Maximum Energy Growth Rate in Dilute Quantum Gases

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In this Letter we study how fast the energy density of a quantum gas can increase in time, when the interatomic interaction characterized by the s-wave scattering length a_s is increased from zero with arbitrary time dependence. We show that, at short time, the energy density can at most increase as $\sqrt{t}$, which can be achieved when the time dependence of a_s is also proportional to $\sqrt{t}$, and especially, a universal maximum energy growth rate can be reached when a_s varies as $2\sqrt{\hbar t/(\pi m)}$. If a_s varies faster or slower than $\sqrt{t}$, it is, respectively, proximate to the quench process and the adiabatic process, and both result in a slower energy growth rate. These results are obtained by analyzing the short time dynamics of the short-range behavior of the many-body wave function characterized by the contact, and are also confirmed by numerically solving an example of interacting bosons with time-dependent Bogoliubov theory. These results can also be verified experimentally in ultracold atomic gases.

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The ability of tuning interactions between particles is a major advantage of ultracold atomic systems [1,2]. Especially, by utilizing magnetic and optical tools, the interaction strength between atoms, usually characterized by the s-wave scattering length a_s, can be tuned over a few thousand of Bohr radius over a few microseconds, which is a timescale much faster than the many-body relaxation time. This has led to a number of interesting ultracold atomic experiments reported in recent years, such as universal quench dynamics observed by quenching interaction to unitarity [3,4], and coherent excitation of the Higgs mode in superfluid Fermi gases and the Bogoliubov quasiparticles in Bose condensates by periodically modulating interactions [5–8]. This experimental progress has also been accompanied by lots of theoretical interest on studying nonequilibrium dynamics driven by time-dependent interactions [9–30].

Energy is a fundamental quantity characterizing quantum matters. It is therefore of broad interest to consider energy dynamics in the nonequilibrium process. Especially, we address a fundamental issue of how fast we can pump energy into a system by changing the interaction parameter and whether there is a universal upper limit for the energy increasing rate. To be concrete, suppose that we start with a noninteracting quantum gas with $a_s = 0$ and then vary a_s in time, and suppose that $a_s(t)$ can be controlled in any function form, the question is whether there is an upper bound for the rate of how fast the total energy can increase as a function of time. In this Letter we show that there does exist such a universal rate limit, as far as the initial growth rate is concerned. This result is quite counterintuitive, because normally the interaction energy increases as the interaction strength increases. Thus, intuitively, one would think that a faster increasing of interaction strength should result in a faster increasing of interaction energy, and consequently, a faster increasing of the total energy. Since we consider that a_s can be increased as fast as one wants, it seems to indicate that there should not exist such a bound.

However, our results show that this intuition is not correct. Before presenting the rigorous mathematical statement, we first emphasize that our result is closely tied to a key quantity in ultracold atomic gases called the contact [31–39]. It is now well known that for quantum gases with zero-range interactions, one can define contact $\mathcal{C}$ through the short-range behavior of the many-body wave function when any two atoms are brought close to each other, or equivalently, through the high-momentum tail of the momentum distribution. It has been shown that the total energy of a quantum gas is directly related to the contact [31–39].

To gain an intuitive understanding of our results, let us first consider two limits. On one limit, the fastest change of the interaction strength is the quench process, during which a_s instantaneously jumps from zero to any non-zero value. However, it can be shown that the contact does not change and retains zero right after the quench [32], and therefore, the total energy also does not change after the quench [27]. This means that the fastest change of interaction actually does not result in a fast change of the total energy, and in contrast, the interaction energy does not change at all. On the opposite limit, we can consider an adiabatic varying of the interaction strength, during which the interaction energy does vary in time but it varies adiabatically with sufficiently slow rate. The physical pictures in these two limits motivate us to expect a universal maximum growth rate driven by intermediate rates of varying the interaction strength. Our studies rigorously establish this physical picture, and this physical picture is quite general and can inspire a similar phenomenon in other systems.

