Designing Codes around Interactions: The Case of a Spin

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I present a new approach for designing quantum error-correcting codes guaranteeing a physically natural implementation of Clifford operations. Inspired by the scheme put forward by Gottesman, Kitaev, and Preskill for encoding a qubit in an oscillator in which Clifford operations may be performed via Gaussian unitaries, this approach yields new schemes for encoding a qubit in a large spin in which single-qubit Clifford operations may be performed via spatial rotations. I construct all possible examples of such codes, provide universal-gate-set implementations using quadratic angular-momentum Hamiltonians, and derive criteria for when these codes exactly correct physically relevant errors.

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Great quantum error-correcting codes shield quantum information from a noisy environment while simultaneously making it easily accessible to the programmer. The very name of these structures betrays an emphasis on the former goal, prioritizing the exact correction of the most likely errors. In this Letter, I develop an alternative approach to finding new codes that begins by ensuring straightforward logical manipulation of the encoded quantum information.

The encoding of a qubit in an oscillator described by Gottesman, Kitaev, and Preskill [1] is an example of a great error-correcting code. By construction, it protects against unwanted shifts in position and momentum up to a certain threshold. This protection also optimally corrects damping errors [2], which are the most prevalent sources of noise in the optical, superconducting, and mechanical systems for which the code is designed. One can also straightforwardly perform logical operations, since the full set of Clifford operations—the largest set of unitary gates that can be implemented easily—are realized by Hamiltonians at most quadratic in position and momentum—the largest set of Hamiltonians that are easy to engineer in an oscillator. For these reasons, the Gottesman-Kitaev-Preskill (GKP) code attracts considerable theoretical and experimental attention [3–7].

Other physical systems deserve their own great error-correction codes. While others have successfully adapted the stabilizer approach of GKP codes to protect against rotational errors [8], alternative single-system codes with easy Cliffords remain unexplored. I design such codes by starting with an algebra of physical Hamiltonians that are natural to the system at hand. The construction guarantees that a suitably large and discrete set of unitary gates—such as logical Clifford operations—can be implemented using only these natural physical interactions. As a consequence, these codes naturally offer resilience against relevant noise channels since environmental fluctuations typically take the form of such natural Hamiltonians. This approach therefore succeeds in allowing desired manipulations to be performed in a straightforward way while suppressing unwanted environmental interference.

To put this philosophy into practice, I demonstrate the construction for large single spins, such as atomic nuclei. Natural physical operations correspond to spatial rotations of the spin, so I construct all qubit codes on which maximal discrete sets of logical single-qubit unitaries can be implemented via these spatial rotations. Within this family of codes, I identify the examples that exactly correct relevant experimental noise such as dephasing to first order, including a code realizable in spin-$7/2$ systems such as antimony nuclei, a promising experimental platform [9,10]. The success of the construction in this particular case builds confidence that the same approach will bear fruit in additional physical systems.

Encoding qubits in spins.—The physics of a system dictates which transformations are straightforward. For large single spins the relevant physics is angular momentum, and the easy transformations are generated by Hamiltonians linear in the angular-momentum operators J_x, J_y, and J_z. These Hamiltonians arise naturally in practice, for example as the result of driving the spin with a resonant ac magnetic field. The physical unitaries generated by these Hamiltonians form a representation of the special unitary group SU(2) on the spin's Hilbert space. The explicit map from an abstract SU(2) element to its representative physical unitary is

$$D\colon\ \exp(-i\theta\hat{\mathbf{n}}\cdot\boldsymbol{\sigma}/2)\mapsto\exp(-i\theta\hat{\mathbf{n}}\cdot\mathbf{J}),\tag{1}$$

where $\boldsymbol{\sigma}$ is the vector of abstract Pauli matrices, $\mathbf{J}$ is the vector of the spin's angular-momentum operators, and $\hat{\mathbf{n}}$ is a unit vector defining the axis of rotation. These representative unitaries are a significantly restricted subgroup of the most general physical unitaries that can act on the large

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spin's Hilbert space. Since these restricted unitaries are straightforward to implement, the goal is to find a codespace where the maximum number of logical unitaries can be implemented by physically applying the SU(2) representatives.

