Entangling Power and Quantum Circuit Complexity

J. Eisert

Dahlem Center for Complex Quantum Systems, Freie Universität Berlin, 14195 Berlin, Germany
and Helmholtz-Zentrum Berlin für Materialien und Energie, 14109 Berlin, Germany

(Received 14 April 2021; accepted 7 June 2021; published 7 July 2021)

Notions of circuit complexity and cost play a key role in quantum computing and simulation where they capture the (weighted) minimal number of gates that is required to implement a unitary. Similar notions also become increasingly prominent in high energy physics in the study of holography. While notions of entanglement have in general little implications for the quantum circuit complexity and the cost of a unitary, in this work, we discuss a simple such relationship when both the entanglement of a state and the cost of a unitary take small values, building on ideas on how values of entangling power of quantum gates add up. This bound implies that if entanglement entropies grow linearly in time, so does the cost. The implications are twofold: It provides insights into complexity growth for short times. In the context of quantum simulation, it allows us to compare digital and analog quantum simulators. The main technical contribution is a continuous-variable small incremental entangling bound.

DOI: 10.1103/PhysRevLett.127.020501

The circuit complexity of a computation captures the number of elementary steps it minimally takes to determine its outcome. A reading of the famous Church-Turing thesis states that all reasonable models of computation give rise to the same class of "easy" problems computable in polynomial time, a statement that can presumably also be applied to processes occurring in nature. Alas, ultimately the world is quantum. Indeed, notions of quantum circuit complexity have long been considered in quantum information science: They provide a quantitative account on the shortest quantum computation that implements a given unitary. Similarly, one can think of the complexity of a quantum state as the circuit complexity of the quantum circuit preparing it, starting from a given fiducial state. Such notions play a similarly central role in quantum as classical circuit complexities do in classical computing. Seminal work [1–4] has introduced a geometric picture of circuit complexities, showing that finding the shortest circuit amounts to identifying the shortest path between two points in a curved geometry. In fact, this program has become so successful that the cost associated with a unitary in such a geometric picture has itself been identified with a notion of circuit complexity.

Yet, it was relatively recently that notions of circuit including those of costs rose to prominence outside the field of quantum computing [5–15]. Again eluding to the physical Church Turing thesis, such an approach is well motivated: One can think of a quantum state—say, one that is being generated by a quantum chaotic Hamiltonian evolution—being highly complex if the quantum circuit that could have prepared it on a quantum computer would have to be long. Since one can argue about how many quantum gates one would have needed to emulate a given

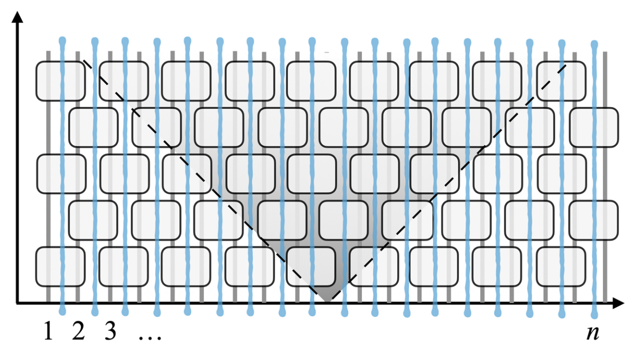

FIG. 1. A schematic picture relating the cost of a circuit with the entanglement over cuts for a system of n constituents. The dark gray triangle represents the Lieb-Robinson cone [32,33,36] that depicts at what rate one expects a linear growth of the entanglement entropy over all cuts in nonequilibrium dynamics generated by a local Hamiltonian.

Hamiltonian time evolution, such notions also immediately allow us to compare the effort in digital and analog quantum simulation [16]. The possibly most compelling application of quantum circuit complexity is in the realm of high energy physics in the context of holography [5–15].

These thoughts provide fuel for a motivation to actually compute quantum circuit complexities and circuit costs. Yet, to actually quantitatively determine any variant of these quantities is not obvious. After all, there are many ways to decompose a given unitary into a quantum circuit, with the best known algorithms for decomposing given circuits in Clifford and T gates featuring an exponential run-time in the circuit size [17], and the computation of the complexity requires the optimization over such decompositions. In any decomposition, one may expect cancellations of some sort, with the impact of a unitary gate being partially compensated by the later action of another, rendering naive combinatoric arguments involved. The geometrically motivated notion of a cost of a quantum circuit substantially lessens the technical burden [1–4], but it is still not obvious how to come up with meaningful lower bounds.

