Optimal Entanglement Certification from Moments of the Partial Transpose

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For the certification and benchmarking of medium-size quantum devices, efficient methods to characterize entanglement are needed. In this context, it has been shown that locally randomized measurements on a multiparticle quantum system can be used to obtain valuable information on the so-called moments of the partially transposed quantum state. This allows one to infer some separability properties of a state, but how to use the given information in an optimal and systematic manner has yet to be determined. We propose two general entanglement detection methods based on the moments of the partially transposed density matrix. The first method is based on the Hankel matrices and provides a family of entanglement criteria, of which the lowest order reduces to the known p_3-positive-partial-transpose criterion proposed in A. Elben *et al.* [Phys. Rev. Lett. **125**, 200501 (2020)]. The second method is optimal and gives necessary and sufficient conditions for entanglement based on some moments of the partially transposed density matrix.

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Introduction.—Intermediate-scale quantum devices involving a few dozen qubits are considered a stepping stone toward the ultimate goal of achieving fault-tolerant quantum computation [1]. For such devices, the standard method of tomography is no longer feasible for gauging the performance in actual experiments [2]. As a result, efficient and reliable characterization methods of such multiparticle systems are indispensable for current quantum information research [3,4]. As entanglement is a key ingredient in quantum computation and other quantum information processing tasks, many efforts have been devoted to its characterization and quantification [5–7].

If an experiment aims at producing a specific quantum state with few particles, entanglement witnesses or Bell inequalities provide mature tools for entanglement detection. For larger and noisy systems, however, these methods require significant measurement efforts; moreover, some of the standard constructions of witnesses are not very powerful. To overcome this, methods using locally randomized measurements have been put forward. In these schemes, one performs on the particles measurements in random bases and determines the moments from the resulting probability distribution. It was noted early that this approach allows one to detect entanglement [8,9] or evaluate the moments of the density matrix [10]. Recently, this approach has become the center of attention and found experimental applications. For instance, it was shown that, with these methods, entropies can be estimated [11,12], different forms of multiparticle entanglement can be characterized [13–16], and bound entanglement as a weak form of entanglement can be detected [17]. Many efforts have been devoted to verify the positive partial transpose (PPT) condition [18] from the moments of the randomized measurements [19–21].

To explain this approach, let ρ_{AB} be a quantum state in a bipartite quantum system $\mathcal{H}_A \otimes \mathcal{H}_B$; the PPT criterion then states that for any separable state $\rho_{AB}^{T_A} \geq 0$, where T_A denotes the partial transposition on subsystem $\mathcal{H}_A$. For a given quantum state ρ_{AB}, it is straightforward to check whether the PPT criterion is violated, and if so, the state must be entangled. However, in actual experiments, the quantum state is unknown unless resource-inefficient quantum state tomography is performed. Recently, researchers found that the PPT condition can also be studied by considering the so-called partial transpose moments (PT moments)

$$p_k := \mathrm{Tr}[(\rho_{AB}^{T_A})^k], \tag{1}$$

which can be efficiently measured from randomized measurements [20,22]. To see the basic idea behind the PT-moment-based entanglement detection, suppose that we know all the PT moments $\boldsymbol{p} = (p_0, p_1, p_2, \ldots, p_d)$, where $d = d_A d_B$ is the dimension of the global system $\mathcal{H}_A \otimes \mathcal{H}_B$. Then, all the eigenvalues of $\rho_{AB}^{T_A}$ can be directly calculated [23], from which we can verify whether the PPT criterion is violated. Hereafter, we always assume that $p_0 = d$ and $p_1 = 1$, which are trivial but included for convenience.

