Universal Constraint for Relaxation Rates for Quantum Dynamical Semigroup

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A general property of relaxation rates in open quantum systems is discussed. We find an interesting constraint for relaxation rates that universally holds in fairly large classes of quantum dynamics, e.g., weak coupling regimes, as well as for entropy nondecreasing evolutions. We conjecture that this constraint is universal, i.e., it is valid for all quantum dynamical semigroups. The conjecture is supported by numerical analysis. Moreover, we show that the conjectured constraint is tight by providing a concrete model that saturates the bound. This universality marks an essential step toward the physical characterization of complete positivity as the constraint is directly verifiable in experiments. It provides, therefore, a physical manifestation of complete positivity. Our conjecture also has two important implications: it provides (i) a universal constraint for the spectra of quantum channels and (ii) a necessary condition to decide whether a given channel is consistent with Markovian evolution.

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Introduction.—An understanding of the general aspects of the dynamics of open quantum systems is of fundamental importance in the study of the interaction between a quantum system and its environment that causes dissipation, decay, and decoherence [1]. This understanding is important for such fundamental issues as the measurement problem (see, e.g., [2,3]) and for applications in modern quantum technologies such as quantum communication, cryptography, and computation [4]. Very often one infers information about the quantum system by measuring a spectrum of some operator representing physical objects (quantum observables, quantum maps, etc.). In this Letter, we analyze the spectral properties of the celebrated Gorini-Kossakowski-Lindblad-Sudarshan (GKLS) generator of the quantum Markovian semigroup [5,6]

$$\dot{\rho} = \mathcal{L}(\rho), \tag{1}$$

where $\mathcal{L}$ has the following well-known form:

$$\mathcal{L}(\rho) = -i[H,\rho] + \sum_k \gamma_k \left(L_k \rho L_k^\dagger - \frac{1}{2}\{L_k^\dagger L_k, \rho\} \right) \tag{2}$$

with arbitrary noise operators L_k and positive rates γ_k (we put $\hbar = 1$). This is the most general structure of the generator that guaranties that the dynamical map $\Lambda_t = e^{t\mathcal{L}}$ is completely positive and trace-preserving (CPTP) [1, 5–7]. Solutions of Eq. (1) define very good approximations of a real system's evolution, provided the system-environment interaction is sufficiently weak and there is separation of timescales for the system and environment [1,8–10]. It is well known that eigenvalues of $\mathcal{L}$ provide information about the rate of relaxation and the dissipation and decoherence processes and hence define key physical properties of the physical process. In this Letter, we ask about the physical meaning of complete positivity. In particular, we analyze how this mathematical requirement affects physical quantities measured in the laboratory. Surprisingly, apart from the fact that all relaxation rates are non-negative, not much more is known about the structure of the spectrum of a GKLS generator except for the qubit case (2-level system) [11]. In this Letter, we find an interesting constraint for relaxation rates that universally holds in fairly large and physically important classes of any d-level quantum dynamics, e.g., in general dynamics derived in the weak coupling limit from the proper microscopic model and the general entropy nondecreasing dynamics (which corresponds to the unital semigroup). We conjecture that this constraint is universally valid in all completely positive Markovian evolutions. Moreover, we find a concrete model that saturates the bound, proving that (provided that the conjecture is true) this constraint is tight and cannot be improved.

Let ℓ_α be the corresponding (complex) eigenvalues of $\mathcal{L}$, that is, $\mathcal{L}(X_\alpha) = \ell_\alpha X_\alpha$ for $\alpha = 0, \ldots, d^2 - 1$, where $d = \dim \mathcal{H}$. Since $\mathcal{L}$ does preserve Hermiticity, one has $\mathcal{L}(X_\alpha^\dagger) = \ell_\alpha^* X_\alpha^\dagger$, that is, if ℓ_α is complex, then ℓ_α^* is also an eigenvalue. It is well known [7] that $\ell_0 = 0$ and the corresponding eigenvector (zero mode of $\mathcal{L}$) X_0 give rise to the invariant state of the evolution $\omega = X_0/\mathrm{Tr}X_0$, that is, $\Lambda_t(\omega) = \omega$. The corresponding eigenvalues $\lambda_\alpha(t)$ of the dynamical map $\Lambda_t = e^{t\mathcal{L}}$ read $\lambda_\alpha(t) = e^{t\ell_\alpha}$, and hence necessarily the relaxation rates Γ_α defined by