General expression for the contact growth.—Here we consider a uniform Bose gas or spin-1/2 Fermi gas starting from any noninteracting state $|\Psi_0\rangle$ at $t = 0$, and then the s-wave scattering length $a_s(t)$ can vary with arbitrary time dependence. Below we use n and n_σ to denote the densities of bosons and fermions with spin-σ ($\sigma = \uparrow, \downarrow$), respectively, and $\hat{\psi}$ and $\hat{\psi}_\sigma$ to denote boson operator and fermion operator with spin-σ, respectively. One of the main results of this work states as follows:

In the short-time limit, the dynamics of the contact $\mathcal{C}(t)$ is given by

$$\mathcal{C}(t) = g_2(0)|\eta(t)|^2. \tag{1}$$

Here, $g_2(\mathbf{r})$ is defined as $\langle\Psi_0|\hat{\psi}^\dagger(\mathbf{r}/2)\hat{\psi}^\dagger(-\mathbf{r}/2)\hat{\psi}(-\mathbf{r}/2)\hat{\psi}(\mathbf{r}/2)|\Psi_0\rangle$ for bosons and $\langle\Psi_0|\hat{\psi}^\dagger_\uparrow(\mathbf{r}/2)\hat{\psi}^\dagger_\downarrow(-\mathbf{r}/2)\hat{\psi}_\downarrow(-\mathbf{r}/2)\hat{\psi}_\uparrow(\mathbf{r}/2)|\Psi_0\rangle$ for fermions, and $g_2(0)$ means $g_2(\mathbf{r})$ evaluated at $\mathbf{r} = 0$. Especially, if $|\Psi_0\rangle$ is the noninteracting ground state, then $g_2(\mathbf{r}) = n^2$ or $n_\uparrow n_\downarrow$ for the Bose or the spin-1/2 Fermi gas. The key result is that the function $\eta(t)$ obeys the following integral equation

$$\left[\hat{L} + \frac{1}{4\pi a_s(t)}\right]\eta(t) = -1, \tag{2}$$

where $\hat{L}$ is a linear operator acting on $\eta(t)$ as

$$\hat{L}\eta(t) = \left(\frac{m}{\hbar}\right)^{\frac{1}{2}} \frac{1}{8\pi^{3/2}\sqrt{i}} \times \lim_{\epsilon\to 0^+}\left[\int_0^{t-\epsilon} \frac{\eta(\tau)}{(t-\tau)^{\frac{3}{2}}} d\tau - \frac{2\eta(t)}{\sqrt{\epsilon}}\right]. \tag{3}$$

This result is motivated by solving the two-body problem, which satisfies the following Schrödinger equation in the relative coordinate $\mathbf{r}$ frame as

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2\nabla^2\psi}{m} + \frac{4\pi\hbar^2 a_s(t)}{m}\delta(\mathbf{r})\frac{\partial}{\partial r}r\psi. \tag{4}$$

Starting from an initial state $\psi(\mathbf{r}) = 1/\sqrt{V}$ (V is the total volume of the system), the time evolution of the wave function always obeys the following asymptotic form in the short-range $r \to 0$ limit, that is

$$\psi(\mathbf{r}, t) = \frac{\eta(t)}{4\pi\sqrt{V}}\left[\frac{1}{r} - \frac{1}{a_s(t)}\right] + O(r), \tag{5}$$

and it can be shown that $\eta(t)$ satisfies Eq. (2) [40]. Generalizing this result from the two-body problem to the many-body problem utilizes the short-time expansion and is quite straightforward, which yields Eq. (1) [40]. Here we note that for the two-body problem, $\eta(t)$ satisfies Eq. (2) for all timescales, but for the many-body problem, the result is only valid for the short timescale. Here short time is defined as the timescale much shorter than the typical many-body timescale $t_n = \hbar/E_n$, where $E_n = \hbar^2 k_n^2/(2m)$ and $k_n = (6\pi^2 n)^{1/3}$ (with n replaced by n_σ for fermions). In other words, in such a short timescale, the short-range behavior of the many-body wave function is still dominated by the two-body physics.

