Generating Spin Polarization from Vorticity through Nonlocal Collisions

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We derive the collision term in the Boltzmann equation using the equation of motion for the Wigner function of massive spin-1/2 particles. To next-to-lowest order in $\hbar$, it contains a nonlocal contribution, which is responsible for the conversion of orbital into spin angular momentum. In a proper choice of pseudogauge, the antisymmetric part of the energy-momentum tensor arises solely from this nonlocal contribution. We show that the collision term vanishes in global equilibrium and that the spin potential is, then, equal to the thermal vorticity. In the nonrelativistic limit, the equations of motion for the energy-momentum and spin tensors reduce to the well-known form for hydrodynamics for micropolar fluids.

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Introduction.—The study of polarization phenomena in heavy-ion collisions has attracted significant attention in the past [1–4]. Experimental studies show that the spin of hadrons emitted in noncentral collisions is aligned with the direction of the global angular momentum [5–7]. The magnitude of the global polarization of Λ baryons can be very well described by models based on relativistic hydrodynamics and assuming local thermodynamic equilibrium of the spin degrees of freedom [4,8–14]. Particles get polarized through the rotation of the medium in a way which resembles the Barnett effect [15]. Unfortunately, the same models [16,17] are not able to describe the experimentally measured longitudinal Λ polarization [18]. This problem is currently the focus of intense work [19–26], however, a convincing solution does not yet exist.

In order to address this problem, one needs to understand how the orbital angular momentum of the strongly interacting matter created in noncentral heavy-ion collisions is converted into the spin angular momentum of its constituents. In order to account for the nontrivial dynamics of the spin degrees of freedom, it has been proposed to introduce the rank-three spin tensor as an additional dynamical variable, promoting relativistic hydrodynamics to a theory of spin hydrodynamics [27–33] (for a different approach based on the Lagrangian formalism, see Refs. [34–37]). Since spin is inherently a quantum feature, any rigorous derivation of spin hydrodynamics must be based on quantum field theory. However, in order to see how orbital angular momentum can be converted into spin and vice versa on a microscopic level, it is advantageous to, first, derive a Boltzmann equation from quantum field theory, and then obtain hydrodynamics from the former, for instance, by applying the method of moments [38].

In the nonrelativistic case, this direction was pursued in a seminal paper [39], where it was already pointed out that a local collision term in the Boltzmann equation will not be able to describe polarization through rotation. Only if one accounts for the nonlocality of the microscopic collision process, can orbital angular momentum be converted into spin. A nonrelativistic nonlocal collision term was derived in Ref. [40] and for spinless particles, e.g., in Refs. [41–43]. In Ref. [21], a microscopic model based on collisions of partons as wave packets was proposed to link the spin polarization to the vorticity, but without considering spin equilibration. A first attempt to systematically incorporate nonlocal collisions in a kinetic framework based on quantum field theory was recently made in Ref. [44], however, without giving an explicit expression for the nonlocal collision term. The aim of this Letter is to present the main steps of a systematic, explicit derivation of such a term, and a discussion of its impact on spin hydrodynamics. For details of the calculation we refer to Ref. [45].

Our notation and conventions are $a \cdot b = a^\mu b_\mu$, $a_{[\mu} b_{\nu]} \equiv a_\mu b_\nu - a_\nu b_\mu$, $g_{\mu\nu} = \mathrm{diag}(+,-,-,-)$, $\epsilon^{0123} = -\epsilon_{0123} = 1$.

Quantum transport.—The Wigner-function formalism provides a first-principle formulation of kinetic theory and is a very powerful tool for the description of anomalous transport in heavy-ion collisions (see, e.g., Refs. [46–53]). The Wigner function for spin-1/2 particles reads [54–56]

$$W_{\alpha\beta}(x,p)=\int\frac{d^4y}{(2\pi\hbar)^4}e^{-\frac{i}{\hbar}p\cdot y}\langle:\bar{\psi}_\beta(x_1)\psi_\alpha(x_2):\rangle, \quad (1)$$

with $x_{1,2}=x\pm y/2$ and the spinor field $\psi(x)$. Extending our previous work [57] (see, also, Refs. [58–63]), which was valid in the free-streaming limit, now, we include interactions and, thus, account for the effect of collisions. Using the Dirac equation in the presence of a general interaction term