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$$\Gamma_\alpha = -\mathrm{Re}\ell_\alpha \tag{3}$$

are non-negative $\Gamma_\alpha \geq 0$ for all $\alpha = 1, \ldots, d^2 - 1$. Eigenvalues $\lambda_\alpha(t)$ of $\Lambda_t = e^{t\mathcal{L}}$ belong to the unit disk on the complex plane, that is, $|\lambda_\alpha(t)| \leq 1$. This is a quantum analog of the celebrated Frobenius-Perron theorem for stochastic matrices. Complete positivity provides additional constraint for the spectrum of $\mathcal{L}$ and in particular for the relaxation rates Γ_α. The relaxation properties of GKLS generators were further studied in [7,12] and more recently in, e.g., [13,14]. Some constraints for relaxation rates for 3- and 4-level systems were presented in [15–17]. Interestingly, the authors of a seminal paper [5] already observed that, for a qubit evolution governed by the following well-studied generator

$$\mathcal{L}(\rho) = -i\frac{\epsilon}{2}[\sigma_z, \rho] + \mathcal{L}_D(\rho) \tag{4}$$

with the dissipative part $\mathcal{L}_D = \gamma_+\mathcal{L}_+ + \gamma_-\mathcal{L}_- + \gamma_z\mathcal{L}_z$ consisting of pumping $\mathcal{L}_+(\rho) = \sigma_+\rho\sigma_- - \frac{1}{2}\{\sigma_-\sigma_+, \rho\}$, damping $\mathcal{L}_-(\rho) = \sigma_-\rho\sigma_+ - \frac{1}{2}\{\sigma_+\sigma_-, \rho\}$, and dephasing $\mathcal{L}_z(\rho) = \sigma_z\rho\sigma_z - \rho$, complete positivity implies the following well-known relation for the relaxation times $T_\alpha = 1/\Gamma_\alpha$:

$$2T_L \geq T_T, \tag{5}$$

where the longitudinal rate $\Gamma_L = \Gamma_3 = \gamma_+ + \gamma_-$ and the transversal rate $\Gamma_T = \Gamma_1 = \Gamma_2 = \frac{1}{2}(\gamma_+ + \gamma_-) + 2\gamma_z$. Equation (5) has been experimentally demonstrated to be true [7,18]. Clearly, the very condition [Eq. (5)] provides only partial information about the corresponding qubit generator. However, violation of Eq. (5) shows that the generator does not provide a legitimate CPTP evolution. The generator [Eq. (4)] is very special and in particular implies that the transversal rates Γ_1 and Γ_2 are the same. Note, that three rates Γ_k $(k = 1, 2, 3)$ satisfy $\Gamma_i + \Gamma_j \geq \Gamma_k$, where $\{i, j, k\}$ are all different. Actually, this relation was proven in [5] for any qubit generator provided the Hamiltonian and dissipative parts [Eq. (2)] commute. Clearly, the Eq. (4) generator belongs to this class. Finally, Kimura [11] showed that commutativity is not essential and that this relation is universally satisfied for any qubit generator.

To go beyond the qubit case, it is instructive to rephrase the relation $\Gamma_i + \Gamma_j \geq \Gamma_k$ as follows:

$$\sum_{j=1}^{3} \Gamma_j \geq 2\Gamma_i, \qquad (i = 1, 2, 3). \tag{6}$$

The condition shown in Eq. (6) is universal for any qubit generator [11]. For a purely dissipative generator, Wolf and Cirac derived the following result (Theorem 6 in [19]):

$$\|\mathcal{L}\| \leq \frac{2}{d}\sum_{\beta=1}^{d^2-1} \Gamma_\beta, \tag{7}$$

where $\|\mathcal{L}\|$ denotes the operator norm. Note that, due to $\|\mathcal{L}\| \geq |\ell_\alpha| \geq \Gamma_\alpha$, the above condition implies $\sum_{\beta=1}^{d^2-1}\Gamma_\beta \geq (d/2)\Gamma_\alpha$ for a purely dissipative generator. Recently, Kimura *et al.* [20] obtained the following universally valid constraint for any GKLS generator:

$$\sum_{\beta=1}^{d^2-1} \Gamma_\beta \geq \frac{d}{\sqrt{2}}\Gamma_\alpha; \quad \alpha = 1, \ldots, d^2 - 1. \tag{8}$$

In this Letter, we conjecture that the bound [Eq. (8)] can still be improved and propose the following.