Any SU(2) representative that realizes a logical unitary must map the codespace to itself. Because the SU(2) representation for a large single spin is an irreducible representation (irrep), the only subspaces mapped to themselves by the full set of SU(2) representatives are the trivial subspace containing only the zero vector and the full Hilbert space of the spin. Neither of these alternatives is a viable codespace. The consequence of this observation is that one must limit oneself to a proper subset of SU(2) representatives when searching for easy physical implementations of logical operations.

I consider two particularly relevant subsets that are representations of finite subgroups of SU(2). The subgroup to which I dedicate the most attention is known to quantum-information scientists as the single-qubit Clifford group [11], also called the binary octahedral group 2O [12][Ch. 7] because it is the double cover of the rotational symmetry group of the octahedron in the same way SU(2) is the double cover of SO(3). The techniques used for 2O are easily adapted to other finite subgroups of SU(2), and I additionally comment on an example from the binary icosahedral group 2I that offers an attractive experimental implementation.

For the sake of clarity, I now specialize to the subgroup 2O. The advantage of restricting the set of physical operations to the representatives of 2O is that these physical operations map nontrivial subspaces to themselves, and these subspaces provide candidate codespaces. Specifically, the desired qubit codespaces are two-dimensional subspaces of the spin's Hilbert space that are mapped to themselves by 2O representatives and on which nontrivial representative unitaries act nontrivially (since the point is for these physical unitaries to act as logical Clifford gates). In the language of representation theory, the codespaces should be faithful two-dimensional irreps of 2O obtained by restricting the SU(2) irrep to the 2O representatives.

The criteria for the desired codespaces having been established, I now present the representation theory of 2O needed to establish their existence.

Identifying binary-octahedral irreps.—The generators for 2O, concretely realized as 2×2 special-unitary matrices, are the phase and Hadamard gates

$$S=\exp\left(-i\frac{\pi}{2}\hat{\mathbf{z}}\cdot\boldsymbol{\sigma}/2\right)=\frac{1}{\sqrt{2}}(\mathbb{1}-i\sigma_z) \tag{2}$$

$$H=\exp\left(-i\pi\frac{\hat{\mathbf{x}}+\hat{\mathbf{z}}}{\sqrt{2}}\cdot\boldsymbol{\sigma}/2\right)=\frac{1}{\sqrt{2}}(-i\sigma_x-i\sigma_z). \tag{3}$$

The unusual phases are a consequence of the convention to enforce the unit-determinant constraint of special unitaries.

TABLE I. Multiplicities of the irreps of interest, ϱ_4 and ϱ_5, in the reducible 2O representation derived from the even-dimensional SU(2) irreps. Because these irreps only appear in even dimensions and their multiplicities follow a pattern that repeats every 24 dimensions, the dimension is presented in the form $24q+2p$, where q is any non-negative integer and $0\le p\le 11$.

SU(2)-irrep dimension	ϱ_4 multiplicity	ϱ_5 multiplicity
$24q$	$2q$	$2q$
$24q+2$	$2q+1$	$2q$
$24q+4$	$2q$	$2q$
$24q+6$	$2q$	$2q+1$
$24q+8$	$2q+1$	$2q+1$
$24q+10$	$2q+1$	$2q$
$24q+12$	$2q+1$	$2q+1$
$24q+14$	$2q+1$	$2q+2$
$24q+16$	$2q+1$	$2q+1$
$24q+18$	$2q+2$	$2q+1$
$24q+20$	$2q+2$	$2q+2$
$24q+22$	$2q+1$	$2q+2$

Being a finite group of 48 elements, 2O possesses only a finite number of irreps. As detailed in the Supplemental Material [13], only two of these irreps satisfy the criteria of being two dimensional and acting as logical Clifford gates. Label these two irreps ϱ_4 and ϱ_5 in recognition of their place among the other irreps of 2O. These irreps are inequivalent as complex representations, ϱ_4 straightforwardly mapping $S\mapsto S$ and $H\mapsto H$ but ϱ_5 mapping $S\mapsto -S$ and $H\mapsto -H$. This inequivalence means that codespaces cannot be split between these two irreps, but since the projective action of a unitary $U:\rho\mapsto U\rho U^\dagger$ is all that is relevant from a quantum perspective, the two representations behave identically when considered separately.