$$(i\hbar\gamma\cdot\partial-m)\psi(x)=\hbar\rho(x), \quad (2)$$

where $\rho=-(1/\hbar)\partial\mathcal{L}_I/\partial\bar{\psi}$, with $\mathcal{L}_I$ being the interaction Lagrangian, the transport equation for the Wigner function reads [54]

$$\left[\gamma\cdot\left(p+i\frac{\hbar}{2}\partial\right)-m\right]W=\hbar\mathcal{C}[W], \quad (3)$$

where

$$\mathcal{C}_{\alpha\beta}[W]\equiv\int\frac{d^4y}{(2\pi\hbar)^4}e^{-\frac{i}{\hbar}p\cdot y}\langle:\rho_\alpha(x_2)\bar{\psi}_\beta(x_1):\rangle. \quad (4)$$

We decompose the Wigner function in terms of a basis of the generators of the Clifford algebra

$$W=\frac{1}{4}\left(\mathcal{F}+i\gamma^5\mathcal{P}+\gamma\cdot\mathcal{V}+\gamma^5\gamma\cdot\mathcal{A}+\frac{1}{2}\sigma^{\mu\nu}\mathcal{S}_{\mu\nu}\right), \quad (5)$$

and substitute it into Eq. (3) to obtain equations of motion for the coefficient functions

$$\frac{\hbar}{2}\partial\cdot\mathcal{A}+m\mathcal{P}=-\hbar D_{\mathcal{P}}, \quad (6)$$

$$p^\mu\mathcal{F}-\frac{\hbar}{2}\partial_\nu\mathcal{S}^{\nu\mu}-m\mathcal{V}^\mu=\hbar D^\mu_{\mathcal{V}}, \quad (7)$$

$$\frac{\hbar}{2}\partial^{[\mu}\mathcal{V}^{\nu]}-\epsilon^{\mu\nu\alpha\beta}p_\alpha\mathcal{A}_\beta-m\mathcal{S}^{\mu\nu}=\hbar D^{\mu\nu}_{\mathcal{S}}, \quad (8)$$

$$\hbar\partial\cdot\mathcal{V}=2\hbar C_{\mathcal{F}}, \quad (9)$$

$$p\cdot\mathcal{A}=\hbar C_{\mathcal{P}}, \quad (10)$$

$$p^{[\mu}\mathcal{V}^{\nu]}+\frac{\hbar}{2}\epsilon^{\mu\nu\alpha\beta}\partial_\alpha\mathcal{A}_\beta=-\hbar C^{\mu\nu}_{\mathcal{S}}, \quad (11)$$

where we defined $D_i=\mathrm{Re}\,\mathrm{Tr}(\tilde{\Gamma}_i\mathcal{C})$, $C_i=\mathrm{Im}\mathrm{Tr}(\tilde{\Gamma}_i\mathcal{C})$, $i=\mathcal{F},\mathcal{P},\mathcal{V},\mathcal{S}$, $\tilde{\Gamma}_{\mathcal{F}}=1$, $\tilde{\Gamma}_{\mathcal{P}}=-i\gamma_5$, $\tilde{\Gamma}_{\mathcal{V}}=\gamma^\mu$, $\tilde{\Gamma}_{\mathcal{S}}=\sigma^{\mu\nu}$.

Following Refs. [57,60,61], we employ an expansion in powers of $\hbar$ for the functions $\mathcal{F},\mathcal{P},\mathcal{V}^\mu,\mathcal{A}^\mu,\mathcal{S}^{\mu\nu}$, and the collision terms D_i, C_i in Eqs. (6)–(11), e.g., for the scalar part

$$\mathcal{F}=\mathcal{F}^{(0)}+\hbar\mathcal{F}^{(1)}+O(\hbar^2). \quad (12)$$

Since gradients are always accompanied by factors of $\hbar$, this is effectively a gradient expansion.