This work provides a compellingly simple lower bound for the cost of a quantum circuit that is tight for small values of the cost. It has indeed rightfully been argued that complexity is not entanglement [14], and neither is the cost of a circuit. No quantity based on entanglement can accommodate the presumed linear growth of state complexity until a time exponential in the system size [5], for obvious reasons. That said, for small values of the circuit cost and entanglement there is a simple connection: One can basically add up—if properly put together—potential entangling powers of quantum gates to arrive at tight bounds. The bounds presented are rooted in notions of entanglement capabilities of quantum gates: The argument captures the insight that quantum gates that are close to the identity in operator norm have little capability to create entanglement from product states (which is very easy to show). They can also add very little entanglement to a given entangled state (which is less obvious to prove), but grasped in terms of the small incremental entangling property [18,19], and which is here freshly proven for Gaussian continuous-variable systems. As such, the simple bound applies both to spin systems as to Gaussian bosonic continuous-variable settings, which are specifically important when approximating noninteracting bosonic quantum fields. Simple as the bound is, it is easily stated and proven (with some of the arguments delegated to the Supplemental Material [20]). It can be also straightforwardly applied to important cases of quantum evolutions, for which quenched Hamiltonian many-body dynamics constitute an example.

Quantum circuit complexity and cost.—The exact circuit complexity basically counts the number of quantum gates from a given gate set that is needed to exactly match the given unitary. An approximate reading thereof merely asks for an approximation in operator norm to a given small error. Lower bounds of the circuit complexity are provided by the *cost of a given circuit*, which is increasingly commonly seen as a notion of circuit complexity in its own right [1,2]. For n quantum systems of local dimension d ($d = 2$ for spins or qubits), one chooses a collection of 2-local traceless Hamiltonian terms $O_1, \ldots, O_J$, normalized in operator norm as $\|O_j\| = 1$ for $j = 1, \ldots, J$. We consider both the situation in which $\{O_j\}$ are geometrically local and the situation where they are merely local in their support. For a given $U \in \mathrm{SU}(d^n)$, one sees the unitary as being generated by a path-ordered integral

$$U = \mathcal{P}\exp\left(-i\int_0^1 ds H(s)\right), \tag{1}$$

with

$$H(s) = \sum_{j=1}^{J} y_j(s) O_j, \tag{2}$$

where $y_j : [0,1] \to \mathbb{R}$ are appropriate continuous cost functions. This path ordered integral can in operator norm be arbitrarily well approximated by

$$V_N = \prod_{k=1}^{N} \exp\left(-\frac{i}{N}\sum_{j=1}^{J} y_j(k/N) O_j\right), \tag{3}$$

in the limit of $N \to \infty$, as follows immediately from the definition of the path-ordered integral. The cost of a unitary $U \in \mathrm{SU}(d^n)$ can then be defined in such terms [1,2].

Definition 1.—(Circuit cost [1]) For a given set $\{O_1, \ldots, O_J\}$ in the Lie algebra $\mathrm{su}(d^n)$ of traceless Hermitian matrices normalized as $\|O_j\| = 1$ for all $j = 1, \ldots, J$, the *cost of a quantum circuit* $U \in \mathrm{SU}(d^n)$ is the infimum

$$C(U) \coloneqq \inf \int_0^1 \sum_{j=1}^{J} |y_j(s)| ds \tag{4}$$

over all continuous functions $y_j : [0,1] \to \mathbb{R}$ so that Eqs. (1), (2) are satisfied. We call it the geometrically local circuit cost $C_{\mathrm{g}}(U)$ if all $\{O_j\}$ are geometrically local.

That is to say, the cost of a quantum circuit expressed in terms of the limit

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \sum_{j=1}^{J} |y_j(k/N)| \tag{5}$$

of many time steps. In what follows, the notion of a potential entangling power of a quantum gate provides some useful intuition. It captures the "coupling strength" and simply takes into account the fact that gates that are close to the identity cannot create much entanglement from products. A somewhat related, but integer-valued, notion of entangling power has been invoked in Ref. [24].