Having identified the two relevant irreps, the task now is to determine whether they appear in the decompositions of the reducible 2O representations obtained by restricting the SU(2) irreps to the 2O representatives. The decomposition of an irrep of a group into irreps of a subgroup proceeds according to what are called "branching rules" [16]. These branching rules—worked out in the Supplemental Material [13]—show that the irreps of interest do not appear at all in integer spins (with odd-dimensional Hilbert spaces). The multiplicities of these irreps in the half-integer spins increase according to a pattern that repeats every 24 dimensions, presented in Table I. Spin $1/2$ (dimension 2) contains the standard irrep of 2O, but given that this is the entirety of the Hilbert space it does not provide a code. Spin $3/2$ (dimension 4) does not contain any of the irreps of interest, being instead a four-dimensional irrep of 2O. For spin $5/2$ (dimension 6) and above, however, every half-integer spin contains at least one two-dimensional codespace on which 2O representatives perform logical Clifford operations.

This result identifies how many codespaces exist in each large single spin. The next step is to explicitly construct these codes and determine their additional properties.

Constructing example codes.—Producing explicit codewords proceeds by building projectors P_ϱ onto irreps ϱ_4 and ϱ_5 using standard expressions from representation theory reproduced in the Supplemental Material [13]. The codeword $|\bar{0}\rangle$ is taken to be an element of the $+1$ eigenspace of the irrep Pauli $\bar{\sigma}_z$, where irrep Paulis are defined by

$$\bar{\sigma}_w \coloneqq P_\varrho[i\exp(-i\pi J_w)]P_\varrho, \qquad w\in\{x,y,z\}. \tag{4}$$

To obtain $|\bar{1}\rangle$, simply apply $\bar{\sigma}_x$ to $|\bar{0}\rangle$. If the irrep ϱ occurs with multiplicity 1, then the $+1$ eigenspace of $\bar{\sigma}_z$ is one dimensional, and no further choices are required. If the irrep ϱ occurs with higher multiplicity, further properties of the code can be engineered as explored in the discussion of the error-correction conditions by making an appropriate choice for $|\bar{0}\rangle$ within the multidimensional $+1$ eigenspace of $\bar{\sigma}_z$.

As an illustration, the logical 0 state for the code in spin 5/2—the smallest nontrivial example—is

$$|\bar{0}\rangle = \sqrt{\frac{1}{6}}\left|\frac{5}{2},\frac{5}{2}\right\rangle - \sqrt{\frac{5}{6}}\left|\frac{5}{2},-\frac{3}{2}\right\rangle. \tag{5}$$

More explicit codes are presented in the Supplemental Material [13].

Computing with encoded qubits.—Employing these codes in the service of quantum computation requires the ability to do more than single-qubit logical Clifford operations. I focus now on the following minimal set of logical operations required for universal quantum computation:

$$\{\mathcal{P}_{|\bar{0}\rangle}, \mathcal{M}_{\bar{\sigma}_z}, \bar{S}, \bar{H}, \overline{\mathrm{CZ}}\} \cup \{\bar{T}\}, \tag{6}$$

where the bars denote logical operators, $\mathcal{P}$ denotes state preparation, and $\mathcal{M}$ denotes operator measurement. In this set, the single-qubit Cliffords are generated by $\bar{S}$ and $\bar{H}$, multiqubit Cliffords are obtained by the addition of $\overline{CZ}$, and $\bar{T}$ supplies a non-Clifford gate. Since all logical unitaries can be efficiently approximated to arbitrary precision by these operations, the ability to prepare at least one logical state (here chosen to be $\mathcal{P}_{|\bar{0}\rangle}$) and perform at least one measurement (here chosen to be $\mathcal{M}_{\bar{\sigma}_z}$) results in universal quantum computation.