In order to simplify the following discussion, now, we make the assumption that all effects arising from spin are at least of first order in $\hbar$. Therefore, since $\mathcal{A}^\mu$ is the spin density in phase space [57], its zeroth-order contribution vanishes, $\mathcal{A}^{(0)\mu}=0$, and consequently, from Eq. (8), $\mathcal{S}^{(0)\mu\nu}=0$. From Eq. (6), we also immediately conclude that $\mathcal{P}^{(0)}=0$. Thus, at zeroth order, all pseudoscalar quantities vanish, which also implies that the collision terms which carry pseudoscalar quantum numbers must vanish at zeroth order, $D^{(0)}_{\mathcal{P}}=C^{(0)}_{\mathcal{P}}=0$. Using Eqs. (6) and (10), this, in turn, implies that

$$\mathcal{P}=\mathcal{O}(\hbar^2), \qquad p\cdot\mathcal{A}=\mathcal{O}(\hbar^2). \quad (13)$$

For the vector part, the only vector at our disposal at zeroth order is p^μ, i.e.,

$$D^\mu_{\mathcal{V}}=p^\mu\delta V+\mathcal{O}(\hbar), \quad (14)$$

with a scalar function δV. Thus, from Eq. (7), we obtain

$$\mathcal{V}^\mu=\frac{1}{m}p^\mu\bar{\mathcal{F}}+\mathcal{O}(\hbar^2), \quad (15)$$

where we defined $\bar{\mathcal{F}}\equiv\mathcal{F}-\hbar\delta V$. From Eqs. (9) and (11), we then derive

$$p\cdot\partial\bar{\mathcal{F}}=mC_F, \qquad p\cdot\partial\mathcal{A}^\mu=mC^\mu_A, \quad (16)$$

with $C_F=2C_{\mathcal{F}}$ and $C^\mu_A\equiv-\frac{1}{m}\epsilon^{\mu\nu\alpha\beta}p_\nu C_{\mathcal{S}\alpha\beta}$. Equations (13) and (16) form a closed system of equations for $\bar{\mathcal{F}}$ and $\mathcal{A}^\mu$.

The next step is to introduce spin as an additional variable in phase space [31,33,64–66]. We define the distribution function

$$\mathfrak{f}(x,p,\mathfrak{s})\equiv\frac{1}{2}[\bar{\mathcal{F}}(x,p)-\mathfrak{s}\cdot\mathcal{A}(x,p)]. \quad (17)$$

Now, we employ the quasiparticle approximation, i.e., we assume that $\mathfrak{f}$ is of the form

$$\mathfrak{f}(x,p,\mathfrak{s})=m\delta(p^2-M^2)f(x,p,\mathfrak{s}), \quad (18)$$

where $f(x,p,\mathfrak{s})$ is a function without singularity at $p^2=M^2\equiv m^2+\hbar\delta m^2$, and $\delta m^2(x,p,\mathfrak{s})$ is an interaction contribution to the mass-shell condition for free particles [where the $\mathfrak{s}$ dependence enters at $\mathcal{O}(\hbar)$]. We introduce the integration measure

$$\int dS(p)\equiv\frac{1}{\kappa(p)}\int d^4\mathfrak{s}\delta(\mathfrak{s}\cdot\mathfrak{s}+3)\delta(p\cdot\mathfrak{s}), \quad (19)$$

where $\kappa(p)\equiv\sqrt{3}\pi/\sqrt{p^2}$ is determined by requiring

$$\bar{\mathcal{F}} = \int dS(p)\mathfrak{f}(x,p,\mathfrak{s}), \qquad \mathcal{A}^\mu = \int dS(p)\mathfrak{s}^\mu \mathfrak{f}(x,p,\mathfrak{s}). \tag{20}$$

The kinetic equation which we want to solve is now given by combining Eq. (17) with Eq. (16),

$$p\cdot\partial\mathfrak{f}(x,p,\mathfrak{s}) = m\mathfrak{C}, \tag{21}$$

where $\mathfrak{C} \equiv \frac{1}{2}(C_F - \mathfrak{s}\cdot C_A)$. In the next section, we will compute this kernel up to first order in $\hbar$. This implies that we assume that $\mathfrak{f}$ varies slowly in space and time on the microscopic scale corresponding to the interaction range. We will also restrict ourselves to the contribution from particles; the extension to include antiparticles is straightforward.

Collisions.—Up to first order in $\hbar$, $\mathfrak{C}$ has the following structure:

$$\mathfrak{C} = \mathfrak{C}_l^{(0)} + \hbar\{\mathfrak{C}_l^{(1)} + \mathfrak{C}_{nl}^{(1)}\} \equiv \mathfrak{C}_l + \hbar\mathfrak{C}_{nl}^{(1)}. \tag{22}$$

Here, local and nonlocal contributions are denoted by subscripts l and nl, respectively. The zeroth-order contribution is purely local [67], while the first-order contribution has both local and nonlocal parts, the latter arising from gradients.