In practice, however, it is difficult, if not impossible, to measure all the moments of a quantum state. Hence, the problem turns to whether we can detect the entanglement from the moments of limited order. The question of whether knowledge of some moments allows one to draw conclusions about the underlying probability distribution is indeed fundamental and has appeared in quantum information theory before [24–26]. For the case of PT moments, we formulate the *PT-moment problem* as follows: Given the PT moments of order n, is there a separable state compatible with the data? More technically formulated, given the numbers $\boldsymbol{p}^{(n)} = (p_0, p_1, p_2, \ldots, p_n)$, is there a separable state ρ_{AB} such that $p_k = \mathrm{Tr}[(\rho_{AB}^{T_A})^k]$ for $k = 0, 1, \ldots, n$?

In Ref. [20] the PT-moment problem for $n = 3$ was studied and a necessary (but not sufficient) condition was proposed, called the p_3-PPT criterion:

$$\rho_{AB} \in \mathrm{SEP} \Rightarrow p_3 \geq p_2^2, \tag{2}$$

where SEP denotes the set of separable states.

In this Letter, we propose two systematic methods for solving the general PT-moment problem. First, we build a connection between the PT-moment problem and the known moment problems in the mathematical literature. This gives a relaxation of the PT-moment problem, resulting in a family of entanglement criteria in which the p_3-PPT criterion is the lowest order. Second, we show that the p_3-PPT criterion is not sufficient for the PT-moment problem of order 3. By reformulating the PT-moment problem as an optimization problem, we derive an explicit necessary and sufficient criterion for $n = 3$ and further generalize it to the case that $n > 3$. Finally, we illustrate the efficiency of our criteria with physically relevant examples, e.g., thermal states of condensed matter systems.

Relaxation to the classical moment problems.—We start by relaxing the PT-moment problem and establishing a connection to the classical moment problems. Here, instead of defining the classical moment problems with respect to the Borel measure on the real line [27–29], we rewrite them with quantum states and observables.

Given a quantum state σ and an observable (Hermitian operator) X, the kth moment is defined as $m_k \coloneqq \mathrm{Tr}(\sigma X^k)$. The moment problems ask the converse: given a sequence of moments, does there exist a quantum state σ and an observable X (with some restrictions) giving the desired moments? Albeit formulated in a quantum language, this scenario is essentially classical since σ can be taken in diagonal form in the eigenbasis of X. The (truncated) Hamburger and Stieltjes moment problems are defined as follows.

Hamburger moment problem: Given the moments of order n, more precisely, $\boldsymbol{m}^{(n)} = (m_0, m_1, m_2, \ldots, m_n)$, is there a quantum state σ and an observable X such that $m_k = \mathrm{Tr}(\sigma X^k)$ for $k = 0, 1, \ldots, n$?

Stieltjes moment problem: Given the moments of order n, more precisely, $\boldsymbol{m}^{(n)} = (m_0, m_1, m_2, \ldots, m_n)$, is there a quantum state σ and a positive semidefinite observable X such that $m_k = \mathrm{Tr}(\sigma X^k)$ for $k = 0, 1, \ldots, n$?

Clearly, the only difference between these problems is that in the Stieltjes moment problem X has to be positive semidefinite. We define the corresponding two sets of moments as

$$\mathcal{M}_n = \{\boldsymbol{m}^{(n)} | \mathrm{Tr}(\sigma X^k) = m_k, \sigma \geq 0, X^\dagger = X\}, \tag{3}$$

$$\mathcal{M}_n^+ = \{\boldsymbol{m}^{(n)} | \mathrm{Tr}(\sigma X^k) = m_k, \sigma \geq 0, X \geq 0\}. \tag{4}$$

Note that in the above definitions there is no restriction on the dimension of σ and X. Also, since there is no bound on the eigenvalues of X, the sets $\mathcal{M}_n$ and $\mathcal{M}_n^+$ are not closed.