Conjecture 1.—Any GKLS generator [Eq. (2)] for d-level quantum systems implies the following constraint for the relaxation rates:

$$\sum_{\beta=1}^{d^2-1} \Gamma_\beta \geq d\Gamma_\alpha; \quad \alpha = 1, \ldots, d^2 - 1. \tag{9}$$

Equivalently, introducing $\Gamma = \sum_{\beta=1}^{d^2-1}\Gamma_\beta$ and the relative relaxation rates $R_\alpha = \Gamma_\alpha/\Gamma$, we conjecture that $R_\alpha \leq (1/d)$. Moreover, the bound [Eq. (9)] is tight, i.e., cannot be improved.

Unfortunately, we still do not have a complete proof for $d \geq 3$. However, in this Letter, we construct a GKLS model that saturates Eq. (9). We also show in this Letter that this conjecture holds for several important classes of GKLS generators. In particular, any generator giving rise to the unital evolution, that is, $\mathcal{L}(\mathbb{1}) = 0$, satisfies Eq. (9). Unital (often called doubly stochastic) maps characterize decoherence processes that do not decrease entropy [21,22] and provide a direct generalization of unitary maps. A second important class is GKLS generators that display additional symmetry, that is, they are covariant w.r.t. a maximal abelian subgroup of the unitary group $U(d)$.

The formula [Eq. (2)] provides the most general mathematical structure of the generator compatible with the requirement of complete positivity and trace preservation. Note, however, that not every generator constructed according to Eq. (2) has a clear physical interpretation. There exists a natural class of generators of Markovian semigroups derived in the weak coupling limit (the so-called Davies generators) [1,7,23] and these do satisfy Conjecture 1. Hence, we may summarize that physically motivated generators derived from the appropriate microscopic model always satisfy Eq. (9).

This conjecture is also strongly supported by numerical analysis (cf. Fig. 1). Interestingly, the numerical analysis is perfectly consistent with the spectral properties of random GKLS generators in the large d limit [24].

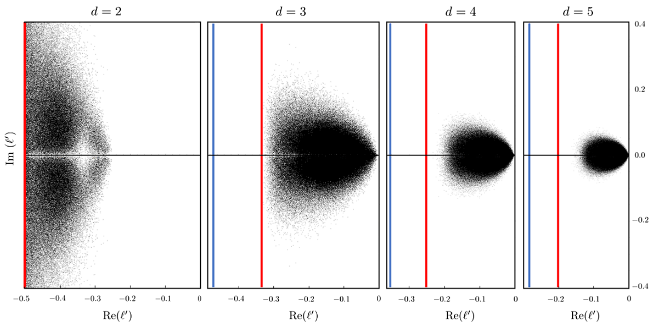

FIG. 1. Distributions of eigenvalues of random Lindbladians. For each $d = 2, 3, 4, 5$, we randomly generated 100000 GKLS generators and plotted the normalized eigenvalues $\ell' := \ell_\alpha/\Gamma$. Red vertical lines denote the bound "$-1/d$" corresponding to our conjecture, while blue ones denote the previously obtained bound, $-\sqrt{2}/d$ [20].

Clearly, the conjecture providing a universal constraint for relaxation rates is interesting by itself since relaxation rates are experimentally accessible and hence such constraints provide a direct method to check the validity of GKLS generators or the completely positive condition. The conjecture has, however, further interesting implications. It allows one to establish a universal constraint for eigenvalues of quantum channels (Conjecture 2). Moreover, it provides the necessary condition for a quantum channel Φ to be represented via $\Phi = e^{\mathcal{L}}$ for some GKLS generators [25]. We found that in this case all eigenvalues are constrained to a ring $r \leq |z_\alpha| \leq 1$, where the inner radius r is fully characterized by the original channel Φ (Conjecture 3).