By construction, these codes have Pauli and single-qubit Clifford operations realizable with Hamiltonians linear in angular-momentum operators [the SU(2) representation]. This construction gives the codes special structure in the J_z basis, detailed in the Supplemental Material [13], which additionally provides explicit recipes for measuring logical Paulis, performing logical $\overline{CZ}$ gates between two encoded qubits, and performing logical $\bar{T}$ gates. The strategy for performing a controlled-Z gate ($\overline{CZ}$) is similar to that used for rotation-symmetric bosonic codes [17]. In the bosonic case, a cross-Kerr interaction $a^\dagger a\otimes a^\dagger a$ generates the CROT gate used to perform $\overline{CZ}$ on the codespaces. In the spin case, the analogous $J_z\otimes J_z$ interaction performs the $\overline{CZ}$ gate (up to individual J_z corrections). As worked out in the Supplemental Material [13], the $\overline{CZ}$ gate takes the following form:

$$\overline{\mathrm{CZ}} = \exp\left(i\frac{\pi}{2}J_z\otimes\mathbb{1}\right)\exp\left(i\frac{\pi}{2}\mathbb{1}\otimes J_z\right) \times\exp(-i\pi J_z\otimes J_z). \tag{7}$$

Again, like in rotation-symmetric bosonic codes, a slightly more complicated single-system Hamiltonian yields a more exotic gate. A self-Kerr interaction $(a^\dagger a)^2$ allows one to perform an $\bar{S}$ gate on the bosonic codes. The 2O-irrep codes already have an $\bar{S}$ gate using linear Hamiltonians, so adding the analogous J_z^2 interaction allows one to perform a $\bar{T}$ gate (again up to a J_z correction). The $\bar{T}$ gate so obtained, as worked out in the Supplemental Material [13], takes the following forms for the two different $|\bar{0}\rangle$ supports:

$$\bar{T} = \begin{cases} \exp(-i\frac{\pi}{4}J_z)\exp(-i\frac{\pi}{4}J_z^2) & m_0=\frac{1}{2} \\ \exp(-i\frac{5\pi}{4}J_z)\exp(-i\frac{\pi}{4}J_z^2) & m_0=-\frac{3}{2}. \end{cases} \tag{8}$$

The Hamiltonians required for $\overline{CZ}$ and $\bar{T}$ gates are admittedly more complicated than those required for Clifford operations. Experiments routinely modulate quadrupolar terms such as needed for $\bar{T}$ [10], though it may be that a different technique will ultimately be required, as happened to be the case for the original $\bar{T}$-gate proposal for GKP [18].

Destructive measurement in the $\bar{\sigma}_z$ eigenbasis is possible via projecting onto the J_z eigenbasis. Since any J_z eigenstate has nonzero overlap with at most one logical computational-basis state—as explained in the Supplemental Material [13]—each possible outcome will unambiguously indicate a $\bar{\sigma}_z$ eigenstate.

Because of the octahedral symmetry of these codes, all the above constructions hold when replacing z with x or y.

Correcting errors.—As alluded to in the introduction, the fact that only a finite subset of SU(2) representatives preserve the codespace suggests that these codes might correct errors taking the form of small random SU(2) representatives in much the same way that GKP codes protect from small random displacements. Such noise is generated by the Lindblad master equation

$$d\rho = \gamma dt\sum_{w\in\{x,y,z\}}\left(J_w\rho J_w - \frac{1}{2}J_w^2\rho - \frac{1}{2}\rho J_w^2\right), \tag{9}$$

where γ is the depolarizing rate. For $\gamma dt \ll 1$, the following Kraus operators map $\rho \mapsto \rho + d\rho$:

$$E_0 = \mathbb{1} - \frac{1}{2}\gamma\, dt \|\mathbf{J}\|^2 = \left(1 - \frac{j(j+1)}{2}\gamma dt\right)\mathbb{1} \quad (10)$$

$$E_w = \sqrt{\gamma dt} J_w, \qquad w \in \{x, y, z\}. \quad (11)$$

Correcting the errors corresponding to these Kraus operators is equivalent to correcting random rotations to lowest order.