In order to explicitly calculate the collision term, we follow Ref. [54]. This calculation is based on an expansion in particle-scattering states and makes the following assumptions: (i) low-density approximation, i.e., the interacting Wigner function in the collision term is identified with the free-streaming Wigner function (also containing contributions up to order $\hbar$), and only binary scattering processes of the form $(p_1,\mathfrak{s}_1),(p_2,\mathfrak{s}_2)\to(p,\mathfrak{s}),(p',\mathfrak{s}')$ are considered, and (ii) initial correlations are neglected.

The details of the calculation are shown in Ref. [45]. Here, we only report the most important intermediate steps. First, it can be shown that the off-shell terms on both sides of Eq. (21) cancel, and we are left with the on-shell Boltzmann equation

$$\delta(p^2-m^2)p\cdot\partial f(x,p,\mathfrak{s}) = \delta(p^2-m^2)\mathfrak{C}_{\text{on-shell}}[f]. \tag{23}$$

where $\mathfrak{C}_{\text{on-shell}}[f]$ is the on-shell contribution of Eq. (22). In this way, we obtain the local part of the on-shell collision term

$$\mathfrak{C}_{\text{on-shell},l}[f] \equiv \mathfrak{C}_{p+s}[f] + \mathfrak{C}_{\text{s}}[f], \tag{24}$$

where

$$\mathfrak{C}_{p+s}[f] \equiv \int d\Gamma_1 d\Gamma_2 d\Gamma' dS_1'(p)\mathcal{W} \times [f(x,p_1,\mathfrak{s}_1)f(x,p_2,\mathfrak{s}_2) - f(x,p,\mathfrak{s}_1')f(x,p',\mathfrak{s}')] \tag{25a}$$

describes momentum- and spin-exchange interactions, while

$$\mathfrak{C}_{\text{s}}[f] \equiv \int d\Gamma_2 dS_1(p)\mathfrak{W}f(x,p,\mathfrak{s}_1)f(x,p_2,\mathfrak{s}_2) \tag{25b}$$

describes purely spin exchange without momentum exchange. Here, we defined the phase-space measure $\int d\Gamma \equiv \int d^4p\delta(p^2-m^2)\int dS(p)$ as well as

$$\begin{aligned}\mathcal{W} \equiv{}& \frac{1}{32}\sum_{s,r,s_1'}[h_{ss_1'}(p,\mathfrak{s}_1')h_{s_1'r}(p,\mathfrak{s}) + h_{ss_1'}(p,\mathfrak{s})h_{s_1'r}(p,\mathfrak{s}_1')] \\ &\times \sum_{s',r',s_1,s_2,r_1,r_2} h_{s'r'}(p',\mathfrak{s}')h_{s_1r_1}(p_1,\mathfrak{s}_1)h_{s_2r_2}(p_2,\mathfrak{s}_2) \\ &\times \langle p,p';r,r'|t|p_1,p_2;s_1,s_2\rangle\langle p_1,p_2;r_1,r_2|t^\dagger|p,p';s,s'\rangle \\ &\times \delta^4(p+p'-p_1-p_2),\end{aligned} \tag{26}$$

and

$$\begin{aligned}\mathfrak{W} \equiv{}& \hbar\frac{\pi}{4m}\sum_{s_1,s_2,r,r_2}\epsilon_{\mu\nu\alpha\beta}\mathfrak{s}^\mu\mathfrak{s}_1^\nu p^\alpha n_{s_1r}^\beta(p)h_{s_2r_2}(p_2,\mathfrak{s}_2) \\ &\times \langle p,p_2;r,r_2|t+t^\dagger|p,p_2;s_1,s_2\rangle,\end{aligned} \tag{27}$$

where

$$h_{sr}(p,\mathfrak{s}) \equiv \delta_{sr} + \mathfrak{s}\cdot n_{sr}(p), \quad n_{sr}^\mu(p) \equiv \frac{1}{2m}\bar{u}_s(p)\gamma^5\gamma^\mu u_r(p), \tag{28}$$

