Editors' Suggestion

Thermal Buckling Transition of Crystalline Membranes in a Field

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Two-dimensional crystalline membranes in isotropic embedding space exhibit a flat phase with anomalous elasticity, relevant, e.g., for graphene. Here we study their thermal fluctuations in the absence of exact rotational invariance in the embedding space. An example is provided by a membrane in an orientational field, tuned to a critical buckling point by application of in-plane stresses. Through a detailed analysis, we show that the transition is in a new universality class. The self-consistent screening method predicts a second-order transition, with modified anomalous elasticity exponents at criticality, while the RG suggests a weakly first-order transition.

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Introduction.—Experimental realization of freely suspended graphene [1,2] and other exfoliated crystals launched a renaissance in the statistical mechanics of elastic membranes [3–13] and their electronic properties [14,15]. Theoretical interest is also motivated by the opportunity to explore rich interplay between field theory and geometry [10].

A striking prediction is the existence of a low-temperature stable "flat" phase of a tensionless *crystalline* membrane [3] that spontaneously breaks rotational symmetry of the embedding space. This is in stark contrast to canonical two-dimensional field theories where the Hohenberg-Mermin-Wagner theorems [16–18] preclude the spontaneous breaking of a continuous symmetry.

In such elastic membranes, in a spectacular phenomenon of order from disorder, thermal fluctuations instead stiffen the long-wavelength (k^{-1}) bending rigidity $\kappa_0 \to \kappa_0 k^{-\eta}$, $\eta > 0$, via a universal power-law "corrugation" effect, with membrane roughness scaling as $h_{\rm rms} \sim L^{\zeta}$, with $\zeta = (4 - D - \eta)/2$ [3,10], where D is membrane's internal dimension, with $D = 2$ for the physical case. The resulting anomalous elasticity is characterized by universal exponents, η, ζ, and $\eta_u = 4 - D - 2\eta$ is determined exactly by the underlying rotational invariance, with a scale dependent Young modulus $K_0 \to K_0 q^{\eta_u}$. This was predicted, together with the values of the exponents, by a variety of methods [3,4,6,7,13], verified in simulations [19] and continues to be explored experimentally [20].

Most theoretical studies to date have focused on stress-free fluctuating membranes in an isotropic embedding environment [3,4,6–8,12,13,21–25], as appropriate for, e.g., soft matter realizations in an isotropic fluid (but see spherical shells [26,27]). However, many experiments on solid-state membranes may be subjected to embedding space anisotropy and/or external stresses due to substrate [28–30], clamping [31–33], electric and magnetic fields [34,35], or by a nematic solvent [36–39]. In all previous theoretical descriptions, the rotational invariance in the embedding space was assumed and the response to boundary stress σ found to be controlled by the thermal tensionless membrane fixed point [4]. The case of weak field or stresses was treated as a cutoff for the isotropic critical fluctuations, beyond a large scale $\xi \sim (\kappa/\sigma)^{\nu}$, that diverges with a vanishing σ, where ν is a universal exponent that we compute below. Such perturbations lead to an anomalous response, which in the context of tension predicts a non-Hookean stress-strain relation $\varepsilon \sim \sigma^{\alpha}$, with $\alpha = (D-2+\eta)/(2-\eta) =_{D=2} \eta/(2-\eta)$ [5,6,12,13,24,40–42].

In this Letter we study geometries (Fig. 1), where the imposed stress and anisotropy lead to richer and universal buckling phenomenology. Generic buckling is a complex out-of-plane instability of a sheet subjected to compression, which results in a strongly distorted, nonperturbative state. Recent studies of isotropic buckling focused on effects of thermal fluctuations on Euler buckling, stabilized only by

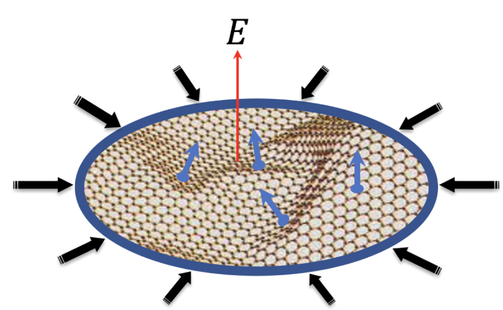

FIG. 1. A critical membrane tuned to a buckling transition by external in-plane isotropic stress $\sigma_{ij} = \frac{1}{2}\sigma\delta_{ij}$, stabilized by an external field $\vec{E}$, that aligns the normal.