Contact growth rate.—Here, without loss of generality, we consider that $a_s(t)$ grows from zero to a positive value in a power-law function as

$$a_s(t) = \sqrt{2}\beta l_0\left(\frac{t}{t_0}\right)^\alpha, \tag{6}$$

where l_0 is an arbitrary length unit and t_0 is the time units, and l_0 and t_0 are both related to the same energy unit as $\hbar/t_0 = \hbar^2/(2ml_0^2)$. For example, when l_0 is taken as a few thousands of Bohr radius, t_0 varies from a few tenths to a thousand of microseconds, depending on the mass of atoms. α, β are two constants describing the power and the coefficient, respectively, and a factor $\sqrt{2}$ is introduced just for the later convenience. The operator $\hat{L}$ has an important property that

$$\hat{L}t^\alpha = \left(\frac{m}{\hbar}\right)^{\frac{1}{2}} B(\alpha)t^{\alpha-\frac{1}{2}}, \tag{7}$$

where $B(\alpha)$ is a constant given by $B(\alpha) = i^{3/2}\Gamma(\alpha+1)/[4\pi\Gamma(\alpha+1/2)]$. That is to say, suppose $\eta(t)$ is a power-law function in t, when $\hat{L}$ acts on $\eta(t)$, it lowers the power of $\eta(t)$ by 1/2. This property plays a crucial role in the following analysis because it means whether α in Eq. (6) is greater or smaller than 1/2 makes significant difference.

Case I: $\alpha > 1/2$. In this case, the $1/(4\pi a_s)$ term dominates Eq. (2), and thus, to the leading order of t, $\eta(t)$, and $\mathcal{C}(t)$ are given by

$$\eta(t) = -4\pi a_s(t); \qquad \mathcal{C}(t) = 16\pi^2 a_s^2(t) g_2(0). \tag{8}$$

This is consistent with the adiabatic regime where the physical quantities only depend on the instantaneous scattering length at time t.

Case II: $\alpha < 1/2$. In this case, the $\hat{L}$ term dominates Eq. (2), and thus, to the leading order of t, $\eta(t)$, and $\mathcal{C}(t)$ are given by

$$\eta(t) = -\left(\frac{\hbar}{m}\right)^{\frac{1}{2}} \frac{1}{B(1/2)}\sqrt{t}; \quad \mathcal{C}(t) = \frac{\hbar}{m}\frac{g_2(0)}{|B(1/2)|^2}t, \tag{9}$$

where $B(1/2) = -1/(8\sqrt{i\pi})$. Surprisingly, in this case this result shows that the growth of contact at the short time is independent of parameters l_0, t_0, α, and β in Eq. (6). That is to say, it is independent of how fast a_s varies in time. Even if l_0 or β is infinitely large, or α is infinitesimally small, and then $a_s(t)$ initially grows infinitely fast, the contact always grows linearly in time with a constant rate. This means that as long as $\alpha < 1/2$, the short-range physics at the short time is the same as a quench process where the scattering length instantaneously jumps to unitarity.

Case III: $\alpha = 1/2$. In this case, Eq. (6) becomes

$$a_s(t) = \beta\sqrt{\frac{\hbar t}{m}}. \tag{10}$$

By dimension analysis, it is easy to see that l_0 and t_0 cancel each other and only the coefficient β enters the expression. In this case, both the $\hat{L}$ term and the $1/(4\pi a_s)$ term are equally important. Also to the leading order of t, we obtain

$$\eta(t) = -A(\beta)\sqrt{t}; \qquad \mathcal{C}(t) = |A(\beta)|^2 g_2(0)t, \tag{11}$$

where $A(\beta)$ is also a constant given by

$$A(\beta) = \left(\frac{\hbar}{m}\right)^{\frac{1}{2}} \frac{1}{B(\frac{1}{2}) + \frac{1}{4\pi\beta}}. \tag{12}$$

As one can see from here, this is a critical case. In case III, by taking $\beta \to \infty$, Eq. (11) recovers Eq. (9), consistent with the quench limit, and by taking $\beta \to 0$, Eq. (11) recovers Eq. (8), consistent with the adiabatic limit.