with the standard particle spinors $u_r(p)$, $\bar{u}_s(p)$. Moreover,

$$\begin{aligned}&\langle p,p';r,r'|t|p_1,p_2;s_1,s_2\rangle \\ &\quad\equiv -\sqrt{\frac{(2\pi\hbar)^7}{2}}\bar{u}_r(p)_{\text{out}}\langle p';r'|\rho(0)|p_1,p_2;s_1,s_2\rangle_{\text{in}}\end{aligned} \tag{29}$$

denotes the amplitude for the scattering of two particles with momenta p_1, p_2 and spin projections s_1, s_2 into two particles with momenta p, p' and spin projections r, r'.

If the distribution functions do not depend on the spin variables, i.e., $f(x,p,\mathfrak{s}) \equiv f(x,p)$, we recover the familiar Boltzmann collision term, where the spin averaging and summation is done directly in the cross section [54]. However, if the distribution functions depend on spin, the two terms on the right-hand side of Eq. (24) require further discussion. Considering $\mathfrak{C}_{p+s}[f]$, the term $\sim f(x,p_1,\mathfrak{s}_1)f(x,p_2,\mathfrak{s}_2)$ has the form of a gain term for

particles with $(p,\mathfrak{s})$, while $\sim f(x,p,\mathfrak{s}_1')f(x,p',\mathfrak{s}')$ does not have an obvious interpretation as a loss term, because the spin variable is $\mathfrak{s}_1'$, not $\mathfrak{s}$.

However, we can modify the definition of distribution function and collision term such that the standard interpretation of gain and loss terms in the latter is recovered. To this end, first, we note that the physically relevant quantities are obtained after integrating over the phase-space spin variable, see, e.g., Eq. (20). Therefore, if we can replace the distribution function $f(x,p,\mathfrak{s})$ by another distribution function $\tilde{f}(x,p,\mathfrak{s})$ and, similarly, the collision term $\mathfrak{C}[f]$ by $\tilde{\mathfrak{C}}[\tilde{f}]$, where

$$p\cdot\partial\tilde{f}(x,p,\mathfrak{s})=\tilde{\mathfrak{C}}[\tilde{f}], \tag{30a}$$

$$\int dS(p)b\tilde{Q}(x,p,\mathfrak{s})=\int dS(p)bQ(x,p,\mathfrak{s}) \tag{30b}$$

is fulfilled for $Q\in\{f,\mathfrak{C}[f]\}$, $\tilde{Q}\in\{\tilde{f},\tilde{\mathfrak{C}}[\tilde{f}]\}$, $b\in\{1,\mathfrak{s}^\mu\}$, then, the physically relevant quantities will not be changed. Equations (30) constitute a kind of "weak equivalence principle," stating that f and $\tilde{f}$ formally obey the same equation of motion and give identical results when integrating over the spin variable.

One can show that the choice $\tilde{f}\equiv f$ and

$$\begin{aligned}\tilde{\mathfrak{C}}_{p+s}[f]\equiv&\int d\Gamma_1 d\Gamma_2 d\Gamma'\tilde{\mathcal{W}}\\&\times[f(x,p_1,\mathfrak{s}_1)f(x,p_2,\mathfrak{s}_2)-f(x,p,\mathfrak{s})f(x,p',\mathfrak{s}')],\end{aligned} \tag{31}$$

with

$$\begin{aligned}\tilde{\mathcal{W}}\equiv&\,\delta^4(p+p'-p_1-p_2)\frac{1}{8}\sum_{s,r}h_{sr}(p,\mathfrak{s})\\&\times\sum_{s',r',s_1,s_2,r_1,r_2}h_{s'r'}(p',\mathfrak{s}')h_{s_1r_1}(p_1,\mathfrak{s}_1)h_{s_2r_2}(p_2,\mathfrak{s}_2)\\&\times\langle p,p';r,r'|t|p_1,p_2;s_1,s_2\rangle\\&\times\langle p_1,p_2;r_1,r_2|t^\dagger|p,p';s,s'\rangle,\end{aligned} \tag{32}$$

satisfies the weak equivalence principle (30) up to $\mathcal{O}(\hbar)$.