If we set $\sigma = \mathbb{1}$ and $X = \rho_{AB}^{T_A}$, the PT moments $\boldsymbol{p}^{(n)} = (p_0, p_1, \ldots, p_n)$ defined by Eq. (1) always satisfy that $\boldsymbol{p}^{(n)} \in \mathcal{M}_n$; furthermore, the PT moments given by the PPT states satisfy that $\boldsymbol{p}^{(n)} \in \mathcal{M}_n^+$. Hence, if we can characterize the set $\mathcal{M}_n^+$, or the difference between $\mathcal{M}_n^+$ and $\mathcal{M}_n \backslash \mathcal{M}_n^+$, we get a family of necessary conditions for the PT-moment problem. This is a relaxation, as in the definition of $\mathcal{M}_n$ and $\mathcal{M}_n^+$ more general σ are allowed.

To proceed, we introduce the notion of Hankel matrices. The Hankel matrices $H_k(\boldsymbol{m})$ and $B_k(\boldsymbol{m})$ are $(k+1) \times (k+1)$ matrices defined by

$$[H_k(\boldsymbol{m})]_{ij} = m_{i+j}, \qquad [B_k(\boldsymbol{m})]_{ij} = m_{i+j+1} \tag{5}$$

for $i, j = 0, 1, \ldots, k$. Hereafter, we will often suppress the argument ($\boldsymbol{m}$ or $\boldsymbol{p}$) in the notation when there is no risk of confusion. For example,

$$H_1 = \begin{bmatrix} m_0 & m_1 \\ m_1 & m_2 \end{bmatrix}, \qquad B_1 = \begin{bmatrix} m_1 & m_2 \\ m_2 & m_3 \end{bmatrix}, \tag{6}$$

$$H_2 = \begin{bmatrix} m_0 & m_1 & m_2 \\ m_1 & m_2 & m_3 \\ m_2 & m_3 & m_4 \end{bmatrix}, \qquad B_2 = \begin{bmatrix} m_1 & m_2 & m_3 \\ m_2 & m_3 & m_4 \\ m_3 & m_4 & m_5 \end{bmatrix}. \tag{7}$$

From the definition of the Hankel matrices, one can prove the following result on the relations between $\mathcal{M}_n$, $\mathcal{M}_n^+$ and H_k, B_k; see Appendix A in the Supplemental Material for details [30].

Lemma 1.—(a) A necessary condition for $\boldsymbol{m}^{(n)} = (m_0, m_1, \ldots, m_n) \in \mathcal{M}_n$ is that $H_{\lfloor n/2 \rfloor} \geq 0$. (b) A necessary condition for $\boldsymbol{m}^{(n)} = (m_0, m_1, \ldots, m_n) \in \mathcal{M}_n^+$ is that $H_{\lfloor n/2 \rfloor} \geq 0$ and $B_{\lfloor (n-1)/2 \rfloor} \geq 0$.

By applying Lemma 1 to the PT-moment problem, we obtain a family of criteria for entanglement detection.

Theorem 1.—Let $p_k = \mathrm{Tr}[(\rho_{AB}^{T_A})^k]$ for $k = 1, 2, \ldots, n$, then a necessary condition for ρ_{AB} being a separable state is that $B_{\lfloor (n-1)/2 \rfloor}(\boldsymbol{p}) \geq 0$.

Before preceding, we have a few remarks on Lemma 1 and Theorem 1. First, the conditions are almost sufficient in Lemma 1. If we consider the moments $\boldsymbol{m}^{(n)}$ in the closure of $\mathcal{M}_n$ or $\mathcal{M}_n^+$, then the conditions of the positivity of the

Hankel matrices are also sufficient in Lemma 1; see Appendix A in the Supplemental Material for details [30]. Because of the finite precision in actual experiments, this also means that Theorem 1 is the best criterion when relaxing the PT-moment problem to the classical moment problems.

Second, although the condition $H_{\lfloor n/2 \rfloor} \geq 0$ is also necessary for ρ_{AB} being separable, it does not give an entanglement criterion as this condition is satisfied by any (separable or entangled) state, according to Lemma 1(a).

