dq}{2\pi}\frac{s}{s^2+q^2}\log[1-g\tilde{p}(q)]\right), \quad (17)$$

where $\tilde{p}(k)$ is the Fourier transform of $p(z)$, here $\tilde{p}(k)=e^{-k^2}$. Going from Laplace to Fourier, from Eq. (11) one finds $r(k)=e^{k^2}\tilde{r}(-k)=\hat{B}_1(s)|_{s=-\mathbf{i}k+0^+}$. Using $[1/(s+\mathbf{i}q)]\to PV[\mathbf{i}/(k-q)]+\pi\delta(k-q)$, we obtain from Eq. (17) the reflection coefficient $r(k)$ and its phase $\varphi(k)$:

$$r(k)=\frac{e^{\mathbf{i}\varphi(k)}}{\sqrt{1-ge^{-k^2}}}, \qquad \varphi(k)=\fint_{\mathbb{R}}\frac{dq}{2\pi}\frac{k\log(1-ge^{-q^2})}{q^2-k^2}, \quad (18)$$

which, together with Eqs. (11)–(13), completes the solution of (7) for droplet IC. Plots of the optimal height $\log Q(x,t)$ are shown in Fig. 1 and Ref. [72]. Note that for $t=1$ a simpler formula, $Q(x,1)=A_1(|x|)$, holds.

To extract $\Phi(H)\equiv g^2\int_{\mathbb{R}}\int_0^1 dxdtP^2Q^2$ from our solution requires the computation of a difficult integral. This is overcome by relating it, as well as $\Psi(z)$, to conserved quantities. We use the construction of ZS [66] to generate all conserved quantities C_n for the $\{P,Q\}$ system; see [72]. We find $C_1=g\int_{\mathbb{R}}dxPQ$ and $C_3=g(\int_{\mathbb{R}}dxP\partial_x^2Q+gP^2Q^2)$. The *values* $C_n(g)$ taken by these conserved charges can be retrieved from the Laurent expansion of $\log a(k)=\sum_{n\geq 1}[C_n(g)/(\mathbf{i}k)^n]$. Until now, this is general for any initial condition of the $\{P,Q\}$ system. Now recall that for the droplet IC we obtained $\log a(k)=-\mathbf{i}\varphi(k)+\frac{1}{2}\log(1-ge^{-k^2})$. Since the second term has vanishing Laurent expansion, we find that $-\mathbf{i}\varphi(k)=\sum_{n\geq 1}[C_n(g)/(\mathbf{i}k)^n]$. Expanding in powers of $1/k$ in Eq. (18), we obtain [92]

$$C_1(g)=\frac{1}{\sqrt{4\pi}}\mathrm{Li}_{3/2}(g), \quad C_3(g)=\frac{-1}{\sqrt{16\pi}}\mathrm{Li}_{5/2}(g). \quad (19)$$

Since $C_1=g\int_{\mathbb{R}}dxPQ$ is time independent, evaluated at $t=1$ it leads to $C_1(g)=gQ(0,1)=ge^H$. On the other hand, differentiating the Legendre transform in Eq. (5) with respect to z gives $\Psi'(z)=e^H$. This implies that $C_1(-z)=-z\Psi'(z)$, and by integration

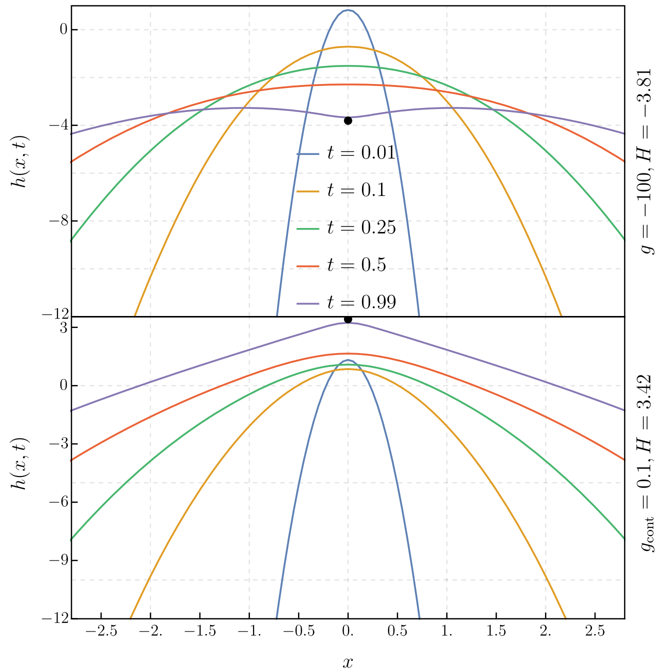

FIG. 1. The optimal height $h(x,t)=\log Q(x,t)$ for the droplet initial condition at various times t for final values (black dot) $H=-3.81$ (top) and $H=3.42$ (bottom).

$$\Psi(z)=\Psi_0(z)\coloneqq-\frac{1}{\sqrt{4\pi}}\mathrm{Li}_{5/2}(-z), \quad (20)$$

which allows one to determine $\Phi(H)$ parametrically as $\Phi(H)=\Psi(z)-z\Psi'(z)$, $e^H=\Psi'(z)$ [it can also be obtained from $C_3(g)$; see Ref. [72]]. Our WNT result (20) agrees with Ref. [47], without relying on an exact solution of the KPZ equation.