Here an important point is that $|A(\beta)|^2$ is *not* a monotonic function in β. For a given initial state, $g_2(0)$ is fixed, and we can then define the initial growth rate for contact as $v_{\mathcal{C}} = \lim_{t\to 0} d\mathcal{C}(t)/dt$. As one can see from Eq. (8), $v_{\mathcal{C}} = 0$ for case I. And for both case II and case III, $v_{\mathcal{C}}$ is a constant, given by $|A(\beta)|^2 g_2(0)$ for case III and $|A(\beta = \infty)|^2 g_2(0)$ for case II. It turns out that $|A(\beta)|^2$ reaches its maximum at $\beta_{c1} = 2\sqrt{2/\pi} \approx 1.596$, at which $v_{\mathcal{C}}^{\max} = (\hbar/m)128\pi g_2(0)$.

Energy growth rate.—The total energy density of a uniform zero-range interacting quantum gas can be measured through its momentum distribution $n_{\mathbf{k}}$. For example, for spin-1/2 fermions, it is given by

$$\mathcal{E} = \int \frac{d^3k}{(2\pi)^3}\epsilon_{\mathbf{k}}\left(n_{\mathbf{k}} - \frac{2\mathcal{C}}{k^4}\right) + \frac{\mathcal{C}}{4\pi m a_s}, \tag{13}$$

where $n_{\mathbf{k}} = n_{\mathbf{k}\uparrow} + n_{\mathbf{k}\downarrow}$, $\epsilon_{\mathbf{k}} = \hbar^2\mathbf{k}^2/(2m)$, and the contact $\mathcal{C}$ is related to $n_{\mathbf{k}\sigma}$ through $\mathcal{C} \equiv \lim_{k\to\infty} k^4 n_{\mathbf{k}\sigma}$ [31]. The same expression, replacing all $\mathcal{C}$ by $\mathcal{C}/2$, also holds for the spinless Bose gas as long as the three-body contact can be ignored [38].

On the other hand, there is a direct relation between the time evolution of the energy and the contact. For spin-1/2 fermions it is given as

$$\frac{d}{dt}\mathcal{E}(t) = \frac{\hbar^2\mathcal{C}(t)}{4\pi m a_s^2(t)}\frac{da_s}{dt}. \tag{14}$$

For spinless bosons, an extra 1/2 factor should also be added in the right-hand side of Eq. (14). Therefore, based on the contact growth discussed above, we can determine the energy growth.

Case I: $\alpha > 1/2$. With the help of Eq. (8), one can obtain that

$$\delta\mathcal{E}(t) = \frac{4\pi\hbar^2 a_s(t)}{m} g_2(0), \tag{15}$$

where $\delta\mathcal{E}(t) = \mathcal{E}(t) - \mathcal{E}(t=0)$. This result again shows that the physics in this regime is consistent with the adiabatic regime where the energy is determined by the instantaneous scattering length. Since $\alpha > 1/2$, the energy increases slower than $\sqrt{t}$ at the short time.

Case II: $\alpha < 1/2$. In this regime, Eq. (9) gives rise to

$$\delta\mathcal{E}(t) = \frac{16\sqrt{2}\alpha}{\beta(1-\alpha)}\frac{\hbar^2}{m} g_2(0) l_0 \left(\frac{t}{t_0}\right)^{1-\alpha}. \tag{16}$$

Since $\alpha < 1/2$, the energy also increases slower than $\sqrt{t}$ at the short time. When taking the limit of $\alpha \to 0$, or $\beta \to \infty$ or $l_0 \to \infty$, $\delta\mathcal{E}(t) \to 0$, and it is consistent with the fact there is no energy change for the quench process as discussed above.

