Fractionalization and Dynamics of Anyons and Their Experimental Signatures in the $\nu = n + 1/3$ Fractional Quantum Hall State

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We show the low-lying excitations at filling factor $\nu = n + 1/3$ with realistic interactions can be understood as quantum fluids with "Gaffnian quasiholes" as the proper elementary degrees of freedom. Each Laughlin quasihole can thus be understood as a bound state of two Gaffnian quasiholes, which in the lowest Landau level (LLL) behaves like "partons" with "asymptotic freedom" mediated by neutral excitations acting as "gluons." Near the experimentally observed nematic FQH phase in higher LLs, quasiholes become weakly bound and can fractionalize with rich dynamical properties. By studying the effective interactions between quasiholes, we predict a finite temperature phase transition of the Laughlin quasiholes even when the Laughlin ground state remains incompressible, and derive relevant experimental conditions for its possible observations.

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Strongly interacting topological systems in low dimensions open doors to many exotic physics, particularly from the topological and geometric properties of low-lying excitations [1–3]. The fractional quantum Hall effect (FQHE) [4] is one such example, where a wide variety of topological phases can be realized from interactions between electrons under a strong magnetic field perpendicular to a two-dimensional manifold [5,6]. A number of innovative techniques have been developed to understand both the universal topological properties and their dynamical robustness [7–14]. These are all fundamentally nonperturbative approaches, because the kinetic energy of the system is completely quenched by the magnetic field, leaving behind only the effective interaction in a single Landau level (LL) [15].

Given that the electrons themselves are no longer good degrees of freedom, the challenge is to find suitable "elementary particles" not perturbatively connected to electrons. Approaches along this line include the hierarchical pictures of the FQHE [7–9,16], and later the composite fermion (CF) theory successfully explaining many experimental observations, especially in the lowest LL (LLL) for Abelian FQH states [10–12]. Extension to higher LLs (e.g., with the parton theory) and non-Abelian FQH states are also possible [17–20], though they are technically more involved presumably because CFs also start interacting strongly. Alternative approaches with the Jack polynomial formalism [13,21,22] and the local exclusion constraint (LEC) as a generalization [14,23,24] seek a more microscopic understanding of the FQHE. There many universal topological properties of the FQHE can be determined algebraically without involving specific local operators (e.g., Hamiltonians). Interestingly, after identifying the model wave functions from this algebraic approach, we can in many cases construct model Hamiltonians for which the model wave functions are exact zero energy states [9,25,26]. The algebraic approach is particularly useful in understanding non-Abelian FQHE, and has fundamental connections to the conformal field theory [27,28]. In this Letter, we propose to understand and analyze the dynamics of the FQH phases using the quasiholes of apparently unrelated FQH phases, justified by the microscopic and algebraic relations revealed by the LEC construction [14] and easily verifiable with numerical calculations. We focus on the familiar Laughlin phase at filling factor $\nu = n + 1/3$, and predict rich dynamical phenomena with realistic interactions. In addition, the experimentally observed nematic FQH phase [29] can be identified as the quantum critical point (QCP) separating the conventional Laughlin phase and a Haffnian-like phase. Near the QCP, Laughlin quasiholes can fractionalize into "Gaffnian" [30] quasiholes carrying $e/6$ charge each. This is interesting given that the Gaffnian state is the subject of numerous studies due to its gapless nature [30–34]. Detailed analysis of the interactions between quasiholes predicts a finite-temperature quasihole phase transition with several experimental signatures.