Third, by noting that $p_1 = 1$, the lowest-order criterion from Theorem 1, $B_1 \geq 0$, gives that $p_3 \geq p_2^2$, which is exactly the p_3-PPT condition in Eq. (2) from Ref. [20]. When $k > 1$, B_k gives stronger criteria for entanglement detection. Accordingly, we call the condition

$$\rho_{AB} \in \mathrm{SEP} \Rightarrow B_{\lfloor (n-1)/2 \rfloor}(\boldsymbol{p}) \geq 0 \tag{8}$$

the p_n-PPT criterion for $n = 3, 5, 7, \ldots$. The power of the p_n-PPT criteria will be illustrated with examples after we describe the optimal method for the PT-moment problem.

Last, we would like to point out that although higher-order criteria $p_n^{n-2} \geq p_{n-1}^{n-1}$ were also proposed in Ref. [20], they usually cannot detect more entangled states than the p_3-PPT criterion. In Appendix B in the Supplemental Material [30], we show that these inequalities are strictly weaker than the p_n-PPT criteria from Theorem 1 and explain why these inequalities are usually much weaker.

Optimal solution to the PT-moment problem.—Theorem 1 already provides a family of strong entanglement criteria, but they are not optimal. This is because in Eqs. (3), (4) σ can be arbitrary, but in the PT-moment problem σ is always $\mathbb{1}$. In the following, we give an optimal solution to the PT-moment problem.

By writing the spectrum of $\rho_{AB}^{T_A}$ as $(x_1, x_2, \ldots, x_d)$, one can easily see that the PT-moment problem is equivalent to characterizing the set

$$\mathcal{T}_n^+ = \left\{ \boldsymbol{p}^{(n)} \middle| \sum_{i=1}^{d} x_i^k = p_k, x_i \geq 0 \right\}. \tag{9}$$

Indeed, for any $\boldsymbol{p}^{(n)} \in \mathcal{T}_n^+$, a compatible separable state can be constructed as follows: Relabel x_i for $i = 1, 2, \ldots, d$ as $x_{\alpha\beta}$ for $\alpha = 1, 2, \ldots, d_A$ and $\beta = 1, 2, \ldots, d_B$; then construct a separable state $\rho_{AB} = \sum_{\alpha,\beta} x_{\alpha\beta} |\alpha\rangle\langle\alpha| \otimes |\beta\rangle\langle\beta|$, where $|\alpha\rangle$, $|\beta\rangle$ are states in the computational basis. This state has $p_k = \mathrm{Tr}[(\rho_{AB}^{T_A})^k]$ for $k = 0, 1, \ldots, n$. For convenience, we also define the more general set

$$\mathcal{T}_n = \left\{ \boldsymbol{p}^{(n)} \middle| \sum_{i=1}^{d} x_i^k = p_k, x_i \in \mathbb{R} \right\}. \tag{10}$$

Hereafter, the eigenvalues $(x_1, x_2, \ldots, x_d)$ are assumed to be sorted in descending order unless otherwise stated. In Eqs. (9), (10), the dimension $d = \dim(\mathcal{H}_A \otimes \mathcal{H}_B)$ is considered as fixed, but actually the optimal entanglement criteria in the following, e.g., Eq. (17), do not depend on d anymore.

The key idea of the optimal criteria is to consider the following optimization:

$$\begin{aligned} &\min_{x_i}/\max_{x_i}\ \hat{p}_n := \sum_{i=1}^{d} x_i^n \\ &\quad \text{subject to } \sum_{i=1}^{d} x_i^k = p_k \quad \text{for } k = 1, 2, \ldots, n-1, \\ &\quad x_i \geq 0 \quad \text{for } i = 1, 2, \ldots, d. \end{aligned} \tag{11}$$