Case III: $\alpha = 1/2$. In this regime, Eq. (11) yields

$$\delta\mathcal{E}(t) = \sqrt{\frac{\hbar^3}{m}}\frac{|A(\beta)|^2}{4\pi\beta} g_2(0)\sqrt{t}. \tag{17}$$

It is in this case that the energy growth at the short time is the fastest. Now we can define an energy growth rate $v_E = \lim_{t\to 0} d\mathcal{E}(t)/d\sqrt{t}$. For cases I and II, this rate is zero. In case II, v_E is given by $\sqrt{\hbar^3/m}|A(\beta)|^2 g_2(0)/(4\pi\beta)$, which reaches its maximum at $\beta_{c2} = 2/\sqrt{\pi} \approx 1.128$ with $v_E^{\max} = 4(2+\sqrt{2})\sqrt{\pi} g_2(0)\sqrt{\hbar^3/m} \approx 24.2 g_2(0)\sqrt{\hbar^3/m}$. Note that this value of $v_E^{\max}$ applies for the spin-1/2 Fermi gas, and for the spinless Bose gas an extra 1/2 factor should be added.

This maximum energy growth rate is the main result of this work. We note that, although this result is obtained by assuming a power-law function of $a_s(t)$ and by considering positive $a_s(t)$, it can be extended to other function forms, such as including the logarithmic function corrections and starting from nonzero scattering length, and including the

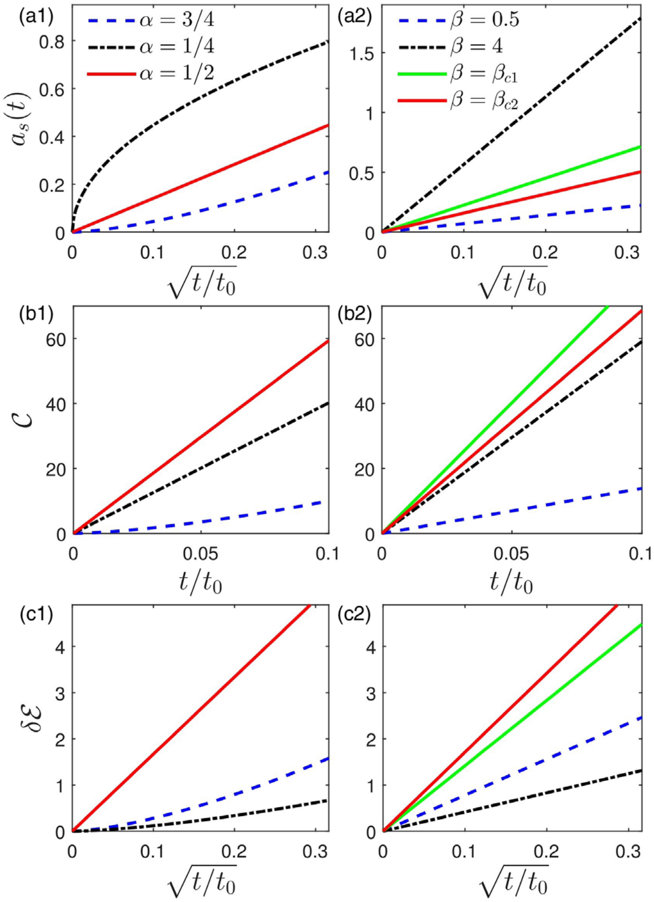

FIG. 1. (a1)–(a2) The time dependence of the scattering length $a_s(t)$ (in units of l_0) with different power-law functions of Eq. (6). (a1) $\alpha = 1/4, 1/2, 3/4$, and β is fixed at $\beta = 1$. (a2) α is fixed at $\alpha = 1/2$ and $\beta = 1/2, 1.128, 1.596$, and 4. (b1)–(b2) The short time behavior of the contact $\mathcal{C}$ [in units of $g_2(0)l_0^2$] with $a_s(t)$ plotted in (a1) and (a2), respectively. (c1)–(c2) The time dependence of the energy density change $\delta\mathcal{E}$ [in units of $g_2(0)l_0\hbar^2/m$] with $a_s(t)$ plotted in (a1) and (a2), respectively.