Low-lying excitations in the Laughlin phase.—In the LLL we can understand the physics of the Laughlin phase from its model Hamiltonian (the $\hat{V}_1^{\mathrm{2bdy}}$ Haldane pseudopotential). However, it is less clear what can happen when the Hall plateau is observed with realistic interactions far away from $\hat{V}_1^{\mathrm{2bdy}}$. We first establish here for a wide range of interactions, the ground state and the low-lying excitations at $\nu = n + 1/3$ live (almost) entirely within the Gaffnian

TABLE I. The overlap of the 1LL ground state with the Laughlin state, $\mathcal{O}^L = |\langle\psi_{1LL}|\psi_{\text{Laughlin}}\rangle|$ and the total overlap with the GQ subspace $(\mathcal{O}_G(|\psi\rangle) = \sqrt{\sum_{|\phi\rangle\in\mathcal{H}_G}|\langle\psi_k|\phi\rangle|^2})$ for the Laughlin ground states and one-quasihole states. The last row shows the dimension of the Gaffnian subspace used for calculation compared to the dimension of the full Hilbert space in the corresponding L sector.

(N_e, N_o)	(9,25)	(9,26)	(10,28)	(10,29)	(11,31)
$\mathcal{O}_L$	0.48	0.45	0.54	0.47	0.70
$\mathcal{O}_G$	0.97	0.97	0.97	0.89	0.97
$\dim(\mathcal{H}_G)/\dim(\mathcal{H})$	0.143	0.135	0.091	0.077	0.039

quasihole (GQ) subspace. This subspace is defined algebraically with LEC condition $\{2,1,2\}\vee\{5,2,5\}$ [14,23]. It coincides with the null space (the GQs) of the Gaffnian model Hamiltonian $\hat{H}_g$, so we denote the subspace as $\mathcal{H}_G$.

We demonstrate this by calculating the ground state of the 1LL Coulomb interaction $\hat{V}_{1\text{LL}}$, and show that its cumulative overlap with $\mathcal{H}_G$ is close to unity (see Table I). One should note in the thermodynamic limit, the GQ subspace is of measure zero in the full Hilbert space. Even for finite systems, it is quite nontrivial to have such high overlap decreasing slowly with system sizes within a subspace containing a small fraction of states. In contrast, the model wave functions for the Laughlin ground state and quasihole states have rather poor overlap to the true eigenstates of the interaction [17,20,35].

We can also look at the spectrum of $\hat{V}_{1LL}$ within $\mathcal{H}_G$, and find the low lying energies to approximate the exact energies from the full Hilbert space very well, unlike the variational energies from Laughlin model wave functions [36]. The numerical evidence strongly suggests in a wide range of realistic interactions where interesting physics are observed at $\nu = n + 1/3$, the low-lying excitations are quantum fluids made of "GQs." At this filling factor, each GQ carries a charge of $e/6$ with respect to the Laughlin ground state (i.e., a charge of $e/5$ with respect to the Gaffnian ground state) [36]. Thus a Laughlin quasihole (LQ) can be viewed as a bound state of two GQs.

Dynamics of Gaffnian quasiholes.—We know that $\mathcal{H}_G$ is spanned by microscopic wave functions in the form of fermionic Jack polynomials $J_\lambda^\alpha(z_1, z_2, \ldots, z_{N_e})$ with $\alpha = -3/2$ and root configurations λ satisfying no more than two electrons for every five consecutive orbitals [13,39,40]. The Gaffnian ground state has the root configuration $\lambda = 1100011000\ldots110001100011$. Here each digit corresponds to an electron orbital on the Haldane sphere [9], with the left most digit corresponding the north pole and the right most digit the south pole. The digit "1" implies the orbital is occupied by an electron, while the digit "0" means the orbital is empty.

Many physical properties of the state can be read off from the root configuration [36]. The quasiholes can be created by inserting fluxes (or 0's) to the ground state root configuration. Adding one flux creates two quasiholes, with two examples as follows:

$$ {}_{\circ\circ}011000\cdots1100011, \qquad {}_{\circ}10100\cdots10100101_{\circ} \tag{1}$$

Here the positions of the quasiholes are marked by empty circles [41]. The two quasiholes can either form a bound state or be separated, given by the root configuration on the left and right in Eq. (1), respectively.