Note that this may also be viewed as a minimization or maximization of the Rényi or Tsallis entropy of order n under the constraint that the entropies for lower integer orders are fixed. Suppose that the solutions are given by $\hat{p}_n^{\min}$ and $\hat{p}_n^{\max}$, respectively; then $p_n \in [\hat{p}_n^{\min}, \hat{p}_n^{\max}]$ provides a necessary condition for ρ_{AB} being separable. If one can further show that all $p_n \in [\hat{p}_n^{\min}, \hat{p}_n^{\max}]$ are attainable by some $(x_1, x_2, \ldots, x_d)$ from a separable state, this will imply the sufficiency of the condition. As Eq. (11) is a polynomial optimization, the sum-of-squares hierarchy can, in principle, be used for approximating the bounds [35,36]. Remarkably, an alternative sum-of-squares method was used in Ref. [37] for bounding the negative eigenvalues from moments. Here, instead of using these approximation methods, we propose an exact method for solving Eq. (11) analytically.

We start from the simplest case $n = 3$. As shown in Appendix C of the Supplemental Material [30], the maximum and minimization are achieved by

$$\boldsymbol{x}_3^{\max} = (x_1, x_2, x_2, \ldots, x_2), \tag{12}$$

$$\boldsymbol{x}_3^{\min} = (x_1, x_1, \ldots, x_1, x_{\alpha+1}, 0, 0, \ldots, 0), \tag{13}$$

respectively, where x_1 appears $\alpha = \lfloor 1/p_2 \rfloor$ times in Eq. (13). Thus, we obtain the following necessary and sufficient condition for the PT-moment problem of order 3.

Theorem 2.—(a) There exists a d-dimensional separable state ρ_{AB} satisfying that $p_k = \mathrm{Tr}[(\rho_{AB}^{T_A})^k]$ for $k = 1, 2, 3$, if and only if

$$p_1 = 1, \qquad \frac{1}{d} \leq p_2 \leq 1, \tag{14}$$

$$p_3 \leq [1 - (d-1)y]^3 + (d-1)y^3, \tag{15}$$

$$p_3 \geq \alpha x^3 + (1 - \alpha x)^3, \tag{16}$$

where $\alpha = \lfloor 1/p_2 \rfloor$, $x = [\alpha + \sqrt{\alpha[p_2(\alpha+1) - 1]}]/[\alpha(\alpha+1)]$, and $y = [d - 1 - \sqrt{(d-1)(p_2 d - 1)}]/$

$[d(d-1)]$. (b) More importantly, suppose that the p_k for $k=1,2,3$ are PT moments from a quantum state. Then, they are compatible with a separable state if and only if

$$p_3 \geq \alpha x^3 + (1-\alpha x)^3, \tag{17}$$

where α and x are as above.

Mathematically speaking, Theorem 2(a) fully characterizes the set $\mathcal{T}_3^+$, while Theorem 2(b) characterizes the difference between $\mathcal{T}_3^+$ and $\mathcal{T}_3 \backslash \mathcal{T}_3^+$. In other words, Eqs. (14), (15) are satisfied by any (separable or entangled) state. In practice, p_k are usually obtained from experiments; hence, Eq. (17) should be used for entanglement detection. Thus, we will refer to Eq. (17) as the p_3-OPPT (optimal PPT) criterion. Again, we emphasize that the p_3-OPPT criterion is dimension independent.

According to Eq. (11), this method is not restricted to the case $n=3$. For example, when $n=4$, the maximum and minimum are achieved by

$$\boldsymbol{x}_4^{\max} = (x_1, x_2, x_2, \ldots, x_2, x_{\beta+2}, 0, 0, \ldots, 0), \tag{18}$$

$$\boldsymbol{x}_4^{\min} = (x_1, x_1 \cdots, x_1, x_{\gamma+1}, x_{\gamma+2}, x_{\gamma+2}, \ldots, x_{\gamma+2}), \tag{19}$$

respectively, where β and γ are some fixed integers. However, an important difference to the case $n=3$ is that, although solving the problem analytically is still possible, writing down the optimal values is no longer straightforward. This is because the roots of higher-order polynomials are much more complicated [38]. In Appendix C in the Supplemental Material [30], we describe the general procedure for solving the optimization problems in Eq. (11). We also provide the computer code for $n = 3, 4, 5$ [30].