Now, let us focus on $\mathfrak{C}_s[f]$. This corresponds to collisions where the momentum of each particle is conserved, but the spin can change: $(p,\mathfrak{s}_1),(p_2,\mathfrak{s}_2)\to(p,\mathfrak{s}),(p_2,\mathfrak{s}')$ [54]. Here, the distribution functions $f(x,p,\cdot)$ and $f(x,p',\cdot)$ describe the particles before and after the collision, which means that they contribute to both the gain and the loss term. We see, from Eq. (27), that the interchange of $\mathfrak{s}^\mu$ and $\mathfrak{s}_1^\nu$ flips the sign of $\mathfrak{W}$. This means that a net gain of particles with $(p,\mathfrak{s})$ corresponds to a net loss of particles with $(p,\mathfrak{s}_1)$.

Now, turning to the nonlocal collision term, first, let us give an intuitive argument. If we assume that particles scatter with a finite impact parameter, the distribution functions entering Eq. (25) have to be evaluated at different space-time points. Hence, the simplest extension of the full collision term (22), and modified using the weak equivalence principle (30), should have the form

$$\begin{aligned}\tilde{\mathfrak{C}}_{\text{on-shell}}[f]=&\int d\Gamma_1 d\Gamma_2 d\Gamma'\tilde{\mathcal{W}}[f(x+\Delta_1,p_1,\mathfrak{s}_1)\\&\times f(x+\Delta_2,p_2,\mathfrak{s}_2)\\&-f(x+\Delta,p,\mathfrak{s})f(x+\Delta',p',\mathfrak{s}')]\\&+\int d\Gamma_2 dS_1(p)\mathfrak{W}f(x+\Delta_1,p,\mathfrak{s}_1)\\&\times f(x+\Delta_2,p_2,\mathfrak{s}_2),\end{aligned} \tag{33}$$

where the position shifts $\Delta,\Delta',\Delta_1,\Delta_2$ are of first order in $\hbar$. Note that, in the collision term where only spin is exchanged, $p_1=p,p'=p_2$. We show, in Ref. [45], that Eq. (33) can, indeed, be derived by an explicit calculation. The only additional assumption that we need to make is that the scattering amplitude is constant over scales of order Δ. This is consistent with the low-density approximation, see, e.g., Ref. [68]. The position shifts are functions of momentum and spin given by

$$\Delta^\mu\equiv-\frac{\hbar}{2m(p\cdot\hat{t}+m)}\epsilon^{\mu\nu\alpha\beta}p_\nu\hat{t}_\alpha\mathfrak{s}_\beta, \tag{34}$$

where $\hat{t}^\mu=(1,\mathbf{0})$ is the timelike unit vector in the frame where p^μ is measured, for details see Ref. [45].

Equilibrium.—Now, we will consider the conditions necessary to reach equilibrium. For the sake of simplicity, we consider uncharged particles, implying zero chemical potential. The standard form of the local equilibrium distribution function is [9,27,31]

$$f_{eq}(x,p,\mathfrak{s})=\frac{1}{(2\pi\hbar)^3}\exp\left[-\beta(x)\cdot p+\frac{\hbar}{4}\Omega_{\mu\nu}(x)\Sigma_{\mathfrak{s}}^{\mu\nu}\right], \tag{35}$$

where $\beta^\mu=u^\mu/T$ (u^μ is the fluid velocity and T the temperature, respectively) and $\Omega^{\mu\nu}$ is the spin potential [27,30]. Note that the exponent of the distribution function has to be a linear combination of the conserved quantities momentum and total angular momentum. Here, we absorbed the orbital part of the angular momentum into the definition of $\beta^\mu(x)$ [9], and we defined the dipole-moment tensor

$$\Sigma_{\mathfrak{s}}^{\mu\nu}\equiv-\frac{1}{m}\epsilon^{\mu\nu\alpha\beta}p_\alpha\mathfrak{s}_\beta. \tag{36}$$