We now turn our attention to the Laughlin ground state, which has the root configuration

$$ {}_{\circ\circ}100{}_{\circ}1{}_{\circ}00{}_{\circ}1{}_{\circ}00{}_{\circ}1{}_{\circ}00\cdots{}_{\circ}100{}_{\circ}100{}_{\circ}100{}_{\circ}1_{\circ\circ} \tag{2}$$

The Laughlin ground state can be seen as a rotationally invariant quantum fluid of GQs on the sphere. It has an excess of $N_e/2+1$ orbitals (where N_e is the electron number) as compared to the Gaffnian ground state, implying the number of GQs it contains is $N_{qh}^G = N_e + 2$. Adding one flux into Eq. (2) at the north pole yields one LQ as follows [41]:

$$ \overset{\triangledown}{\otimes\otimes}{}_{\circ\circ}0100{}_{\circ}1{}_{\circ}00{}_{\circ}1{}_{\circ}00{}_{\circ}100\cdots{}_{\circ}100{}_{\circ}100{}_{\circ}100{}_{\circ}1_{\circ\circ} \tag{3}$$

Here the LQ is denoted by the empty triangle. Recall that adding one flux is also equivalent to adding two GQs—the *additional* GQs are denoted by red crossed circles. In this configuration, the two GQs are on top of each other, forming the bound state at the north pole that is the LQ.

Splitting the GQ pair can be achieved by violating the admissibility rule of the Laughlin state, while still satisfying the admissibility rule of the Gaffnian state. One example is given as follows:

$$ {}_{\otimes\circ\circ}\overset{\triangledown}{0}1\overset{\blacktriangledown}{0}1\overset{\triangledown}{0}0\overset{\blacktriangledown}{0}1\overset{\triangledown}{0}1\overset{}{0}0\overset{\triangledown}{0}\cdots1\overset{\blacktriangledown}{0}1\overset{\triangledown}{0}0\overset{\blacktriangledown}{0}1\overset{\triangledown}{0}1\overset{}{0}{}_{\circ\circ\otimes} \tag{4}$$

The solid triangles give the positions of Laughlin quasiparticles [41]. Given any microscopic Hamiltonian, the variational energies of the corresponding many-body states show if the two GQs are attractive or repulsive. We can clearly see that such interaction is mediated by neutral excitations, or LQ-quasiparticle pairs. Thus with $\hat{V}_1^{\text{2bdy}}$, the GQs are strongly attractive with the interaction energy *increasing* with the quasihole separation, mimicking the "asymptotic freedom" for quark system where the neutral excitations play the role of "gluons." With $\hat{V}_3^{\text{2bdy}}$, however, GQs are repulsive (see Fig. 1). Thus with realistic interactions, the GQs can be either bound, weakly bound, or unbound.

Nematic FQH state as the quantum critical point.—Let us first compare the Laughlin model state in Eq. (2) with the Haffnian model state with the following root configuration [42,43]:

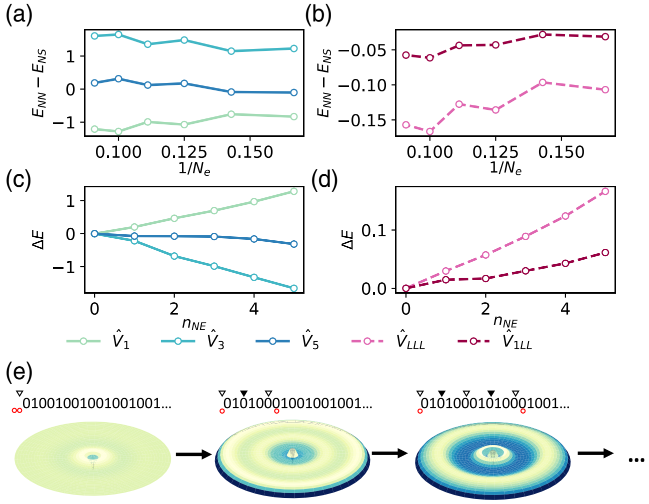

FIG. 1. (a)–(b) Variational energy difference between bound (E_{NN}) and unbound (E_{NS}) GQs, plotted against system size. (c)–(d) The energy cost to separate the two GQs plotted against the number of neutral excitations between them, for systems with 10 electrons and 28 orbitals. (e) Electron density of the quasihole wave functions. The center of the disk corresponds to the north pole of the sphere. The leftmost Laughlin state has a bound quasihole at the center, shown by the dip in electron density. As the GQs are pulled apart, neutral excitations are formed, shown by the ripples around the center.

$$1100\underset{\circ\circ}{00}1100\underset{\circ\circ}{00}\cdots 1100\underset{\circ\circ}{00}11 \tag{5}$$

Both the many-body states of Eqs. (2) and (5) are zero energy states of $\hat{H}_g$, and are thus linear combinations of the Gaffnian Jack polynomials. From the reduced density matrix near the north pole, we can see that the Laughlin state is made of bound GQs, but the Haffnian state is made of unbound GQs. From Fig. 1 we thus expect the Haffnian state to have *extensive* variational energy with respect to $\hat{V}_1^{\mathrm{2bdy}}$.

We can now have a better understanding of the dynamics of the magnetoroton modes for the Laughlin phase at $\nu = 1/3$, with the following root configurations [22,44]:

$$11\underset{\circ\circ\circ\circ}{0000}1\underset{\circ}{0}01\underset{\circ}{0}0100\cdots \qquad L = 2 \tag{6}$$

$$1100\underset{\circ}{0}1\underset{\circ\circ}{00}0\underset{\circ}{1}00100\cdots \qquad L = 3 \tag{7}$$

$$\vdots$$

where L is the total angular momentum quantum number on the sphere. These are no longer Jack polynomials, but we know immediately from LEC that the entire branch of the magnetoroton mode lives in $\mathcal{H}_G$ (i.e., zero energy states of $\hat{H}_g$). The quadrupole excitation at $L = 2$ is also the zero energy state of the Haffnian Hamiltonian $\hat{H}_h$ [45]. With the $\hat{V}_1^{\mathrm{2bdy}}$ interaction, the quadrupole excitation has higher energy because it consists of an unbound pair of GQs. In contrast, the dipole excitations in the limit of large L define the incompressibility gap and consist of LQs as bound GQs. Thus the energy difference between quadrupole and dipole excitations results from the $\hat{V}_1^{\mathrm{2bdy}}$ favoring bound states of GQs.

The nematic FQH state, an experimentally observed phase where the quantum Hall plateau coexists with the anisotropic longitudinal transport at low temperature [29,46], is believed to result from the quadrupole excitations going soft at $\nu = 2 + 1/3$ [47,48]. Its underlying microscopic mechanism, however, is still not fully understood [45,48]. Here, we show that the quadrupole excitations going soft results from Hamiltonians favoring unbound GQ pairs. The physics can be captured by the following toy model *within* $\mathcal{H}_G$:

$$\hat{H}(\lambda_1, \lambda_2) = \hat{H}_{\mathrm{h}} + \lambda_1 \hat{V}_1^{\mathrm{2bdy}} + \lambda_2 \hat{V}_3^{\mathrm{2bdy}}. \tag{8}$$

Given the assumption that $\hat{V}_1^{\mathrm{2bdy}}$ is incompressible at $\nu = 1/3$, we know $\hat{H}(\lambda_1, 0)$ is also incompressible for any positive λ_1. With $\lambda_2 = 0$ and small λ_1, the dipole excitations (and thus the charge excitations) are gapped by both $\hat{H}_h$ and $\hat{V}_1^{\mathrm{2bdy}}$. The lowest energy excitation is given by the quadrupole excitation in the $L = 2$ sector. A "linear" dispersion in the even L sector can be seen [Fig. 2]. They correspond to the multiple quadrupole excitation states with unbound GQs, with the following root configurations [49]:

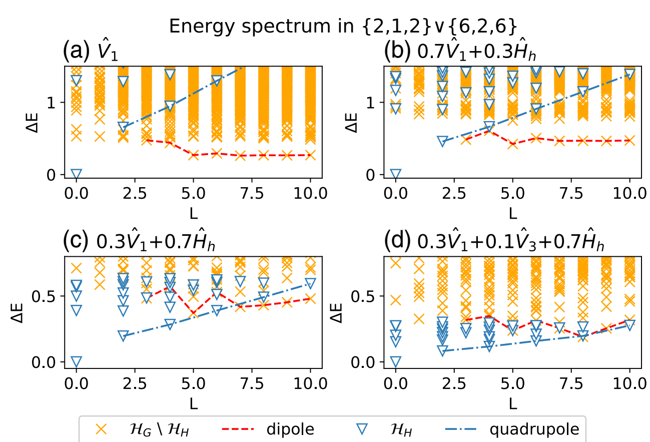

FIG. 2. Energy spectrum for system in the Haffnian subspace (blue triangles) and the complement of the Haffnian subspace in the Gaffnian subspace (orange crosses). The dipole and quadrupole excitation branches are highlighted in red and blue dotted lines, respectively. Results on a system with 10 electrons and 28 orbitals evaluated is shown here. The important features of this numerics are robust and consistent with the full ED spectrum [36].

$$1100001001001001001001001\cdots \qquad L=2, \tag{9}$$

$$1100001100001001001001001\cdots \qquad L=4. \tag{10}$$

$$\vdots$$

Let us denote the neutral gap of the system to be Δ_n, and the charge gap to be Δ_c (both with respect to the lowest energy state in the $L=0$ sector). The nematic FQH is realized in the regime of $\Delta_n \ll \Delta_T \ll \Delta_c$ as given by the $\lambda_1 \ll 1$, $\lambda_2 = 0$ model, where Δ_T is the energy scale of the temperature or disorder. We can also make the nematic FQH phase more robust by increasing λ_2. This is because $\hat{V}_3^{2\text{bdy}}$ punishes the Laughlin ground state and dipole excitations (they consist of bound GQs), while energetically favoring the quadrupole excitation [see Fig. 2(d)].

By increasing λ_2 from zero, we enter the regime where the quadrupole excitation becomes gapless. Here the dispersion of multiple quadrupoles becomes truly linear [50] in the long wavelength limit with an effective velocity v_g. At the QCP, $v_g = 0$, implying the Laughlin state and the Haffnian state become degenerate. We thus expect that tuning of λ_2 allows us to access a Haffnian-like phase at $\nu = 1/3$ with topological shift $S = -4$ (in contrast to $S = -2$ for the Laughlin phase) [51].

While Eq. (8) is artificial, it is actually more realistic than it appears. The mean-field interaction $\hat{H}_h^{2\text{bdy}} = \hat{H}_h + \hat{H}_h^*$, where $\hat{H}_h^*$ is the particle-hole conjugate, is a rather physical short range two-body interaction (consisting of $\hat{V}_1^{2\text{bdy}}$, $\hat{V}_3^{2\text{bdy}}$, $\hat{V}_5^{2\text{bdy}}$). In the thermodynamic limit, replacing $\hat{H}_h$ with $\hat{H}_h^{2\text{bdy}}$ in Eq. (8) may retain the qualitative features of the original model [52,53]. In higher LLs, three-body interactions also arise from LL mixing, which can play an important role to the physics near the QCP [54].