finite size effects [32]. Instead, here we focus on a continuous anisotropic buckling, where the instability is controlled by a stabilizing external field. Specifically, we consider an externally oriented membrane tuned to a buckling transition by a compressional boundary stress applied within the plane explicitly selected by the orientational field [43]. The compressive stress can be tuned to a critical value, σ_c to cancel out at quadratic order the (embedding-space) rotational symmetry breaking fields. Our key observation is that, at this new buckling critical point (to which the isotropic flat membrane critical point [4] is unstable), although at harmonic order the membrane appears to be rotationally invariant and stress free, thus exhibiting strong thermal fluctuations, it admits new important elastic nonlinearities that are *not* rotationally invariant [44]. These lead to a critical membrane, tuned to the buckling point, that is qualitatively distinct from the conventional tensionless membrane [45].

Results.—Subjecting a crystalline membrane to a lateral compressive isotropic boundary stress σ, tuned to a critical tensionless buckling point σ_c and stabilized by an orienting field, we find a new buckling universality class, distinct from the isotropic tensionless membrane [3,4,6,7,13]. We use two complementary approaches to analyze the properties of the resulting critical state. The first is the self-consistent screening approximation (SCSA), which was found to provide an accurate description for the isotropic case [7,13]. Thermal fluctuations and elastic nonlinearities at the buckling transition lead to a universal anomalous elasticity with exponent

$$\eta^{\mathrm{anis}} = 0.754, \tag{1}$$

characterizing the divergence of the effective length-scale dependent bending rigidity $\kappa(k) \sim k^{-\eta}$. The in-plane elastic moduli remain finite at the critical point, i.e., $\eta_u^{\mathrm{anis}} = 0$ [47]. This is despite the fact that the five eigencouplings $w_i(q) \sim q^{4-D-2\eta}$ renormalize nontrivially, vanishing in the long wavelength limit. This is distinct from the tensionless isotropic membrane for which SCSA predicts universal exponents $\eta \approx 0.821$, $\eta_u \approx 0.358$ [7]. The corresponding roughness $h_{\mathrm{rms}} \sim L^{\zeta}$ of the critically buckled membrane is characterized by a universal roughness exponent

$$\zeta^{\mathrm{anis}} = 0.623, \tag{2}$$

and it is thus rougher than a tensionless isotropic membrane, with a roughness exponent $\zeta \approx 0.59$ [7].

We complement this SCSA calculation by an RG analysis in an expansion in $\epsilon = 4 - D$. It confirms the instability of the standard anomalous elasticity fixed point of the isotropic, tensionless membrane, under breaking of the embedding space rotational symmetry. As for the isotropic membrane, the elastic nonlinearities destabilize the harmonic theory beyond the length scale $\xi_{\mathrm{NL}}^{\mathrm{iso}} \sim (\kappa^2/TK_0)^{1/(4-D)}$. If the anisotropy perturbation is very weak, e.g., $w \sim \mu_{1,2}$, $\lambda_{1,2} \ll K_0$, the membrane still experiences the standard isotropic anomalous elasticity up to scales $\xi_{\mathrm{NL}}^{\mathrm{iso}}$, crossing over to the new anisotropic critical behavior beyond the crossover length

$$\xi_{\mathrm{NL}}^{\mathrm{anis}} = \xi_{\mathrm{NL}}^{\mathrm{iso}} \left(\frac{K_0}{w}\right)^{1/\rho}, \quad \rho = \frac{\epsilon d_c}{d_c + 24} + O(\epsilon^2), \tag{3}$$

where ρ is the crossover exponent obtained from linearization of the RG flow around the isotropic fixed point. If the anisotropy perturbation is stronger, the thermal fluctuations and elastic nonlinearities directly destabilize the harmonic theory at scales of order $\xi_{\mathrm{NL}}^{\mathrm{iso}}$. Beyond these scales, the RG flows to a new stable buckling critical point, which, within the ϵ expansion, is however accessible only for space codimension $d_c = d - D > 219$, analogous to the crumpling transition found by Paczuski *et al.* [48]. For the physical case, $d_c = 1$, we interpret the resulting runaway flows as a weakly first-order transition. The SCSA is exact for large d_c, and the two methods match in their common regime of validity.