Inserting Eq. (35) into Eq. (33) and expanding to first order in $\hbar$, we obtain

$$\tilde{\mathfrak{C}}_{\text{on-shell}}[f_{eq}] = -\int d\Gamma' d\Gamma_1 d\Gamma_2 \widetilde{\mathcal{W}} e^{-\beta\cdot(p_1+p_2)} \times \Big[\partial_\mu\beta_\nu(\Delta_1^\mu p_1^\nu + \Delta_2^\mu p_2^\nu - \Delta^\mu p^\nu - \Delta'^\mu p'^\nu) - \frac{\hbar}{4}\Omega_{\mu\nu}(\Sigma^{\mu\nu}_{\mathfrak{s}_1} + \Sigma^{\mu\nu}_{\mathfrak{s}_2} - \Sigma^{\mu\nu}_{\mathfrak{s}} - \Sigma^{\mu\nu}_{\mathfrak{s}'})\Big] - \int d\Gamma_2 dS_1(p) dS'(p_2) \mathfrak{W} e^{-\beta\cdot(p+p_2)} \times \Big\{\partial_\mu\beta_\nu[(\Delta_1^\mu - \Delta^\mu)p^\nu + (\Delta_2^\mu - \Delta'^\mu)p_2^\nu] - \frac{\hbar}{4}\Omega_{\mu\nu}(\Sigma^{\mu\nu}_{\mathfrak{s}_1} + \Sigma^{\mu\nu}_{\mathfrak{s}_2} - \Sigma^{\mu\nu}_{\mathfrak{s}} - \Sigma^{\mu\nu}_{\mathfrak{s}'})\Big\}, \quad (37)$$

where the zeroth-order contribution to the collision term was omitted, as it vanishes for the distribution function (35). Since $L^{\mu\nu} = \Delta^{[\mu}p^{\nu]}$ is the orbital angular momentum tensor of the particle with $(p, \mathfrak{s})$, the parentheses in the second and fifth line contain the balance of orbital angular momentum in the respective collision.

Introducing the total angular momentum $J^{\mu\nu} = L^{\mu\nu} + (\hbar/2)\Sigma^{\mu\nu}_{\mathfrak{s}}$ of the particle and assuming that this is conserved in a collision, $J^{\mu\nu} + J'^{\mu\nu} = J_1^{\mu\nu} + J_2^{\mu\nu}$, the collision term vanishes for any $\widetilde{\mathcal{W}}$, $\mathfrak{W}$ if

$$\partial_\mu\beta_\nu + \partial_\nu\beta_\mu = 0, \quad (38)$$

$$\Omega_{\mu\nu} = \varpi_{\mu\nu} \equiv -\frac{1}{2}\partial_{[\mu}\beta_{\nu]} = \text{const}, \quad (39)$$

which corresponds to the conditions for global (and not just local) equilibrium. Our calculation derives condition (39) for the first time in a kinetic-theory approach, confirming a known result from statistical quantum field theory [69]. In previous works, condition (39) was found in the massless case [70], but for massive particles, only in the presence of an electromagnetic field [57,63].

Spin hydrodynamics from kinetic theory.—The equation of motion for the spin tensor $S^{\lambda,\mu\nu}$ is derived from the conservation of the total angular momentum tensor $J^{\lambda,\mu\nu} \equiv x^\mu T^{\lambda\nu} - x^\nu T^{\lambda\mu} + \hbar S^{\lambda,\mu\nu}$ [27–33]. Among all possible pseudogauges [30,71,72], we use the energy-momentum and spin tensors proposed by Hilgevoord and Wouthuysen (HW) [54,73,74] which, unlike the canonical currents, were shown to give a covariant description for the spin of free fields. Defining $dP \equiv d^4p\delta(p^2 - M^2)$, one obtains

$$T^{\mu\nu}_{\text{HW}} = \int dP dS(p) p^\mu p^\nu f(x, p, \mathfrak{s}) + \mathcal{O}(\hbar^2), \quad (40)$$

$$S^{\lambda,\mu\nu}_{\text{HW}} = \int dP dS(p) p^\lambda \left(\frac{1}{2}\Sigma^{\mu\nu}_{\mathfrak{s}} - \frac{\hbar}{4m^2}p^{[\mu}\partial^{\nu]}\right) f(x, p, \mathfrak{s}) + \mathcal{O}(\hbar^2). \quad (41)$$