Definition 2.—(Potential entangling power) A unitary $U \in \mathrm{SU}(d^2)$ has the potential entangling power

$$e(U) \coloneqq \log(d)\ \min\{\|H\| : U = e^{-iH}, H = H^\dagger\}. \tag{6}$$

It is indeed perfectly meaningful to refer to this quantity as the potential entangling power: If $\rho = \rho_A \otimes \rho_B$, both ρ_A and ρ_B being pure and supported on $\mathbb{C}^d$ each, then the resulting degree of entanglement $S[\mathrm{tr}_B(U\rho U^\dagger)]$ as quantified in terms of the von-Neumann *entanglement entropy* over the cut $A:B$ is expected to be small if $e(U)$ is small, and converging to zero for $e(U) \to 0$. Notions of entangling powers of quantum gates have long been connected to coupling strengths of interactions [25–28]. As is well known, notions of entangling power of unitary gates are altered depending on whether or not auxiliary quantum systems are allowed for: The swap gate obviously has no entangling power, if no auxiliary systems are made use of, while it has $2\log_2(d)$ when auxiliary systems are included. It is less obvious to see how much entanglement can be generated, however, if one initially already encounters an intricate entangled state and the unitary acts only on a small subsystem of the total system. The question of how much entanglement can be generated in this fashion has been largely settled in Ref. [18], however, which we can make use of here.

Entanglement bounds circuit costs.—Such notions of potential entanglement power can be related to tight bounds of circuit costs. In what follows, we denote for a pure state ρ defined on a spatially one-dimensional system of n constituents for $s \in \{1, \ldots, n-1\}$ with

$$E(\rho : s) \coloneqq S[\mathrm{tr}_B(\rho)], \tag{7}$$

being the *entanglement entropy* over the cut $A = \{1, \ldots, s\}$ and $B = \{s+1, \ldots, n\}$.

Observation 1.—(Entanglement lower bounds the cost) The geometrically local circuit cost of a $U \in \mathrm{SU}(d^n)$ is lower bounded by

$$C_{\mathrm{g}}(U) \geq \frac{1}{c\log(d)} \sum_{s=1}^{n-1} E(U|\phi\rangle\langle\phi|U^\dagger : s), \tag{8}$$

for an absolute constant $c > 0$, where $|\phi\rangle \in (\mathbb{C}^d)^{\otimes n}$ is a product state vector. For the cost one finds

$$C(U) \geq \frac{1}{c\log(d)} \max_s E(U|\phi\rangle\langle\phi|U^\dagger : s). \tag{9}$$

The *potential cancellation* of gates in notions of complexity is faithfully captured in this bound: If most gates in a circuit commute, they will give rise to a lower circuit cost, but at the same time also to a smaller entanglement. So even if the bound is simple indeed, it does capture a key feature of the relationship of the circuit cost to those of entanglement.

Proof.—The proof of this observation is straightforward, acknowledging the results of Ref. [18]. We start by decomposing the circuit in a convenient manner. Making use of a Trotter decomposition, we find that U can in operator norm $\|U - W_N\|$ arbitrarily well approximated as a product

$$W_N \coloneqq \prod_{k=1}^{N} V_k \tag{10}$$

with each term being given by

$$V_k \coloneqq \exp\left(-\frac{i}{N}\sum_{j=1}^{J} y_j(k/N) O_j\right) = \lim_{m\to\infty} (V_{k,1}^{1/m} \ldots V_{k,J}^{1/m})^m, \tag{11}$$

where

$$V_{k,j} \coloneqq \exp\left(-\frac{i}{N} y_j(k/N) O_j\right). \tag{12}$$

Building upon this, let

$$|\psi_l\rangle \coloneqq \prod_{k=1}^{l} V_k |\psi\rangle, \tag{13}$$

be the state vector after $l \in \{1, \ldots, N\}$ temporal layers, with $|\psi_0\rangle \coloneqq |\phi\rangle$. Then, for $l = 1, \ldots, N$, using the time integrated instance of Lemma 1, one finds that the entanglement growth over the cut $A = \{1, \ldots, s\}$ and $B = \{s+1, \ldots, n\}$ in each step can at most be

$$\begin{aligned} &E(|\psi_l\rangle\langle\psi_l| : s) - E(|\psi_{l-1}\rangle\langle\psi_{l-1}| : s) \\ &\quad = E(V_l|\psi_{l-1}\rangle\langle\psi_{l-1}|V_l^\dagger : s) - E(|\psi_{l-1}\rangle\langle\psi_{l-1}| : s) \\ &\quad \leq \frac{mc}{N}\sum_{j=1}^{J} \frac{1}{m} y_j(l/N)\|O_j\| \log(d) \end{aligned} \tag{14}$$

which gives

$$E(|\psi_l\rangle\langle\psi_l| : s) - E(|\psi_{l-1}\rangle\langle\psi_{l-1}| : s) \leq \frac{c\log(d)}{N}\sum_{j=1}^{J} |y_j(l/N)|. \tag{15}$$