Model of anisotropic membrane buckling.—The coordinates of the atoms in the d-dimensional embedding space are denoted $\vec{r}(\mathbf{x}) \in \mathbb{R}^d$, with the atoms labeled by their position $\mathbf{x} \in \mathbb{R}^D$ in the internal space. For graphene $D = 2$, and atoms span a triangular lattice, described here in the continuum limit. The deformations with respect to the flat sheet are described by D phonon fields $u_\alpha(\mathbf{x})$, and $d_c = d - D$ height fields $\vec{h} \in \mathbb{R}^{d_c}$ (orthogonal to the $\vec{e}_\alpha$) as $\vec{r}(\mathbf{x}) = [x_\alpha + u_\alpha(\mathbf{x})]\vec{e}_\alpha + \vec{h}(\mathbf{x})$, where the $\vec{e}_\alpha$ are a set of D orthonormal vectors. While the physical case corresponds to $d = 3$ and $d_c = 1$, it is useful to study the theory for a general d_c. The nonlinear strain tensor measures the deformation of the induced metric relative to the preferred flat metric, $u_{\alpha\beta} = \frac{1}{2}(\partial_\alpha \vec{r} \cdot \partial_\beta \vec{r} - \delta_{\alpha\beta}) \simeq \frac{1}{2}(\partial_\alpha u_\beta + \partial_\beta u_\alpha + \partial_\alpha \vec{h} \cdot \partial_\beta \vec{h})$ to the accuracy needed here, with the $O[(\partial u)^2]$ phonon nonlinearities irrelevant and therefore neglected. The tensor $u_{\alpha\beta}$ encodes full rotational invariance in the embedding space, its approximate form being invariant under infinitesimal rotations by θ, i.e., the $O(\theta^2)$ term vanishes under the (apparent) distortion $u_1 = x_1(\cos\theta - 1)$, $h_1 = x_1 \sin\theta$ of a rigid rotation, with a vanishing of the exact strain tensor.

Here we build on the model of a rotationally invariant tensionless membrane. Its Hamiltonian is the sum of curvature energy and in-plane stretching energy

$$\mathcal{F}_1[\vec{h}, u_\alpha] = \int d^D x \left[\frac{\kappa}{2}(\partial^2 \vec{h})^2 + \tau u_{\alpha\alpha} + \mu (u_{\alpha\beta})^2 + \frac{\lambda}{2}(u_{\alpha\alpha})^2\right] \tag{4}$$

where κ is the bending modulus, λ, μ the in-plane Lamé elastic constants. The parameter τ controls the preferred extension of the membrane in the $\vec{e}_\alpha$ plane.

Based on symmetry considerations, complemented by a model-building derivation below, external orientational and boundary stresses introduce new relevant elastic nonlinearities, with five independent couplings, that by symmetry lead to a modified effective Hamiltonian $\mathcal{F} = \mathcal{F}_1 + \mathcal{F}_2$, where $\mathcal{F}_2$ breaks rotational invariance in the embedding space,

$$\mathcal{F}_2[\vec{h}, u_\alpha] = \int d^D x \Big\{ \frac{\gamma}{2}(\partial_\alpha \vec{h})^2 + \frac{\lambda_1}{2}\partial_\alpha u_\alpha (\partial_\beta \vec{h})^2 + \frac{\lambda_2}{8}[(\partial_\alpha \vec{h})^2]^2 + \mu_1 \partial_\alpha u_\beta (\partial_\alpha \vec{h}\cdot\partial_\beta \vec{h}) + \frac{\mu_2}{4}[\partial_\alpha \vec{h}\cdot\partial_\beta \vec{h}]^2 \Big\}, \quad (5)$$

retaining in-plane isotropy and the $h \to -h$ invariance as a feature of our geometry, preserving the equivalence between the two sides of the membrane.