Classical Pauli master equation.—Let us start our analysis with a classical counterpart of a master equation. Consider a Pauli rate equation for a classical system with d states

$$\dot{p}_i = \sum_{j=1}^{d} K_{ij} p_j, \tag{10}$$

where K is the standard classical generator satisfying the following Kolmogorov conditions [26]: $K_{ij} \geq 0$ $(i \neq j)$ and $\sum_{i=1}^{d} K_{ij} = 0$. It is, therefore, clear that K_{ij} can be represented as $K_{ij} = t_{ij} - \delta_{ij} \sum_{m=1}^{d} t_{mj}$, with $t_{ij} \geq 0$. Note that here only t_{ij} with $i \neq j$ are relevant, so in the following we put $t_{ii} = 0$. Equivalently, Eq. (10) can be formulated as follows:

$$\dot{p}_i = \sum_{j=1}^{d} (t_{ij} p_j - t_{ji} p_i). \tag{11}$$

Do we have a classical analog of Eq. (9)? Spectral properties of $d \times d$ matrix K_{ij} are similar to those of $\mathcal{L}$: there are d-complex eigenvalues $\ell_0^{\mathrm{cl}}, \ldots, \ell_{d-1}^{\mathrm{cl}}$ with $\ell_0^{\mathrm{cl}} = 0$. Moreover, $\Gamma_k^{\mathrm{cl}} = -\mathrm{Re}\ell_k^{\mathrm{cl}} \geq 0$, and the spectrum is symmetric w.r.t. the real axis. Interestingly, in the classical case there is no bound on the relative classical rates R_k^{cl}, that is, given a set of classical rates $\Gamma_k^{\mathrm{cl}} \geq 0$, one can construct a classical generator K_{ij} that does display exactly these rates. In particular, any single relative rate R_k^{cl} can be arbitrarily close to "1" [27].

Consider now a quantum evolution $t \to \rho(t)$ such that the diagonal elements $p_k = \langle k|\rho|k\rangle$ evolve according to the classical Pauli equation [Eq. (10)]. The evolution is governed by the following GKLS generator:

$$\mathcal{L}(\rho) = \sum_{i,j=1}^{d} t_{ij} |i\rangle\langle j|\rho|j\rangle\langle i| - \frac{1}{2}\{B, \rho\}, \tag{12}$$

where $B = \sum_k b_k |k\rangle\langle k|$ with $b_k = \sum_{j=1}^{d} t_{jk}$. The spectrum of $\mathcal{L}$ consists of *d-classical* eigenvalues of the classical generator represented by the matrix $K_{ij} = t_{ij} - \delta_{ij} b_j$: $\ell_0 = 0$, $\ell_1^{\mathrm{cl}}, \ldots, \ell_{d-1}^{\mathrm{cl}}$, and the remaining eigenvalues correspond to eigenvectors $|k\rangle\langle l|$

$$\mathcal{L}(|k\rangle\langle l|) = -\frac{1}{2}(b_k + b_l)|k\rangle\langle l|, \qquad (k \neq l). \tag{13}$$

Summarizing, a set of relaxation rates corresponding to $\mathcal{L}$ consists of *classical* rates $\Gamma_1^{\mathrm{cl}}, \ldots, \Gamma_{d-1}^{\mathrm{cl}}$ and the remaining *quantum* rates

$$\Gamma_{kl} = \frac{1}{2}(b_k + b_l), \quad (k \neq l). \tag{14}$$

Proposition 1.—The generator [Eq. (12)] satisfies Eq. (9).For the proof, see [27].