In Fig. 2 we show the transition between the dipole excitations and the quadrupole excitation when increasing λ_1, λ_2. While the mixing of the two branches of excitations complicates the dynamics of the neutral excitations, the softening of the quadrupole excitation at $L=2$ sector is not affected, since there is no competing single-dipole excitation (which starts at $L=3$) in this sector. The condition of $\Delta_n \ll \Delta_c$ is also maintained for a large parameter range.

Unlike the dipole excitations, states containing quadrupole excitations [e.g., Eq. (9) and Eq. (10)] have uniform electron density in the thermodynamic limit. They are thus conjectured to be the nematic Goldstone mode proposed in the effective field theories [47,55]. Thus for the effective field theory to be relevant to the nematic FQH, the microscopic interaction has to gap out all states *not* in $\mathcal{H}_G$. The effective theory also assumes $\Delta_n < 0 < \Delta_c$ with $v_g > 0$ in the nematic phase. Microscopically, since the ground state energy, quadrupole and dipole energies are fundamentally determined by the dynamics of GQs, they cannot really be independently tuned. The likely scenario relevant to the experiments is for $0 < \Delta_n < \Delta_T < \Delta_c$ with $v_g \sim \Delta_n l_B$. For $\Delta_n < 0$ we are no longer in the Laughlin phase and may not have a charge gap with $v_g < 0$.

Experimental signatures of quasihole fractionalization—Unlike $\hat{V}_1^{2\text{bdy}}$ or $\hat{V}_{\text{LLL}}$, the interaction close to the nematic FQH no longer heavily punishes unbound GQs. This can also been seen from the root configuration of an unbound pair of GQs in Eq. (4). The interaction between them is mediated by quadrupolelike neutral excitations (satisfying LEC condition $\{2,1,2\}\vee\{6,2,6\}$), instead of the dipolelike ones (satisfying $\{2,1,2\}\vee\{5,2,5\}$). Indeed, Eq. (8) can serve as the model Hamiltonian both for the quasihole fractionalization and the nematic FQH transition [56]. Thus near the nematic FQH phase, $\Delta_n \ll \Delta_c$ implies an incompressible phase given by Δ_c and thermally excited quasiholes of charge $e/6$ (with excitation energies given by Δ_n). Unlike individual quarks confined to very small length scales and are thus unobservable, at the Laughlin phase $e/6$ GQs are bound at the order of a few magnetic lengths. Near the QCP and at finite temperature the separation of GQs can be significantly larger. We thus expect $e/6$ charge to be observable in the bulk with single electron tunneling experiments [57] near the nematic FQH phase.

We also predict a quasihole phase transition at the incompressible Laughlin phase at some critical temperature, similar to the Berezinskii-Kosterlitz-Thouless (BKT) transition [58,59], evidenced by the softening of the quadrupole modes in Fig. 2. Let n, $n_{e/6}$ be the density of additional magnetic flux to the ground state, and of GQs with charge $e/6$, respectively. The average distance between any two GQs is thus $\bar{d} \sim 1/\sqrt{n_{e/6}}$. From Eq. (4) and the linear dispersion in Fig. 3, the average energy cost is proportional to the number of quadrupole excitations between two fractionalized GQs, with $\Delta E \simeq \bar{\Delta}_n \bar{d}^2$, $\bar{\Delta}_n = \Delta_n/(3\pi^2 l_B^2)$. We can thus define dimensionless quantities $\bar{n} = n_{e/6}/n$ and $\bar{\beta} = \beta n^{-1}\bar{\Delta}_n$, $\beta = (k_B T)^{-1}$, satisfying the following [36]:

$$\bar{n}(1 + e^{\bar{\beta}/\bar{n}}) = 2. \tag{11}$$

There is thus a critical temperature given by $\bar{\beta}_c = 0.55693$. The $\bar{n} > 0$ solutions, implying a finite density of $e/6$ quasiholes, only exist for $T > T_c$ with the following:

$$T_c = T_n \frac{2}{3\pi\bar{\beta}_c}\left(\frac{\delta B}{B_0}\right)^{-1} \sim 0.381 T_n \left(\frac{\delta B}{B_0}\right)^{-1}, \tag{12}$$

where $T_n = \Delta_n/k_B$ is the quadrupole gap temperature; B_0 is the magnetic field at the center of the $\nu = n + 1/3$ plateau, and δB is the deviation of the magnetic field from B_0 on the quasihole side. The solutions to Eq. (11) and the intuitive picture of this phase transition are illustrated in Fig. 3.