We now study the membrane with parameters tuned to the thermal buckling critical point defined by the renormalized $\gamma_R = 0$. Integrating over the in-plane phonon modes u_α and, rescaling for convenience all elastic constants by $1/d_c$, we obtain an effective Hamiltonian for the height field,

$$\mathcal{F}[\vec{h}] = \int d^D x \left[\frac{\kappa}{2}(\partial^2 \vec{h})^2 + \frac{\gamma}{2}(\partial_\alpha \vec{h})^2\right] + \frac{1}{4d_c}\int d^D x d^D y \times \partial_\alpha \vec{h}(\mathbf{x})\cdot\partial_\beta \vec{h}(\mathbf{x}) R_{\alpha\beta,\gamma\delta}(\mathbf{x}-\mathbf{y})\partial_\gamma \vec{h}(\mathbf{y})\cdot\partial_\delta \vec{h}(\mathbf{y}), \quad (6)$$

with a nonlocal quartic tensorial interaction, which in Fourier space is given by [49] $R_{\alpha\beta,\gamma\delta}(\mathbf{q}) = \sum_{i=1}^5 w_i (W_i)_{\alpha\beta,\gamma\delta}(\mathbf{q})$. The W_i are five projectors in the space of four index tensors, equal to bilinear combinations of $P^L_{\alpha\beta}(\mathbf{q}) = q_\alpha q_\beta / q^2$ and $P^T(\mathbf{q}) = \delta_{\alpha\beta} - P^L_{\alpha\beta}(\mathbf{q})$ projectors. The five "bare couplings" w_i are given in the Supplemental Material [53] in terms of the bare elastic moduli in (4) and (5), together with the basis tensors W_i [54]. The important features are the following. When rotational symmetry breaking is absent, $\gamma = 0$, $\mu_1 = \mu_2 = \lambda_1 = \lambda_2 = 0$, the couplings w_2, w_4, w_5 vanish and $w_1 = \mu$, $w_3 = \mu + (D-1)[\mu\lambda/(\lambda + 2\mu)]$, leading to

$$R_{\alpha\beta,\gamma\delta}(\mathbf{q}) = (w_3 - w_1)P^T_{\alpha\beta}P^T_{\gamma\delta} + w_1 \frac{1}{2}(P^T_{\alpha\gamma}P^T_{\beta\delta} + P^T_{\alpha\delta}P^T_{\beta\gamma}), \quad (7)$$

which is the usual quartic coupling associated to $\mathcal{F}_1$. When λ_1 and λ_2 are turned on, while $\mu_1 = \mu_2 = 0$, all w_i are nonzero except $w_2 = 0$. Finally, when all couplings in $\mathcal{F}_2$ are nonzero, all w_i are nonzero.

SCSA analysis.—The form (6) is suitable to apply the SCSA method, which is exact in the limit of large d_c, as detailed in the Supplemental Material [53] and parallels Sec. IVA of [13]. Consider the two point correlation of the height field in Fourier space, $\langle h^i(\mathbf{k})h^j(\mathbf{k}')\rangle = \mathcal{G}(k)(2\pi)^d \delta^d(\mathbf{k}+\mathbf{k}')\delta_{ij}$. If we neglect the quartic nonlinearities in (6) we find $\mathcal{G}(k) = G(k) = 1/(\gamma k^2 + \kappa k^4)$. The nonlinearities lead to a nonzero self-energy $\Sigma(k) = \mathcal{G}(k)^{-1} - \gamma k^2 - \kappa k^4$. Together with the renormalized interaction tensor, $\tilde{R}(\mathbf{q})$, it satisfies the SCSA equation

$$\Sigma(k) = \frac{2}{d_c}\int_q k_\alpha (k_\beta - q_\beta)(k_\gamma - q_\gamma)k_\delta \tilde{R}_{\alpha\beta,\gamma\delta}(\mathbf{q})\mathcal{G}(\mathbf{k}-\mathbf{q})$$

together with $\tilde{R}(\mathbf{q}) = R(\mathbf{q}) - R(\mathbf{q})\Pi(\mathbf{q})\tilde{R}(\mathbf{q})$ where $\Pi(\mathbf{q})$ encodes the screening of the in-plane elasticity by out-of-plane fluctuations

$$\Pi_{\alpha\beta,\gamma\delta}(\mathbf{q}) = \frac{1}{4}\int_p v_{\alpha\beta}(\mathbf{q},\mathbf{q}-\mathbf{p})v_{\gamma\delta}(\mathbf{q},\mathbf{q}-\mathbf{p})\mathcal{G}(\mathbf{p})\mathcal{G}(\mathbf{q}-\mathbf{p})$$