The bound is tight.—For any dimension d, one can construct $\mathcal{L}$ such that the Eq. (9) bound is attained. Indeed, consider the well-studied generator constructed via a double commutator

$$\mathcal{L}(\rho) = -[\Sigma, (\Sigma, \rho)] = 2\Sigma\rho\Sigma - \{\Sigma^2, \rho\} \tag{15}$$

for some Hermitian operator Σ. A well-known example is a qubit dephasing corresponding to $\Sigma = \sigma_z$. Let $\Sigma = \sum_k s_k |k\rangle\langle k|$ and assume that $s_1 \leq \ldots \leq s_d$. Then, one finds for the relaxation rates $\Gamma_{ij} = (s_i - s_j)^2$ with the maximal rate $\Gamma_{\max} = \Gamma_{1d}$. One shows [27] that

$$\sum_{i,j=1}^{d} \Gamma_{ij} \geq d\Gamma_{\max}, \tag{16}$$

which supports Conjecture 1 [Eq. (9)]. Moreover, taking $s_2 = \ldots = s_{d-1} = [(s_1 + s_d)/2]$, one finds $\sum_{i,j=1}^{d} \Gamma_{ij} = d\Gamma_{\max}$, or equivalently $R_{\max} = (1/d)$.

Universal formula for relaxation rates.—It is more convenient to proceed in the Heisenberg picture defined by the dual generator $\mathcal{L}^\ddagger$, which is related to the Schrödinger picture generator $\mathcal{L}$ via $\mathrm{Tr}[X\mathcal{L}(Y)] = \mathrm{Tr}[\mathcal{L}^\ddagger(X)Y]$ for any pair of operators $X, Y \in \mathcal{B}(\mathcal{H})$. Clearly, both $\mathcal{L}$ and $\mathcal{L}^\ddagger$ have the same spectrum ℓ_α but in general different eigenvectors. As was shown by Lindblad [6], any GKLS generator satisfies the dissipativity condition $D(X) \geq 0$ for all $X \in \mathcal{B}(\mathcal{H})$ where

$$D(X) = \mathcal{L}^\ddagger(X^\dagger X) - \mathcal{L}^\ddagger(X^\dagger)X - X^\dagger\mathcal{L}^\ddagger(X). \quad (17)$$

Inserting Eq. (2) for the generator, one finds [27]

$$D(X) = \sum_k \gamma_k [L_k, X]^\dagger [L_k, X]. \quad (18)$$

In particular, taking $X = Y_\alpha$, where $\mathcal{L}^\ddagger(Y_\alpha) = \ell_\alpha Y_\alpha$, one derives the following formula for relaxation rates [27]:

$$\Gamma_\alpha = \frac{1}{2\|Y_\alpha\|_\omega^2} \sum_k \gamma_k \|[L_k, Y_\alpha]\|_\omega^2, \quad (19)$$

where we introduced the inner product $(A, B)_\omega = \mathrm{Tr}(\omega A^\dagger B)$ and the corresponding ω norm $\|A\|_\omega^2 = (A, A)_\omega$. This formula is universal, that is, it holds for any GKLS generator (with faithful invariant state). Clearly, to compute Γ_α one has to know the corresponding eigenvector Y_α and the invariant state ω. In particular, since $Y_0 = \mathbb{1}$, one recovers $\Gamma_0 = 0$.

Unital semigroups: Starting from the universal formula, Eq. (19), we prove Eq. (9) for generators of the unital semigroup, i.e., semigroups satisfying $e^{t\mathcal{L}}(\mathbb{1}) = \mathbb{1}$. Unital semigroups enjoy several important properties. One proves [21,22] that $e^{t\mathcal{L}}$ is unital if and only if for any initial state ρ

$$\frac{d}{dt} S[e^{t\mathcal{L}}(\rho)] \geq 0, \quad (20)$$

where $S(\rho)$ stands for the von-Neumann entropy (actually it holds for the Rényi and Tsallis entropies as well). The corresponding generator satisfies $\mathcal{L}(\mathbb{1}) = 0$, which is equivalent to

$$\sum_k \gamma_k L_k^\dagger L_k = \sum_k \gamma_k L_k L_k^\dagger. \quad (21)$$