Iterating this expression, one finds

$$E(U|\phi\rangle\langle\phi|U^\dagger : s) - E(|\phi\rangle\langle\phi| : s) \leq \frac{c\log(d)}{N}\sum_{k=1}^{N}\sum_{j=1}^{J} |y_j(l/N)|. \tag{16}$$

Acknowledging that the right-hand side approximates the circuit cost $C(U)$ arbitrarily well, find finds the latter statement of Observation 1, by applying the argument to the cut $A = \{1, \ldots, s\}$ and $B = \{s+1, \ldots, n\}$ providing the tightest bound. For the geometrically local circuit cost $C_{\mathrm{g}}(U)$, the argument can be applied to each such cut, leading to the proof of the statement of observation 1. ■

In the above statement, the following statement from Ref. [18] has been made use of.

Lemma 1.—(Small incremental entanglement [18]) For a pure state ρ and a Hamiltonian h supported on a $d \times d$-dimensional subspace acting over the cut $\{1, \ldots, s\}$ and $\{s+1, \ldots, n\}$, the entangling rate defined as

$$\Gamma(h,\rho) := \frac{d}{dt}E(e^{-ith}\rho e^{ith}:s)\Big|_{t=0} \tag{17}$$

is upper bounded by

$$\Gamma(h,\rho) \le c\log(d)\|h\|. \tag{18}$$

The constant presented in the proof is $c = 22$, but numerical evidence is presented that rather $c = 2$ actually provides a tight bound. Interpreted in terms of the above notion of a potential entangling power of a unitary $X \in U(d^2)$ acting on two constituents connecting both subsystems over the cut, one can argue that

$$|E(X\rho X^\dagger:s) - E(\rho:s)| \le ce(X), \tag{19}$$

so that up to an absolute constant, the maximum increase of entanglement is indeed nothing but the potential entangling power: In each application, a quantum gate with a certain potential entangling power can increase the value of entanglement only to some extent, no matter how entangled the initial state has been. From the above Trotter decomposition it also follows that the circuit cost is nothing but the weighted quantum circuit complexity, weighted by the potential entangling power of each quantum gate.

Corollary 1.—(Weighted quantum circuit complexity) For a given $U \in \mathrm{SU}(2^n)$, the infimum of the sum of weights $e(U_j)$ of a circuit consisting of quantum gates $\{U_j\}$ generated by $\{O_j\}$ is given by $C(U)$.

Gaussian circuit cost.—In fact, there is a small incremental entanglement bound as well as a harmonic equivalent of the above relationship between entanglement and quantum circuit cost for Gaussian bosonic settings [11,13], including ones motivated by evolutions of noninteracting bosonic quantum fields. For such bosonic systems, characterized by canonical coordinates $R = (x_1, p_1, x_2, p_2, \ldots, x_n, p_n)$, the Supplemental Material [20] present the proof of the following small incremental entanglement statement for such continuous-variable systems.

Theorem 1.—(Gaussian small incremental entanglement) For a pure Gaussian state ρ and a Hamiltonian $H = RhR^T$ supported on one of the modes each of $A = \{1, \ldots, s\}$ and $B = \{s+1, \ldots, n\}$, the entangling rate defined as

$$\Gamma(h,\rho) := \frac{d}{dt}E(e^{-itH}\rho e^{itH}:s)\Big|_{t=0} \tag{20}$$

is upper bounded by

$$\Gamma(h,\rho) \le \|h\| f(\|\gamma(0)\|), \tag{21}$$

where $f:[1,\infty) \to \mathbb{R}$ is a monotone increasing function.

Interestingly, it is not the operator norm of the Hamiltonian as such (which would make little sense anyway and would not be finite) but that of the kernel matrix when expressed as a polynomial in canonical coordinates that features in this small incremental entanglement statement. In the same way as above, and elaborated upon in the Supplemental Material [20], we can conclude the following.