and $v_{\alpha\beta}(\mathbf{p},\mathbf{p}') = p_\alpha p'_\beta + p'_\alpha p_\beta$. One can decompose $\Pi(\mathbf{q}) = \sum_{i=1}^5 \pi_i(q)W_i(\mathbf{q})$ and $\tilde{R}(\mathbf{q}) = \sum_{i=1}^5 \tilde{w}_i(q)W_i(\mathbf{q})$, with $\tilde{w}_i(q)$ the momentum dependent renormalized couplings. Looking for a small-k solution, $\mathcal{G}(k) \simeq Z_\kappa^{-1}/k^{4-\eta}$, and evaluating the integrals $\pi_i(q)$ [53] one finds that they diverge at small q as $\pi_i(q) \simeq Z_\kappa^{-2}a_i(\eta, D)q^{-(4-D-2\eta)}$. We find that the renormalized couplings are softened at small q as $\tilde{w}_i(q) \propto Z_\kappa^2 c_i(\eta, D)q^{\eta_u}$, with $\eta_u = 4 - D - 2\eta$ [53]. When all bare couplings w_i are nonzero, and for a physical membrane, $D = 2$, it reduces to a cubic equation $d_c = [24(\eta-1)^2(2\eta+1)]/[(\eta-4)\eta(2\eta-3)]$ [53]. For $d_c = 1$ we obtain our main result (1). For large d_c we find $\eta = 2/d_c + O(1/d_c^2)$. The roughness of a size L membrane is characterized by $h_{\rm rms} = \langle h^2\rangle^{1/2} \simeq L^\zeta$ where $\zeta = (4 - D - \eta)/2$. Hence for $d_c = 1$ we find $\zeta = 0.623$.

One can define renormalized amplitude ratio as $\lim_{q\to 0}\{[\tilde{w}_i(q)]/[\tilde{w}_j(q)]\} = c_i/c_j$ for any pair (i, j) such that the bare couplings w_i, w_j are nonzero. Near $D = 4$ we find that these renormalized couplings take values such that the interaction energy becomes $v_1/2[(\partial_\alpha \vec{h})^2]^2 + v_2(\partial_\alpha \vec{h}\cdot\partial_\beta \vec{h})^2$, i.e., local in the fields $\partial_\alpha \vec{h}$. This property however does not hold for $D < 4$, e.g., one finds $c_2/c_1 = (D + \eta - 2)/(2 - \eta)$ instead of unity for $D = 4$, $\eta = 0$. Thus the critical point requires a fully nonlocal five-coupling description. In the physical case of $D = 2$ and $d_c = 1$ we find $c_i = \{\frac{1}{2}, 0.302, 0.338, -0.029, 0.173\}$, and the universal $\lambda/\mu = -0.978$ and the Poisson ratio (not to be confused with the stress field),

$$\sigma^{\rm anis} = -0.968, \tag{8}$$

to be contrasted with $\sigma = -1/3$ for an isotropic tensionless membrane [7,13].

There are other fixed points that lie in the invariant subspaces of the SCSA equations. The anomalous flat phase of the isotropic membrane corresponds to bare couplings $w_2 = w_4 = w_5 = 0$, leading for $D = 2$ to $\eta = 4/(d_c + \sqrt{16 - 2d_c + d_c^2})$, and $\eta \simeq 0.821$, $\zeta = 0.590$, for $d_c = 1$ [7,13]. Near $D = 4$ one recovers $\eta = [12/(d_c + 24)]\epsilon + O(\epsilon^2)$ from the ϵ expansion [4]. Another fixed manifold is $w_2 = 0$, i.e., $(\mu + \mu_1)^2 = \mu(\mu + \mu_2)$, which includes the choice $\mu_1 = \mu_2 = 0$, leading to $\tilde{w}_2(q) = 0$ and to yet another fixed point with $c_2 = 0$. For $D = 2$ and $d_c = 1$ we find $\eta = 0.854$ and $\zeta = 0.573$. Near $D = 4$ we find $\eta = [18/(d_c + 36)]\epsilon + O(\epsilon^2)$.