Until now, we assumed $z=-g\geq 0$ corresponding to $H\leq\hat{H}_0=-\frac{1}{2}\log(4\pi)$, the most probable value of H such that $\Phi'(\hat{H}_0)=0$ [93]. However, (7) also holds for any $H>\hat{H}_0$ [55,72], corresponding to the attractive regime $g>0$ of the $\{P,Q\}$ system. Indeed, $\Psi(z)$ can be analytically continued to $z<0$, allowing one to determine $\Phi(H)$ for any H [47]. For $H\in(-\infty,H_c]$, Eq. (20) holds, with $z=-g$ varying from $+\infty$ down to $z=-1$. For $H>H_c=\log\{[\zeta(3/2)]/[\sqrt{4\pi}]\}$, a second continuation is needed, $\Psi(z)=\Psi_0(z)+\Delta(z)$, with $\Delta(z)=\frac{4}{3}\{\log[-(1/z)]\}^{3/2}$ with $z\in[-1,0)$ as $H\in[H_c,+\infty)$. These continuations correspond to two branches of solutions of the $\{P,Q\}$ system for $0<g\leq 1$. One finds [72] that the second branch corresponds to the spontaneous generation of a solitonic part in the solution, of rapidity κ_0 with $g=e^{-\kappa_0^2}$, which dominates the large deviations for $H\to+\infty$. It is described by $A_t(x)=A_t(x)|_{\phi(k)\to\phi(k)+\Delta\phi(k)}+2\kappa_0e^{-\kappa_0x+\kappa_0^2t+\mathbf{i}\varphi(\mathbf{i}\kappa_0)}$, where $\Delta\varphi(k)=2\arctan(\kappa_0/k)$ and $B_t(x)=A_{1-t}(x)$. The values of the odd conserved charges are increased by

$\Delta C_n(g) = (2/n)\kappa_0^n$, which for $n = 1$ induces the additional part $\Delta(z)$.

General initial condition.—For general even $Q_0(x)$, the only difference is that $b(k)$ is nontrivial, with $r(k) = b(k)e^{k^2}\tilde{r}(-k)$, and now $A_t(x) = (\hat{b} * B_{1-t})(x)$, where $\hat{b}(x)$ denotes the Fourier transform of $b(k)$ and $*$ the convolution. Equation (15), replacing $A_1(v) = (p * \hat{b} * B_1)(v)$, is again equivalent to a linear HI equation for $B_1(x)$, obtained by simply replacing p by $p * \hat{b}$ in Eq. (16), with the same random walk interpretation for a new jump probability $p(z) \to (p * \hat{b})(z)$. Thus, this leads to $r(k) = \left\{[b(k)e^{\mathrm{i}\varphi(k)}]\Big/\left[\sqrt{1 - gb(k)e^{-k^2}}\right]\right\}$ and $\varphi(k) = \fint_{\mathbb{R}}(dq/2\pi)\{k\log[1 - gb(q)e^{-q^2}]/(q^2 - k^2)\}$. One now finds $C_1(g) = \int_{\mathbb{R}}(dq/2\pi)\mathrm{Li}_1[gb(q)e^{-q^2}]$, leading to [94]

$$z\Psi'(z) = \int_{\mathbb{R}} \frac{dq}{2\pi} \log[1 + zb(q)e^{-q^2}]. \tag{21}$$

Interestingly, such a form was observed to describe all known exact solutions [95], e.g., flat IC [96]. Thus, for a general initial condition, we reduced the problem to computing a single unknown function $b(k)$ and relating it to $Q_0(x)$, a question left for the future. In Ref. [72], we give a formula relating $b(k)$, $Q_0(x)$, and $P(x,1)$ allowing for expansions around the droplet solution.

In conclusion, our solution allows one to calculate the optimal height and noise for arbitrary values of H, previously inaccessible. The Fredholm approach provides a novel analytical and numerical scheme for the solution of the integrable $\{P, Q\}$ system as shown in Fig. 1; see [72]. The present work demonstrates that inverse scattering methods can successfully address optimal fluctuation theory of stochastic systems, leading to analytic results and interesting phenomena such as spontaneous soliton generation.

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[69] So that $\overline{H}$ remains $\mathcal{O}(1)$ as $T \to 0$. Indeed, from Ito one has $\overline{\exp[h(y,\tau)]} = 1/\sqrt{4\pi\tau}$ for, e.g., droplet IC and, more generally, $\overline{\exp[h(y,\tau)]} = (1/\sqrt{T}) \int_{\mathbb{R}} (dx'/\sqrt{4\pi t}) \times e^{-(x-x')^2/(4t)} Q_0(x')$.

[70] If $\Psi(x,\tau)$ solves NLS, then $Q = \Psi(x,\tau)|_{\tau\to-\mathrm{i}t}$ and $P = \Psi^*(x,\tau)|_{\tau\to-\mathrm{i}t}$ solves the $\{P, Q\}$ system.

[71] The equation for P should, thus, be solved backward in time.

[72] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.127.064101 for the principal details of the calculations described in the Letter and additional information about the results, which includes Refs. [73–77].

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[92] Note that all even conserved quantities $C_{2p} = 0$ by parity of the initial condition $Q_0(x) = \delta(x)$.

[93] From Eq. (5), one has $\Psi'(z) = e^H$ and, in particular, $\Psi'(0) = e^{\hat{H}_0}$, where $\hat{H}_0$ is the most probable value of H,

i.e., such that $\Phi'(\hat{H}_0) = 0$. Since one has, to $\mathcal{O}(z)$, $\Psi(z) \simeq z\overline{e^H}$, it implies from Ref. [69] that $\hat{H}_0 = \log \overline{e^H} = \log \int_{\mathbb{R}} (dx'/\sqrt{4\pi}) e^{-(x')^2/(4)} Q_0(x')$ for a general initial condition and $\hat{H}_0 = -\frac{1}{2}\log(4\pi)$ for the droplet IC.

[94] Where we recall that $b(q)$ depends on z.
[95] See Table 7.1 in Ref. [50] and also Ref. [38].
[96] Deriving it from the WNT requires to extend our method to a nondecaying $Q_0(x)$, a work in progress.