Examples.—Before discussing the examples, we show how to quantify the violation of the p_n-PPT and p_n-OPPT criteria. Analogous to the PPT criterion, we use the negativity [39,40] to quantify the violation of p_n-PPT criteria. For $n = 3, 5, 7, \ldots$, we define

$$\mathcal{N}_n(\rho_{AB}) = \frac{1}{2}\|B_{\lfloor (n-1)/2 \rfloor}(\boldsymbol{p})\| - \frac{1}{2}\mathrm{Tr}[B_{\lfloor (n-1)/2 \rfloor}(\boldsymbol{p})], \tag{20}$$

i.e., the absolute sum of the negative eigenvalues of $B_{\lfloor (n-1)/2 \rfloor}$, where $\|\cdot\|$ denotes the trace norm. For the p_n-OPPT, we quantify the violation via

$$\mathcal{O}_n(\rho_{AB}) = \max\{p_n^{\min} - p_n, p_n - p_n^{\max}, 0\} \tag{21}$$

for $n = 3, 4, 5, \ldots$. Remarkably, although both the p_n-PPT and p_n-OPPT criteria can be viewed as hierarchical entanglement criteria based on PT moments, there are two important distinctions. First, the p_n-PPT criteria only work when n is odd, while the p_n-OPPT criteria work whenever $n \geq 3$. Second, $\mathcal{N}_n(\rho_{AB})$ in Eq. (20) is well-defined for any ρ_{AB}, while $\mathcal{O}_n(\rho_{AB})$ only exists when $\mathcal{O}_{n-1}(\rho_{AB}) = 0$, i.e., the optimization problems in Eq. (11) are feasible.

To show the power of our criteria, we first investigate the entanglement of randomly generated states. Here, we sample the random $D \times D$ states $[\dim(\mathcal{H}_A) = \dim(\mathcal{H}_B) = D]$ with the Hilbert-Schmidt distribution [41]. In Table I, we show the results when D is small ($D = 2$, 3, 4, 5, 6); additional results when D is large ($D = 10$, 20, 30, 40) are shown in Appendix D in the Supplemental Material [30].

From the sampling, one can see a few remarkable advantages of our criteria. First, most of the entangled states can already be detected by the p_5-PPT or the p_4-OPPT criterion. Second, although the p_3-PPT and p_3-OPPT criteria are both based on the PT moments p_2 and p_3, the optimal criterion p_3-OPPT is significantly stronger than the p_3-PPT criterion in Ref. [20]. Furthermore, the optimal criterion not only detects more entangled states but also the violation is more significant, as shown in Appendix D in the Supplemental Material [30]. Third, compared with the usual entanglement witness method, our criteria have the advantage that neither common reference frames nor prior information is needed for the entanglement detection [13,20]. Also, compared with the widely used fidelity-based entanglement witness, many more entangled states can be detected by comparing Table I with the results in Refs. [42,43].

For the second example, we consider the one-dimensional quantum Ising model in a transverse magnetic field,

TABLE I. Fraction of (small) $D \times D$ states in the Hilbert-Schmidt distribution (1×10^6 samples) that can be detected with various criteria. Here, NPT denotes the states violating the PPT criterion, NPTn (NPT3, NPT5) denotes the states violating the p_n-PPTcriterion in Eq. (8), and ONPTn (ONPT3, ONPT4, ONPT5) denotes the states violating the p_n-OPPT criterion.