Observation 2.—(Gaussian entanglement lower bounds Gaussian circuit cost) The geometrically local Gaussian quantum circuit cost of a bosonic Gaussian unitary U that prepares a state vector $U|\phi\rangle$ from the product state vector $|\phi\rangle$ associated with the covariance matrix $\gamma(0)$ is lower bounded by

$$G_{\mathrm{g}} \ge \frac{1}{f(\|\gamma(0)\|)}\sum_{s=1}^{n-1} E(U|\phi\rangle\langle\phi|U^\dagger:s). \tag{22}$$

For the Gaussian quantum circuit cost one finds

$$G \ge \frac{1}{f(\|\gamma(0)\|)}\max_s E(U|\phi\rangle\langle\phi|U^\dagger:s). \tag{23}$$

Making use of these statements, one can infer about noninteracting bosonic theories in largely the same way as for spin systems, despite the presence of unbounded operators.

Quenched quantum many-body systems.—Simple as the above bounds are, they provide tight and relevant bounds to circuit costs and complexities for small times in a number of settings. An interesting insight along these lines of thought is the point that whenever a quantum many-body system undergoing *nonequilibrium dynamics* leads to a linear increase in the entanglement entropy over suitable cuts, so does the quantum state complexity. This is in particular true for quenched quantum many-body systems, for which the linear growth of entanglement entropies is generic [29–31]. In fact, both upper [32,33] and lower bounds [34] for the entanglement entropy as a function of time have readily been established. That is to say, whenever the right-hand side of Eq. (9) grows linearly in time, so does the left-hand side, as an immediate corollary (see Fig. 1). We state this explicitly for the Ising Hamiltonian, but it should be clear that the same behavior is expected for any local Hamiltonian.

Observation 3.—(Growth of circuit cost in dynamics) For any time $T > 0$ there exists a system size n for a translationally invariant Ising Hamiltonian such that the unitary dynamics e^{-itH} applied to a product state vector $|\phi\rangle$ leads to $C(e^{-iHt}) > \delta t$ for an absolute constant $\delta > 0$, for all times $t \in [0, T]$.

The upper bound in time T is merely accommodating the possibility of having a finite system of finitely many degrees of freedom n, for which at some point, the respective entanglement entropies will no longer grow in time (rendering the bound then uninteresting). The result stated here is a corollary of observation 1, together with the results of Ref. [35]. Since the model is translationally invariant, any cut serves to show the linear growth of the quantum state complexity in time. For the geometrically local circuit cost, one also finds a growth linear in time, but now the largest value of $C_g(e^{-iHT})$ attained at intermediate times scales as $\Theta(n^2)$ in the system size n, instead of the essentially linear scaling $\Theta(n)$ in case of the quantity $C(e^{-iHT})$.

Summary and outlook.— In this work, we have carefully and quantitatively revisited the connection between entanglement and notions of circuit cost and complexity. While there is in general no tight connection between these quantities, for small values, there is, as this work shows: Indeed, one arrives at compellingly simple bounds. The usefulness of such bounds is manifest. One can argue, for example, how deep a weighted quantum circuit has to be at least to give rise to a given entanglement pattern in a desired final state; at least for pure states, but it seems perfectly conceivable to establish similar techniques for mixed quantum states. Also, it helps assessing the power and capabilities of analog quantum simulators [16]. Using such tools, one can argue that a digital quantum simulator would have required a precisely defined computational effort to produce the same results as a given analog quantum simulator. In this sense, it makes the computational effort of digital and analog quantum simulators comparable. It is the hope that this simple bound provides a useful and versatile tool in various studies of this kind.

I would like to warmly thank V. Balasubramanian, B. Chairo, S. Chapman, L. Hackl, R. C. Myers, and specifically M. Heller for discussions. This work has been supported by the DFG (CRC 183, project A03 and B01, FOR 2724, EI 519/14-1, and EI 519/15-1), the BMBF (DAQC), and the FQXi. It has also received funding from the European Union's Horizon 2020 research and innovation programme under Grant Agreement No. 817482 (PASQuanS).