In particular Eq. (21) holds when all Lindblad operators L_k are normal ($L_k L_k^\dagger = L_k^\dagger L_k$). Inserting $\omega = \mathbb{1}/d$ into Eq. (19), one obtains

$$\Gamma_\alpha = \frac{1}{2\|Y_\alpha\|^2} \sum_k \gamma_k \|[L_k, Y_\alpha]\|^2, \quad (22)$$

where now $\|A\|^2 = \mathrm{Tr}(A^\dagger A)$. To prove Eq. (9), we use the following intricate inequality [28]:

$$\|[A, B]\|^2 \leq 2\|A\|^2\|B\|^2. \quad (23)$$

Actually, this inequality was conjectured by Böttcher and Wenzel [29] in 2005 (see [28] for more details). A simpler proof can be found in [30]. It should be stressed that the bound in [20] was shown by the direct use of this inequality as well. The inequality presented in Eq. (23) immediately implies

$$\Gamma_\alpha \leq \sum_k \gamma_k \|L_k\|^2. \quad (24)$$

Assuming the normalization $\|L_k\|^2 = 1$ as well as the condition $\mathrm{Tr}L_k = 0$ without loss of generality, one shows [27] that $\sum_k \gamma_k = (1/d)\sum_\alpha \Gamma_\alpha$, and hence Eq. (24) reproduces Eq. (9). Thus, we have shown the following.

Theorem 1.—The GKLS generator of a unital semigroup satisfies Eq. (9).

A class of covariant generators.—Symmetry plays a key role in modern physics. In many cases, it enables one to simplify the problem and often leads to a much deeper understanding and a more elegant mathematical formulation. Let us consider a class of generators covariant w.r.t. the maximal commutative subgroup of the unitary group $U(d)$:

$$U_{\mathbf{x}}\mathcal{L}(X)U_{\mathbf{x}}^\dagger = \mathcal{L}(U_{\mathbf{x}} X U_{\mathbf{x}}^\dagger), \quad (25)$$

where $U_{\mathbf{x}} = \sum_{k=1}^d e^{-ix_k}|k\rangle\langle k|$, and $\mathbf{x} = (x_1, \ldots, x_d) \in \mathbb{R}^d$. Any generator satisfying Eq. (25) has the following form:

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_2, \quad (26)$$

where $\mathcal{L}_0(\rho) = -i[H, \rho]$, together with

$$\mathcal{L}_1(\rho) = \sum_{i \neq j}^d t_{ij}|i\rangle\langle j|\rho|j\rangle\langle i| - \frac{1}{2}\{B, \rho\}$$
$$\mathcal{L}_2(\rho) = \sum_{i,j=1}^d d_{ij}|i\rangle\langle i|\rho|j\rangle\langle j| - \frac{1}{2}\{D, \rho\}, \quad (27)$$

where the Hamiltonian $H = \sum_i h_i|i\rangle\langle i|$, $B = \sum_j b_j|j\rangle\langle j|$, with $b_j = \sum_i t_{ij}$, and $D = \sum_{i=1}^d d_{ii}|i\rangle\langle i|$ [27]. This is a GKLS generator iff $t_{ij} \geq 0$ and the Hermitian matrix $[d_{ij}]_{i,j=1}^d$ is positive definite. Clearly, $\mathcal{L}_1$ is a standard generator considered before and $\mathcal{L}_2$ just adds pure decoherence.

Proposition 2.—The generator, Eq. (26), satisfies Eq. (9). For the proof, see [27].

Markovian semigroup in the weak coupling limit.—Any legitimate generator of CPTP semigroup has a GKLS form [Eq. (2)]. However, not every such generator has a clear physical interpretation. Davies [23,31] showed that, if the (open) quantum system is weakly coupled to the environment, by performing the so-called weak coupling limit one eventually derives a Markovian generator of exactly the GKLS form but with a clear physical meaning derived from the proper microscopic model (cf. [1,7,32,33]). Actually, if the invariant state ω has a nondegenerate spectrum (generic situation), then the corresponding generator derived in the weak coupling limit satisfies Eq. (25), and moreover $[\omega, U_{\mathbf{x}}] = 0$, that is, $\omega = \sum_k \omega_k |k\rangle\langle k|$. An additional property of such a generator is a quantum detailed balance condition [7], which in this case reduces to $t_{ik}\omega_k = t_{ki}\omega_i$, which is, however, not essential for Eq. (9). Hence, we may conclude that a class of physically legitimate GKLS generators defined via a weak coupling limit does satisfy Eq. (9).