The equations of motion for these tensors can be written with the help of the Boltzmann equation (23) and the weak equivalence principle (30) as

$$\partial_\mu T^{\mu\nu}_{\text{HW}} = \int d^4p dS(p) p^\nu \mathfrak{C} = 0, \quad (42)$$

$$\hbar\partial_\lambda S^{\lambda,\mu\nu}_{\text{HW}} = \int d^4p dS(p) \frac{\hbar}{2}\Sigma^{\mu\nu}_{\mathfrak{s}}\mathfrak{C} = T^{[\nu\mu]}_{\text{HW}}, \quad (43)$$

where energy-momentum conservation in a binary collision makes the right-hand side of Eq. (42) vanish. On the other hand, the right-hand side of Eq. (43) is derived from the conservation of total angular momentum and shows that spin is, in general, not conserved in collisions. This is described by the antisymmetric part of the energy-momentum tensor, which contains terms of at least second order in $\hbar$. Moreover, in the HW formulation, $T^{[\nu\mu]}_{\text{HW}}$ is nonzero only in the presence of nonlocal collisions. In local collisions, the orbital angular momentum vanishes and the dipole-moment tensor itself is a collision invariant [31], which makes the right-hand side of Eq. (43) vanish. In general, however, this happens only in global equilibrium. Away from global equilibrium, the collision term is nonzero, and thus, there is always dissipative dynamics. In this sense, "ideal spin hydrodynamics" exists only if one neglects the nonlocality of a microscopic collision.

Nonrelativistic limit.—In the nonrelativistic limit, $p^\mu \to m(1, \mathbf{v})$ with the particle three-velocity $\mathbf{v}$, which implies $\Sigma^{ij}_{\mathfrak{s}} \to \epsilon^{ijk}\mathfrak{s}^k$. With this, the spatial components of the last equality in Eq. (43) read

$$T^{[ji]}_{\text{HW}} = m\epsilon^{ijk}\partial^0\left\langle\frac{\hbar}{2}\mathfrak{s}^k\right\rangle + m\epsilon^{ijk}\partial^l\left\langle v^l\frac{\hbar}{2}\mathfrak{s}^k\right\rangle, \quad (44)$$

where we used the Boltzmann equation (23) to replace the collision term $\tilde{\mathfrak{C}}[f]$ and introduced the notation $\langle\ldots\rangle \equiv (m^2/2\pi\sqrt{3})\int d^3v d^3\mathfrak{s}\delta(\mathfrak{s}^2 - 3)(\ldots)f$. The above result agrees with Eq. (12.11) of Ref. [39] (up to a constant due to a different normalization). We also compare to the results obtained for micropolar fluids. The equation of interest, here, is Eq. (2.2.9) of Ref. [75], which in the absence of external fields reads

$$\rho(\partial^0 + u^j\partial^j)\ell^i = \partial^j C^{ji} + \epsilon^{ijk}T^{jk}, \quad (45)$$

where ρ is the mass density, ℓ^i is the internal angular momentum, C^{ji} is the so-called couple-stress tensor, and T^{jk} is the (conventional) stress tensor. Using the continuity equation $\partial^0\rho + \partial^i(\rho u^i) = 0$ in Eq. (45) and comparing to Eq. (44), we identify $m\langle(\hbar/2)\mathfrak{s}^i\rangle = \rho\ell^i$, $C^{ji} = -\langle(\hbar/2)\mathfrak{s}^i p^j\rangle + m\langle(\hbar/2)\mathfrak{s}^i\rangle u^j$, and $T^{jk} = -T^{jk}_{\text{HW}}$.

Conclusions.—In this Letter, we computed the collision term in the Boltzmann equation up to first order in $\hbar$,

accounting for the nonlocality of the microscopic collision process. Nonlocal collisions are essential to convert orbital into spin angular momentum. We have shown that the collision term vanishes in global equilibrium, where the spin potential is equal to the thermal vorticity. In the approach to equilibrium, a rotating fluid of particles will develop a nonvanishing polarization, while a polarized fluid will develop a nonvanishing vorticity. Furthermore, we have shown that, in a certain pseudogauge choice [74], the antisymmetric part of the energy-momentum tensor arises solely from the nonlocal contribution to the collision term. The equations of motion for the energy-momentum and spin tensors show that, away from global equilibrium, nonlocal collisions always imply dissipative dynamics. In the nonrelativistic limit, we have obtained well-known results for hydrodynamics with internal degrees of freedom, as has been applied to, e.g., micropolar fluids [75], spintronics [76], and chiral active fluids [77]. An interesting extension of our work would be to derive the equations of motion of relativistic dissipative spin hydrodynamics using the method of moments [38].

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