RG analysis.—We complement SCSA with $\epsilon = 4 - D$ RG expansion with five dimensionless couplings $\hat{w}_i = w_i/\kappa^2 C_4 \Lambda_\ell^{-\epsilon}$ governed by $\partial_\ell \hat{w}_i = \epsilon \hat{w}_i + a_{ijk}\hat{w}_j\hat{w}_k$ [53]. The anomalous dimension of the out-of-plane height field h defines the exponent η given by $\eta = \frac{1}{12}(10\hat{w}_1 - 18\hat{w}_2 + 5\hat{w}_3 + 3\hat{w}_5 - 6\hat{w}_{44})$ with $\hat{w}_{44} = \sqrt{3}\hat{w}_4$, and evaluated at the fixed point of interest $\hat{w}_i^*$. The anomalous dimension of the phonon field is given by $\eta_u = \frac{1}{12}(\hat{w}_1 - \hat{w}_2)$. The isotropic membrane corresponds to the space $\hat{w}_2 = \hat{w}_4 = \hat{w}_5 = 0$, which is preserved by the RG flow and along which $\partial_\ell \hat{w}_1 = -\frac{1}{12}\hat{w}_1[(d + 20)\hat{w}_1 + 10\hat{w}_3]$ and $\partial_\ell \hat{w}_3 = -\frac{5}{24}\hat{w}_3[(d + 4)\hat{w}_3 + 8\hat{w}_1]$. The isotropic membrane fixed point is $\hat{w}_1^* = 12\epsilon/(d + 24)$, $\hat{w}_3^* = [24\epsilon/5(d + 24)]$, corresponding to $\hat{\mu}^* = [12\epsilon/(24 + d)]$, $\hat{\lambda}^* = [-4\epsilon/(24 + d)]$ [4]. Diagonalizing the RG flow for $\hat{w}_i = \hat{w}_i^* + \delta\hat{w}_i$ around this fixed point in the larger space of five couplings shows that, in addition to the two negative eigenvalues -1 and $-d_c/(d_c + 24)$ within the plane $\delta\hat{w}_{1,3}$ of the isotropic membrane, (i) there is a marginal direction mixing $\delta\hat{w}_{1,3,4}$ (eigenvalue 0), (ii) there are two unstable directions with eigenvalues $d_c/(d_c + 24)$ with $\delta w_{2,5}$ nonzero (in the large d_c limit this eigenspace is purely along $\delta w_{2,5}$). Hence, consistent with the SCSA, the isotropic membrane fixed point is unstable to anisotropy of the orientational field and external boundary stress.

We found attractive fixed points of the RG equations in the subspace of couplings $\hat{w}_i$ at which the interaction energy is fully local in the gradients $\partial_\alpha \vec{h}$ and parametrized by two couplings v_1, v_2 as defined above. This subspace is preserved by the RG and also arises in the study of the crumpling transition. In fact the RG flow within this subspace is identical to the one obtained in [48] with d replaced by d_c. It admits a stable FP for $d_c > 219$, that is fully attractive in the space of the five couplings. Hence the RG approach is consistent, around $D = 4$, with the SCSA exact for large d_c and any D, predicting a new fixed point for membrane in anisotropic embedding space. For the physical membrane $D = 2$ and $d_c = 1$, while the SCSA predicts this new "anisotropic buckling transition" to be continuous, the RG, if extrapolated from $D = 4$, suggests a weak first-order transition, as argued for the crumpling transition [22,23,48].

To reach the new anisotropic buckling critical point requires tuning $\gamma = \gamma_c$, so that $\gamma_R = 0$. Slightly away from criticality the correlation length is long but finite, $\xi \sim |\delta\gamma|^{-\nu}$, diverging with a vanishing $\delta\gamma = \gamma - \gamma_c$. Linearizing the RG flow around the fixed point yields $\delta\gamma(L) \sim \delta\gamma L^\theta$, where $\theta = -\epsilon/d_c[1 - (66/d_c) + O(1/d_c^2)]$ [53]. By balancing $\kappa(\xi)\xi^{-4} \sim \delta\gamma(\xi)\xi^{-2}$ and using that $\kappa(\xi) \sim \xi^\eta$ we obtain the correlation length exponent, $\nu = 1/(2 + \theta - \eta)$.