D	NPT	NPT3	ONPT3	ONPT4	NPT5	ONPT5
2	75.68%	25.53%	39.97%	75.68%	64.78%	75.68%
3	99.99%	25.32%	39.46%	91.63%	97.51%	98.97%
4	100%	23.29%	33.69%	98.68%	100.00%	100.00%
5	100%	21.80%	34.54%	99.95%	100%	100%
6	100%	20.93%	31.20%	100.00%	100%	100%

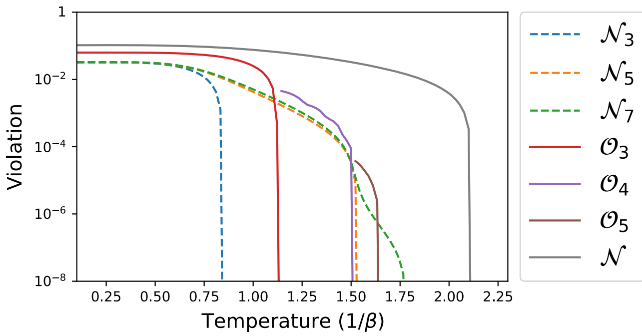

FIG. 1. The strength of different PT-moment-based entanglement criteria. Here, we choose the parameters $J = 1$ and $g = 2.5$ for a 10-qubit system. The entanglement for the bipartition $(1, 2, \ldots, 5|6, 7, \ldots, 10)$ is considered. The violations $\mathcal{N}_n$ and $\mathcal{O}_n$ are defined in Eqs. (20), (21), and $\mathcal{N}$ is the negativity of entanglement [40].

$$H = -J\left(\sum_{i=1}^{N} \sigma_i^z \sigma_{i+1}^z + g \sum_{i=1}^{N} \sigma_i^x\right), \tag{22}$$

with the periodic boundary condition ($\sigma_{N+1}^z = \sigma_1^z$), where J corresponds to the coupling strength and g is the relative strength of the external magnetic field. We study the entanglement of the thermal equilibrium (Gibbs) state $\rho(\beta) = \mathrm{e}^{-\beta H}/Z(\beta)$, where $Z(\beta) = \mathrm{Tr}[\mathrm{e}^{-\beta H}]$ is the partition function and β is the inverse temperature. The strength of different PT-moment-based entanglement criteria for this model is illustrated in Fig. 1.

Finally, we would like to note that the example in Fig. 1 also illustrates an important challenge for testing the PT-moment-based criteria. That is, the violations can become very small for higher-order criteria. Indeed, this is not specific to the PT moments but also the other moment-based methods. The fundamental reason is that the (PT) moments decrease exponentially as n goes large. This can be easily seen from the relation that $|\mathrm{Tr}(X^n)| \leq [\mathrm{Tr}(X^2)]^{(n/2)}$ for any Hermitian operator X and $n \geq 2$ [44]. In the PT-moment problem, $\mathrm{Tr}[(\rho_{AB}^{T_A})^2] = \mathrm{Tr}[\rho_{AB}^2]$, which is the purity of the state. Hence, the violations in Fig. 1 become small (compared to the PPT criterion) if the temperature increases. Still, it should be remembered that a small violation is, in general, not connected to a statistically insignificant violation [46–48]; see Appendix E in the Supplemental Material [30] for more discussions. This difference also means that the numerical values of the violations in Eqs. (20), (21) should not be directly compared with each other.

Conclusion.—We have developed two systematic methods for detecting entanglement from PT moments. The first method is based on the classical moment problems, whose lowest order gives the p_3-PPT criterion in Ref. [20] and higher orders provide strictly stronger criteria. The second method is the optimal method, which gives necessary and sufficient conditions for entanglement detection based on PT moments. We demonstrated that our criteria are significantly better than existing criteria for physically relevant states.

For future research, there are several possible directions. First, one may extend the presented theory by taking, instead of the transposition, other positive but not completely positive maps. This may allow one to characterize entanglement in quantum states that escape the detection by the PPT criterion. Second, for the analysis of current experiments, it would be highly desirable to extend the presented theory to the characterization of multiparticle entanglement. Indeed, potential generalizations of the PPT criterion for the multiparticle case exist [49] but how to evaluate this using randomized measurements remains an open question.

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Note added.—While finishing this manuscript, we became aware of a related work by A. Neven *et al.* [50].

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