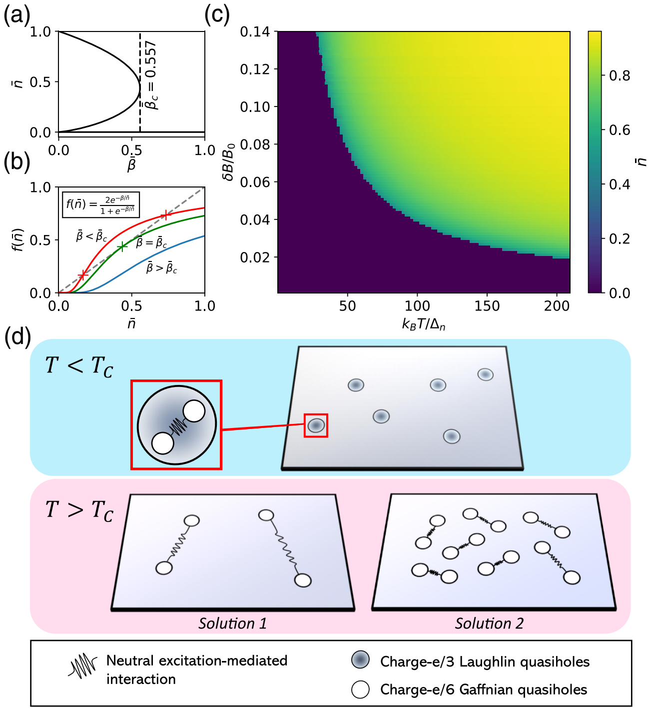

FIG. 3. (a) $\bar{n}$ as a function of reduced temperature $\bar{\beta}$. (b) Solutions of Eq. (11) are marked with crosses. At $\bar{\beta}_c$ there is a unique solution. (c) $\bar{n}$ as a function of δB and temperature. (d) When $T < T_C$, only LQs are present in the system; when $T > T_C$, there can be a lower GQ density solution (each GQ pair has a higher energy cost), and a higher unbound GQ density solution (with shorter average separation distance, hence lower energy cost for each GQ pair).

In experiments, lower T_c is preferred because the $e/6$ quasiholes are only observable if k_BT_c is smaller than the charge gap. We see from Eq. (12) that this can be done by lowering Δ_n, and from Fig. 2 that Δ_n can be lowered by adding long-range interaction (V_3^{2bdy}). Thus, we expect such a window to exist in higher LL and near the nematic FQH phase, where $\Delta_n \sim 0$ can be potentially realized in experiments [29,46–48]. Finally, we emphasize that the robustness of the Hall plateau at $\nu = n + 1/3$ in experiments does not automatically imply well-quantized quasiparticle charge of $e/3$ in shot noise or tunneling experiments.

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[49] The many-body wave functions can be uniquely determined by the root configuration, with details in the Supplemental Material [36].

[50] As long as the quadrupole gap vanishes as $1/N_e$ in the thermodynamic limit.

[51] With $\lambda_1, \lambda_2 > 0$ in Eq. (8), null space conformal invariance is broken and the CFT arguments [27] do not explicitly apply; whether or not this Haffnian-like phase can be incompressible is still an open question.

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[56] In particular, we can now prove analytically the fractionalized Laughlin quasihole state and the quadrupole excitations are exact zero energy states of the Haffnian model Hamiltonian, and the results will be published soon.

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