In spin systems it may be more natural to think of the dominant noise sources in terms of T_2-type dephasing errors J_z, T_1-type relaxation errors J_-, and thermalization errors J_+. Since these error operators are linear combinations of the random-rotation error operators, correcting either family of errors is equivalent. This mirrors the situation in GKP codes, whose manifest protection of random-displacement errors extends to relaxation errors as well [2].

The elements of the quantum-error-correction matrix indicate whether the codes exactly correct such errors. The exact-correction condition [19] is

$$\langle \bar{a}|E_j E_k|\bar{b}\rangle = C_{jk}\delta_{ab}. \quad (12)$$

Because of the octahedral symmetry of the codes, the only independent $E_j E_k$ pairs are J_z^2, $J_x J_y$, and J_z. As detailed in the Supplemental Material [13], these conditions are satisfied if and only if $\langle \bar{0}|J_z|\bar{0}\rangle = 0$.

In general it is not the case that $\langle \bar{0}|J_z|\bar{0}\rangle = 0$. For example, for the spin-5/2 code in Eq. (5), $|\bar{0}\rangle$ has a nonzero J_z expectation value. However, if an irrep appears with higher multiplicity, and the projection of J_z onto the $+1$ eigenspace of $\bar{\sigma}_z$ has both positive and negative eigenvalues (or a 0 eigenvalue), then a propitious choice for $|\bar{0}\rangle$ ensures that the quantum-error-correction criteria are exactly satisfied for these first-order rotation errors. The first spin in which one of the irreps appears with higher multiplicity is spin 13/2. The two eigenvalues of J_z projected onto the $+1$ eigenspace of $\bar{\sigma}_z$ are $-13/6$ and $5/2$, with associated eigenvectors $|\bar{0}_{-(13/6)}\rangle$ and $|\bar{0}_{(5/2)}\rangle$. To get a codeword with zero J_z expectation value, one takes linear combinations of the following form:

$$|\bar{0}_\phi\rangle = \frac{\sqrt{105}}{14}|\bar{0}_{-(13/6)}\rangle + e^{i\phi}\frac{\sqrt{91}}{14}|\bar{0}_{(5/2)}\rangle. \quad (13)$$

Considerations for first-order correction of random-rotation errors make no distinction between different values of the phase ϕ, leaving a free parameter that may be further optimized over.

While satisfying the error-correction conditions guarantees the existence of an error-correction procedure, the highly noncommutative nature of $\{J_x, J_y, J_z\}$ errors makes the definition of physically natural commuting stabilizers difficult, though one can use the structure of the support in the angular-momentum basis to build noncommuting projectors that are analogous to stabilizers. The construction of practical error-correction procedures using such elements is an ongoing project.

Implementing in experiments.—Since nuclear spins are obvious host systems for these codes, it would be nice to have examples with good error-correcting properties in a Hilbert space of dimension at most 10 (corresponding to the largest available nuclei of spin 9/2). As just demonstrated, the 2O codes require a larger Hilbert space to reliably correct errors. This motivates considering an alternative maximal discrete subgroup of SU(2), the binary icosahedral group 2I, consisting of gates corresponding to the symmetries of a regular icosahedron. Using the same tools developed for 2O, one finds a two-dimensional 2I irrep in spin 7/2 that allows for the correction of random-rotation errors to first order:

$$|\bar{0}\rangle = \sqrt{\frac{3}{10}}\left|\frac{7}{2},\frac{7}{2}\right\rangle + \sqrt{\frac{7}{10}}\left|\frac{7}{2},-\frac{3}{2}\right\rangle. \quad (14)$$

Spin 7/2 is the smallest Hilbert space in which one can correct these errors, making this code analogous to a perfect block code. Additionally, the nuclear spin of antimony provides an ideal physical realization of a spin-7/2 system over which impressive experimental control has been obtained [10]. Figure 1 depicts the Wigner functions for this code, defined via a self-dual kernel obeying the Stratonovitch-Weyl postulates for SU(2) [20,21]. See the Supplemental Material [13] for more details.