[1] M. A. Nielsen, arXiv:quant-ph/0502070.
[2] M. A. Nielsen, M. R. Dowling, M. Gu, and A. M. Doherty, Science **311**, 1133 (2006).
[3] M. A. Nielsen, M. R. Dowling, M. Gu, and A. C. Doherty, Phys. Rev. A **73**, 062323 (2006).
[4] M. R. Dowling and M. A. Nielsen, arXiv:quant-ph/0701004.
[5] D. Stanford and L. Susskind, Phys. Rev. D **90**, 126007 (2014).
[6] A. R. Brown, D. A. Roberts, L. Susskind, B. Swingle, and Y. Zhao, Phys. Rev. Lett. **116**, 191301 (2016).
[7] S. Chapman, H. Marrochio, and R. C. Myers, J. High Energy Phys. 01 (2017) 062.
[8] A. R. Brown and L. Susskind, Phys. Rev. D **97**, 086015 (2018).
[9] A. Reynolds and S. F. Ross, Classical Quantum Gravity **34**, 105004 (2017).
[10] A. Reynolds and S. F. Ross, Classical Quantum Gravity **34**, 175013 (2017).
[11] S. Chapman, M. P. Heller, H. Marrochio, and F. Pastawski, Phys. Rev. Lett. **120**, 121602 (2018).
[12] R. A. Jefferson and R. C. Myers, J. High Energy Phys. 10 (2017) 107.
[13] S. Chapman, J. Eisert, L. Hackl, M. P. Heller, R. Jefferson, H. Marrochio, and R. C. Myers, SciPost Phys. **6**, 034 (2019).
[14] S. Aaronson, arXiv:1607.05256.
[15] F. G. S. L. Brandao, W. Chemissany, N. Hunter-Jones, R. Kueng, and J. Preskill, arXiv:1912.04297.
[16] J. I. Cirac and P. Zoller, Nat. Phys. **8**, 264 (2012).
[17] D. Gosset, V. Kliuchnikov, M. Mosca, and V. Russo, Quantum Inf. Comput. **14**, 1261 (2014).
[18] M. Mariën, K. M. R. Audenaert, K. Van Acoleyen, and F. Verstraete, Commun. Math. Phys. **346**, 35 (2016).
[19] K. Van Acoleyen, M. Mariën, and F. Verstraete, Phys. Rev. Lett. **111**, 170501 (2013).
[20] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.127.020501 for further details of the proof of observation 2, which includes Refs. [21–23].
[21] J. Eisert and M. B. Plenio, Int. J. Quantum. Inform. **1**, 479 (2003).
[22] C. Weedbrook, S. Pirandola, R. Garcia-Patron, N. J. Cerf, T. C. Ralph, J. H. Shapiro, and S. Lloyd, Rev. Mod. Phys. **84**, 621 (2012).
[23] J. Eisert, M. Cramer, and M. B. Plenio, Rev. Mod. Phys. **82**, 277 (2010).
[24] V. Balasubramanian, M. DeCross, A. Kar, and O. Parrikar, J. High Energy Phys. 02 (**2019**) 069.
[25] J. Eisert, K. Jacobs, P. Papadopoulos, and M. B. Plenio, Phys. Rev. A **62**, 052317 (2000).
[26] D. Collins, N. Linden, and S. Popescu, Phys. Rev. A **64**, 032302 (2001).
[27] J. I. Cirac, W. Dür, B. Kraus, and M. Lewenstein, Phys. Rev. Lett. **86**, 544 (2001).
[28] P. Zanardi, C. Zalka, and L. Faoro, Phys. Rev. A **62**, 030301 (R) (2000).
[29] J. Eisert, M. Friesdorf, and C. Gogolin, Nat. Phys. **11**, 124 (2015).
[30] A. Polkovnikov, K. Sengupta, A. Silva, and M. Vengalattore, Rev. Mod. Phys. **83**, 863 (2011).
[31] C. Gogolin and J. Eisert, Rep. Prog. Phys. **79**, 056001 (2016).
[32] J. Eisert and T. J. Osborne, Phys. Rev. Lett. **97**, 150404 (2006).
[33] S. Bravyi, M. B. Hastings, and F. Verstraete, Phys. Rev. Lett. **97**, 050401 (2006).
[34] N. Schuch, M. M. Wolf, F. Verstraete, and J. I. Cirac, Phys. Rev. Lett. **100**, 030504 (2008).
[35] N. Schuch, M. M. Wolf, K. G. H. Vollbrecht, and J. I. Cirac, New J. Phys. **10**, 033032 (2008).
[36] M. B. Hastings and T. Koma, Commun. Math. Phys. **265**, 781 (2006).