situations where a_s varies to negative values. The results discussed above are summarized in Figs. 1 and 2. Figures 1(a1) and (a2) show different power-law functions of $a_s(t)$ given by Eq. (6), either with different power α, or with different coefficient β and fixed $\alpha = 1/2$. Figures 1(b1) and 1(b2) show the corresponding contact growth, and Figs. 1(c1) and 1(c2) show the corresponding energy growth, using spinless bosons as an example. It clearly shows that a faster increasing of a_s does not necessarily lead to a faster increasing of the contact and the energy density. One can see that for different powers, $\alpha = 1/2$ gives the fastest contact growth and energy growth at the short time. And for α fixed at $1/2$, $\beta = \beta_{c1}$ yields the fastest contact growth and $\beta = \beta_{c2}$ yields the fastest energy growth, as also shown in Fig. 2. In the inset of Fig. 2(b), we have also shown that a maximum energy growth rate can also be found for negative β, where a_s varies to the negative value.

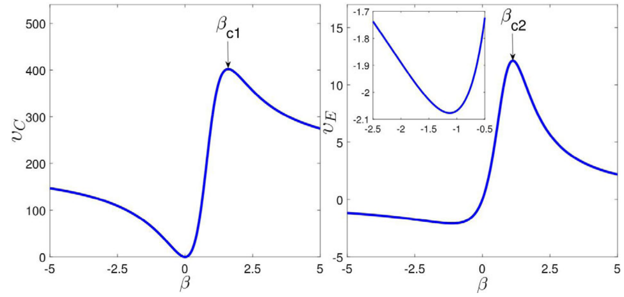

FIG. 2. Initial growth rate for contact (a) and for energy (b) as a function of β for $a_s(t) = \beta\sqrt{\hbar t/m}$. Arrows mark β_{c1} and β_{c2} where the maximum contact growth rate and the maximum energy growth rate are reached. v_C and v_E are plotted in units of $g_2(0)\hbar/m$ and $g_2(0)\sqrt{\hbar^3/m}$, respectively. The inset in (b) highlights the peak at the negative β side where a_s changes to negative values.

Example.—The analysis above is based on the short time expansion. To support the validity of this expansion, here we consider a concrete example of spinless bosons, which can be described by the following time-dependent Hamiltonian

$$\hat{H}(t) = \sum_{\mathbf{k}} \epsilon_{\mathbf{k}} \hat{b}_{\mathbf{k}}^\dagger \hat{b}_{\mathbf{k}} + \frac{U(t)}{2V} \sum_{\mathbf{k},\mathbf{k}',\mathbf{q}} \hat{b}_{\mathbf{k}}^\dagger \hat{b}_{\mathbf{q}-\mathbf{k}}^\dagger \hat{b}_{\mathbf{q}-\mathbf{k}'} \hat{b}_{\mathbf{k}'}, \quad (18)$$

where $\hat{b}_{\mathbf{k}}$ are boson creation operators with momentum $\mathbf{k}$. $U(t)$ is related to $a_s(t)$ through the renormalization relation

$$\frac{1}{U(t)} = \frac{m}{4\pi\hbar^2 a_s(t)} - \frac{1}{V}\sum_{\mathbf{k}} \frac{1}{2\epsilon_{\mathbf{k}}}. \quad (19)$$