Comparing to existing codes.—These spin codes are unique among existing codes in protecting from random SU(2) rotations within an irrep of SU(2). There are some analogous examples worth mentioning, however. The minimal qudit codes of [22] protect against a discrete set of finite J_z-rotation errors but are "classical" in the sense that they offer no protection against J_z or J_y rotations. The qudit analogs of the GKP code [1,23] add protection against some cyclic permutations of the J_z basis elements. Neither of these codes perform well against first-order rotation noise, however, as illustrated by some numerical experiments in the Supplemental Material [13]. Another family of codes designed to protect against rotation errors are molecular codes [8]. In their current formulation, these codes are built in spaces that are direct sums of SU(2) irreps and additionally protect against shifts in total angular

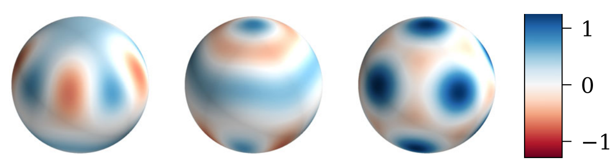

FIG. 1. Wigner functions for $|\bar{0}\rangle$, $|\bar{1}\rangle$, and the codespace projector for the icosahedral code in spin 7/2.

momentum, making direct comparison difficult. Note that decoherence-free subspaces and noiseless subsystems for random-rotation errors do not exist in the Hilbert spaces of large single spins since these errors generate an irrep.

Generalizing to other systems.—The construction presented for spin codes exemplifies a more general procedure. One can replace the representation of the Lie algebra $\mathfrak{su}(2)$ given by angular-momentum operators with any representation of a Lie algebra $\mathfrak{g}$ given by physically natural Hamiltonians on a Hilbert space. Exponentiating these Hamiltonians will generate easily implementable unitaries forming a representation of a Lie group G analogous to SU (2). One will then want to consider a discrete subgroup $K \subset G$ just as I considered $2\mathrm{O} \subset \mathrm{SU}(2)$. The representation of G restricts to a representation of K, and the small-dimensional irreps of K into which this representation decomposes form the candidate codespaces. At this point, one must tailor the procedure to the particular set of errors and the particular discrete subgroup K. When considering random rotations, the error-correction conditions were greatly simplified because the noise was generated by Lindblad operators taken from a subalgebra of $\mathfrak{su}(2)$ and 2O contained a rich set of symmetries of this subalgebra. One expects similar simplifications to take place in the more general case when analogous structure is present. Some obvious candidate Lie-algebra representations are those given by quadratic bosonic and fermionic Hamiltonians. Pursuing the bosonic Hamiltonians brings the prospect of finding additional GKP-like codes in oscillators, though the noncompact nature of the Gaussian unitaries they generate presents qualitatively different challenges than encountered in the SU(2) case. Quadratic fermionic Hamiltonians generate compact Lie groups [[24] Thm. 13.1], and so provide an arena for a much more straightforward application of the techniques presented here.

Conclusion.—In this Letter, I have constructed all single-spin qubit codes admitting Cliffords via SU(2) unitaries. These codes exist for all half-integer spins larger than $3/2$ and admit the entangling gate $\overline{CZ}$ and the non-Clifford gate $\bar{T}$ via Hamiltonians quadratic in angular-momentum operators. I have also exhibited codes in spins as small as $7/2$ that exactly protect against random-rotation errors to first order. In addition to showing how to build better qubits out of large spins, these achievements illustrate the power of the finite-group-representation approach. Adapting these techniques to systems with different algebras of natural Hamiltonians offers a new path by which to discover useful quantum-error-correcting codes.