Implications.—Provided our conjecture is true, for what is it useful? In response, we note that it enables one to characterize the spectra of quantum channels. Indeed, if Φ is a quantum channel (CPTP map), then $\mathcal{L}(\rho) = \Phi(\rho) - \rho$ defines a legitimate GKLS generator [5]. An example of such a generator is just qubit dephasing '$\sigma_z\rho\sigma_z - \rho$'. Now, let $z_\alpha = x_\alpha + iy_\alpha$ denote eigenvalues of Φ. Clearly, they belong to the unit disk $|z_\alpha| \leq 1$ and $z_0 = 1$. It is therefore clear that Conjecture 1 implies the following.

Conjecture 2.—The spectrum $z_\alpha = x_\alpha + iy_\alpha$ of any quantum channel satisfies $\sum_{\beta=1}^{d^2-1} x_\beta \leq d(d-1) - 1 + dx_\alpha$ for $\alpha = 1, \ldots, d^2 - 1$.Since Conjecture 1 holds in the qubit case, one has the following.

Proposition 3.—The spectrum $z_\alpha = x_\alpha + iy_\alpha$ of any qubit channel satisfies

$$|x_1 \pm x_2| \leq 1 \pm x_3. \tag{28}$$

In particular, for the Pauli channel $\Phi(\rho) = \sum_{\alpha=0}^{3} p_\alpha \sigma_\alpha \rho \sigma_\alpha$, one has $z_k = x_k$ and Eq. (28) is equivalent to the celebrated Fujiwara-Algoet conditions [34]. Clearly, Conjecture 2 holds for all unital channels.

A second immediate implication of Conjecture 1 relates to the problem of deciding whether a given quantum channel Φ can be represented as $\Phi = e^{\mathcal{L}}$ for some GKLS generator $\mathcal{L}$ [19,25]. Our original Conjecture 1 implies the following.

Conjecture 3.—If $\Phi = e^{\mathcal{L}}$, then the spectrum z_α of Φ satisfies

$$\det \Phi = z_1 \ldots z_{d^2-1} \leq |z_\alpha|^d, \tag{29}$$

for $\alpha = 1, \ldots, d^2 - 1$.Interestingly, Conjecture 3 shows that all z_α are not only constrained to the unit Frobenius disk but belong to the ring

$$\sqrt[d]{\det \Phi} \leq |z_\alpha| \leq 1. \tag{30}$$

Clearly, Conjecture 3 is satisfied for all qubit channels and all unital channels. In particular, for a qubit Pauli channel, all eigenvalues z_α are real, and hence Eq. (29) reduces to the following simple condition: $z_i z_j \leq z_k$, where $i, j, k \in \{1, 2, 3\}$ are all different. This condition was recently derived in [35,36].

Conclusions.—In this Letter, we propose a conjecture for the universal constraint for relaxation rates Γ_α of a quantum dynamical semigroup. It is shown that the conjecture is supported by several well-studied examples of quantum semigroups, including unital (doubly stochastic) evolution and semigroups derived in the weak coupling limit. Moreover, the conjecture is strongly supported by numerical analysis (cf. Fig. 1). We would like to emphasize that the universality, i.e., the model-independent property, and its experimental accessibility are of particular importance. Any violation of the conjectured constraint in the experiment immediately implies the impossibility of realizing the evolution via a completely positive Markovian semigroup (in the same spirit as a violation of Bell inequalities forbids the explanation in terms of a local realistic model). In this sense, the presented conjecture provides a physical manifestation of complete positivity.

Note that our analysis may be immediately generalized for the time-dependent case where the dynamics is governed by a time-dependent generator $\mathcal{L}_t$. In particular, one may analyze the issue of non-Markovian evolutions (cf. recent reviews [37–40]) having access to local relaxation rates $\Gamma_\alpha(t)$. A violation of Eq. (9) demonstrates the non-Markovianity of the corresponding evolution, that is, it shows that the corresponding dynamical map Λ_t cannot be represented via $\Lambda_t = V_{t,s}\Lambda_s$ with completely positive and trace-preserving propagators $V_{t,s}$ for $t > s$ [41–43].

Finally, our conjecture also has two important implications: it provides (i) a universal constraint for the spectra of quantum channels and (ii) a necessary condition for deciding whether a given channel Φ is consistent with the Markovian evolution $\Phi = e^{\mathcal{L}}$.

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