Model development.—To develop a beyond-symmetry model (4), (5) of a critically buckled membrane we consider an elastic membrane in an external field $\vec{E} = E\hat{z}$ that aligns the membrane's normal $\hat{n}$ along the field (Fig. 1). We thus expect the energy-density to be a monotonic function of $\hat{n} \cdot \vec{E}$, namely of the small tilt angle θ of the membrane's normal away from the preferred z axis, $\mathcal{H}_{\rm orient} = (\alpha_1/2)\theta^2 + (\tilde{\alpha}_2/4)\theta^4 + \cdots$, with $\alpha_1 > 0$, $\tilde{\alpha}_2 > 0$. Combining this orientational field energy with the Hamiltonian for an elastic membrane [10,13], subjected to an in-plane compressional boundary stress $\sigma > 0$, isotropic in the membrane's xy plane, and, using that, to lowest order $\theta \sim |\partial_\alpha h|$, we obtain,

$$\mathcal{H} = \frac{\kappa}{2}(\partial^2 h) + \mu u_{\alpha\beta}^2 + \frac{\lambda}{2}u_{\alpha\alpha}^2 + \sigma\partial_\alpha u_\alpha + \frac{\alpha_1}{2}(\partial_\alpha h)^2 + \frac{\alpha_2}{4}(\partial_\alpha h)^4 + \cdots. \tag{9}$$

We note that the external stress σ is an in-plane boundary term, which induces a stress-dependent inward displacement of the membrane's edges. Observing that $\sigma\partial_\alpha u_\alpha = \sigma u_{\alpha\alpha} - \frac{1}{2}\sigma(\partial_\alpha h)^2$, the rotationally invariant strain component $\sigma u_{\alpha\alpha}$ can be accommodated by simply changing the preferred extension of the membrane without breaking the embedding space rotational symmetry [i.e., a redefinition of the parameter τ in (4), which determines the preferred membrane's projected area [55]]. The negative in-plane strain $\partial_\alpha u_\alpha$ induced by positive σ can also be relieved by a membrane tilt, $(\partial_\alpha h)^2 > 0$, which is stress free in the actual plane of the membrane. The lowering of the energy associated with the membrane tilt is then given by $\mathcal{H}_\sigma = -\frac{1}{2}\sigma(\partial_\alpha h)^2$, which, neglecting bending energy and boundary conditions, is unbounded, since tilt is unconstrained in the absence of the orientational field. Putting these ingredients together and rescaling xy coordinate system, we obtain the Hamiltonian governing a buckling transition of a membrane in an orientational field,

$$\mathcal{H} = \frac{\kappa}{2}(\partial^2 h) + \mu u_{\alpha\beta}^2 + \frac{\lambda}{2}u_{\alpha\alpha}^2 + \frac{\gamma}{2}(\partial_\alpha h)^2 + \frac{\alpha_2}{4}(\partial_\alpha h)^4,$$

where $\gamma = \alpha_1 - \sigma$ is the critical parameter which can be tuned to γ_c to reach the buckling transition (with $\gamma_c = 0$ at $T = 0$), studied here. As detailed in the Supplemental Material [53], we can estimate the buckling stress σ_c based on a model of homeotropic alignment of a membrane in a nematic solvent [39] and a model of a ferroelectric membrane aligned by an electric field. These give $\sigma_c \sim 1$–10 eV/μm^2, with the thermal fluctuation corrections to γ, that we show in the Supplemental Material to be subdominant.

Conclusion.—In contrast to previous works on tensionless crystalline membranes, we studied a thermal elastic sheet tuned by an external boundary stress to a critical point of a buckling transition, stabilized by an orientational field. We find that this breaking of embedding rotational symmetry leads to a new class of anomalous elasticity, that we have explored in detail here using the SCSA and RG analyses. With much recent interest in elastic sheets, most notably graphene and other van der Waals monolayers, we hope to stimulate further experiments to probe the rich universal phenomenology predicted here for an elastic membrane tuned to a buckling transition in an anisotropic environment. We expect that ideas explored here can be extended to richer class of anomalously elastic media [56,57].

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Note added.—We have recently became aware of an ongoing work by S. Shankar and D. R. Nelson on a membrane with a boundary stress, which, in contrast to our work only breaks embedding rotational symmetry at the boundary.

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[55] It is important to contrast this rotational symmetry-breaking stress applied in the xy plane with the "stress" τ imposed in the actual plane of the membrane, without breaking the rotational symmetry of the embedding space [5], by, e.g., confining the membrane in a spherically symmetric potential. Namely, we note that the added $\tau u_{\alpha\alpha}$ (instead of $\sigma \partial_\alpha u_\alpha$) can be eliminated by absorbing it into $u_{\alpha\alpha}^2 \to (u_{\alpha\alpha} + \sigma/\lambda)^2$ and redefining the crumpling order parameter ζ.

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