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[1] D. Gottesman, A. Kitaev, and J. Preskill, Encoding a qubit in an oscillator, Phys. Rev. A **64**, 012310 (2001).
[2] K. Noh, V. V. Albert, and L. Jiang, Quantum capacity bounds of Gaussian thermal loss channels and achievable rates with Gottesman-Kitaev-Preskill codes, IEEE Trans. Inf. Theory **65**, 2563 (2019).
[3] S. Glancy and E. Knill, Error analysis for encoding a qubit in an oscillator, Phys. Rev. A **73**, 012325 (2006).
[4] D. J. Weigand and B. M. Terhal, Generating grid states from Schrödinger-cat states without postselection, Phys. Rev. A **97**, 022341 (2018).
[5] B. Q. Baragiola, G. Pantaleoni, R. N. Alexander, A. Karanjai, and N. C. Menicucci, All-Gaussian Universality and Fault Tolerance with the Gottesman-Kitaev-Preskill Code, Phys. Rev. Lett. **123**, 200502 (2019).
[6] C. Flühmann, T. L. Nguyen, M. Marinelli, V. Negnevitsky, K. Mehta, and J. P. Home, Encoding a qubit in a trapped-ion mechanical oscillator, Nature (London) **566**, 513 (2019).
[7] P. Campagne-Ibarcq, A. Eickbusch, S. Touzard, E. Zalys-Geller, N. E. Frattini, V. V. Sivak, P. Reinhold, S. Puri, S. Shankar, R. J. Schoelkopf, L. Frunzio, M. Mirrahimi, and M. H. Devoret, Quantum error correction of a qubit encoded in grid states of an oscillator, Nature (London) **584**, 368 (2020).
[8] V. V. Albert, J. P. Covey, and J. Preskill, Robust Encoding of a Qubit in a Molecule, Phys. Rev. X **10**, 031050 (2020).
[9] V. Mourik, S. Asaad, H. Firgau, J. J. Pla, C. Holmes, G. J. Milburn, J. C. McCallum, and A. Morello, Exploring quantum chaos with a single nuclear spin, Phys. Rev. E **98**, 042206 (2018).
[10] S. Asaad, V. Mourik, B. Joecker, M. A. I. Johnson, A. D. Baczewski, H. R. Firgau, M. T. Madzik, V. Schmitt, J. J. Pla, F. E. Hudson, K. M. Itoh, J. C. McCallum, A. S. Dzurak, A. Laucht, and A. Morello, Coherent electrical control of a single high-spin nucleus in silicon, Nature (London) **579**, 205 (2020).
[11] D. Gottesman, The Heisenberg representation of quantum computers, arXiv:quant-ph/9807006.
[12] H. S. M. Coxeter, *Regular Complex Polytopes* (Cambridge University Press, Cambridge, England, 1991).
[13] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.127.010504, which includes Refs. [14,15], for more detailed calculations.
[14] T. Dokchitser, GroupNames, https://people.maths.bris.ac.uk/~matyd/GroupNames/.

[15] K. Audenaert and B. De Moor, Optimizing completely positive maps using semidefinite programming, Phys. Rev. A **65**, 030302(R) (2002).

[16] M. Fallbacher, Breaking classical Lie groups to finite subgroups—An automated approach, Nucl. Phys. **B898**, 229 (2015).

[17] A. L. Grimsmo, J. Combes, and B. Q. Baragiola, Quantum Computing with Rotation-Symmetric Bosonic Codes, Phys. Rev. X **10**, 011058 (2020).

[18] J. Hastrup, M. V. Larsen, J. S. Neergaard-Nielsen, N. C. Menicucci, and U. L. Andersen, Unsuitability of cubic phase gates for non-Clifford operations on Gottesman-Kitaev-Preskill states, Phys. Rev. A **103**, 032409 (2021).

[19] E. Knill and R. Laflamme, Theory of quantum error-correcting codes, Phys. Rev. A **55**, 900 (1997).

[20] R. L. Stratonovich, On distributions in representation space, Sov. Phys. JETP-USSR **4**, 891 (1957).

[21] S. Heiss and S. Weigert, Discrete Moyal-type representations for a spin, Phys. Rev. A **63**, 012105 (2000).

[22] S. Pirandola, S. Mancini, S. L. Braunstein, and D. Vitali, Minimal qudit code for a qubit in the phase-damping channel, Phys. Rev. A **77**, 032309 (2008).

[23] C. Cafaro, F. Maiolini, and S. Mancini, Quantum stabilizer codes embedding qubits into qudits, Phys. Rev. A **86**, 022308 (2012).

[24] D. H. Sattinger and O. L. Weaver, *Lie Groups and Algebras with Applications to Physics, Geometry, and Mechanics*, Vol. 61 (Springer Science & Business Media, New York, 2013).