We solve this Hamiltonian by adopting the Bogoliubov-type variational ansatz as

$$|\Psi(t)\rangle = \mathcal{N}(t) \exp\left[g_0(t)\hat{b}_0^\dagger + \sum_{\mathbf{k}\neq 0} g_{\mathbf{k}}(t)\hat{b}_{\mathbf{k}}^\dagger \hat{b}_{-\mathbf{k}}^\dagger\right]|0\rangle, \quad (20)$$

where $\mathcal{N}(t)$ is a normalization factor, $|0\rangle$ is vacuum of particles, and g_0 and $g_{\mathbf{k}}$ are all variational parameters. This approach is not restricted to the short time and has been successfully used in the previous studies of degenerate Bose gas quenched to unitarity [10,13,26]. The evolution of variational parameters $g_0(t)$ and $g_{\mathbf{k}}(t)$ can be obtained from the Euler-Lagrange equation for the Lagrangian $\mathcal{L} = \frac{1}{2}[\langle\Psi(t)|\dot{\Psi}(t)\rangle - \langle\dot{\Psi}(t)|\Psi(t)\rangle] - \langle\Psi(t)|\hat{H}(t)|\Psi(t)\rangle$, which yields a set of differential equations for g_0 and $g_{\mathbf{k}}$. Since we start with a noninteracting Bose condensate, we take $g_0 = 1$ and $g_{\mathbf{k}} = 0$ at $t = 0$ as the initial conditions for these differential equations. We can obtain the variational wave function by solving these equations, and subsequently, we can determine the total energy density with Eq. (13). The results for the total energy density are shown

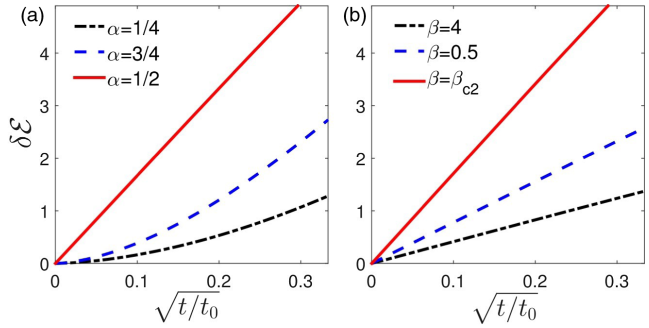

FIG. 3. Dynamics of the total energy density of Bose gas for $a_s(t)$ with different power-law functions of Eq. (6). (a) $\beta = 1$ and $\alpha = 1/4, 1/2, 3/4$. (b) $\alpha = 1/2$ and $\beta = 4$, 0.5, 1.128. $\delta\mathcal{E}$ is plotted in units of $n^2 l_0 \hbar^2/m$ and we have set $t_0 = t_n$ and thus $l_0 = 1/k_n$ in the numerical calculation.

in Fig. 3 for different powers and different coefficients. One can see that the short time behaviors agree very well with that given by Figs. 1(c1) and 1(c2).

Summary.—In summary, we have studied the energy growth rate of degenerate quantum gas driven by increasing the s-wave scattering length a_s from zero, by both analyzing the short time behavior on general situations and numerically solving a concrete example of interacting bosons. Two main results are summarized as follows: (i) At short time, the energy density increases as t^α and α cannot be smaller than $1/2$, and $\alpha = 1/2$ is achieved when $a_s(t)$ varies as $\propto \sqrt{t}$. (ii) When $a_s(t)$ varies as $\propto \sqrt{t}$, the fastest energy increasing is achieved when $a_s(t) = 2\sqrt{\hbar t/(\pi m)}$, with a maximum energy growth given by $4(2+\sqrt{2})\sqrt{\pi\hbar^3 t/m g_2(0)}$ for the spin-1/2 Fermi gas and half of that for the spinless Bose gas. These results also hold for harmonic trapped and finite temperature cases when the trap and the temperature average are performed. This prediction can be directly verified in cold atom experiments.

We emphasize that this maximum energy growth rate is universal, that is, it is independent of any length or energy scale. This is because when a_s varies as $\sqrt{t}$, the entire many-body Schrödinger equation is invariant under a space-time scaling transformation $t \to \lambda^2 t$ and $r \to \lambda r$. Similar examples of such scale invariant many-body dynamics have been studied in Refs. [41–43]. Hence, this result ties together the fastest energy growth with the scaling symmetry, and this is also reminiscent of the equilibrium situation, where the interaction effect is the strongest at unitarity when the system is also scale invariant.

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