Exactness of Mean-Field Equations for Open Dicke Models with an Application to Pattern Retrieval Dynamics

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Open quantum Dicke models are paradigmatic systems for the investigation of light-matter interaction in out-of-equilibrium quantum settings. Albeit being structurally simple, these models can show intriguing physics. However, obtaining exact results on their dynamical behavior is challenging, since it requires the solution of a many-body quantum system with several interacting continuous and discrete degrees of freedom. Here, we make a step forward in this direction by proving the validity of the mean-field semiclassical equations for open multimode Dicke models, which, to the best of our knowledge, so far has not been rigorously established. We exploit this result to show that open quantum multimode Dicke models can behave as associative memories, displaying a nonequilibrium phase transition toward a pattern-recognition phase.

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Since its inception [1], the Dicke model has become a paradigm for the study of light-matter interaction and its equilibrium, as well as its isolated-system dynamical properties, have been widely investigated theoretically and experimentally [2–14]. Today, the interest is in understanding how the presence of an environment leading to dissipative effects affects the behavior of Dicke models. Several arguments indicate the persistence of the Dicke superradiant phase transition [15–18] and those assertions are further supported by numerical [19] and experimental [20] evidence.

Particularly intriguing is the possibility that these nonequilibrium spin-boson systems can feature dynamics akin to associative memories [21,22], i.e., they can display pattern-recognition behavior [23–28], and implementations of this physics are being explored in realistic experimental setups [29]. Couplings between spins and bosons encode different patterns that, in the simplest case, are strings of ± 1 [see Fig. 1(a)]. The overlap ξ_μ of the spin configuration with pattern μ, which plays the role of an order parameter, is defined by means of a generalized "magnetization" [cf. Fig. 1(a)]. Assuming the initial configuration to be close to one pattern, two different regimes may emerge. In the first, the state converges due to dissipation to a stationary one where all information about the initial time is lost. As sketched in Fig. 1(b), this coincides with a regime where the overlaps ξ_μ are all zero. In the other, the state converges instead to a stationary one displaying a finite overlap with the initially stored pattern. In that case, the system "recognizes" the initial condition as a pattern and stores this information in its nonequilibrium steady state. In Dicke models, the observed stationary regime is expected to depend on the spin-boson coupling strength [see Fig. 1(b)].

Understanding whether this pattern-recognition behavior corresponds to a genuine nonequilibrium phase requires the study of quantum systems with a large number of bosons and spins. Simulations in fully quantum regimes beyond perturbative approaches [28–30] are thus infeasible. Analytically, one may study these systems relying on so-called mean-field equations obtained by assuming that the expectation values of products of operators factorize [15,19,31]. However, a proof of the validity of this assumption in nonequilibrium open Dicke models is still missing, and a widespread belief is that a "full quantum treatment" may lead to different results.

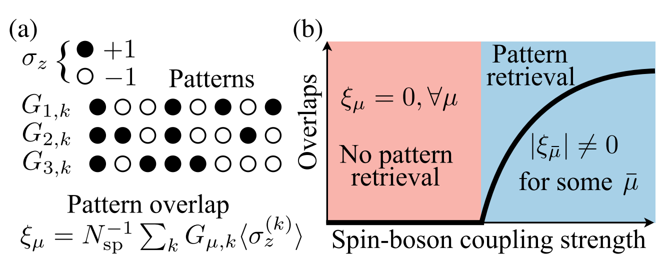


FIG. 1. Pattern recognition in Dicke models. (a) Patterns: strings of ± 1 are encoded in the couplings $G_{\mu,k}$ between $N_{\rm sp}$ spins and M bosonic modes. Each pattern is associated with a mode. The overlap of the quantum state with the patterns is defined as a generalized "magnetization" aligned with the coefficients $G_{\mu,k}$. (b) As a function of the spin-boson coupling strength, the quantum system passes from a disordered phase, in which it cannot store any pattern, to an "ordered" one, in which it can recognize and protect a pattern.

In this Letter, we provide a proof of the exactness of the mean-field assumption for open multimode Dicke models. This result is relevant as it solves an open question on the validity of the semiclassical treatment for these systems. Further, it allows us to establish the existence of a nonequilibrium pattern-recognition phase transition in Dicke models. Our proof, which takes inspiration from Ref. [32], is of broad applicability: it can be adapted to account for the presence of individual spin dissipative processes [19], to account for time-dependent coefficients in the generator [33–35], or even to other models with all-to-all couplings [36–41] and multibody interactions [42–44].

Open multimode Dicke models.—Our Dicke model consists of an ensemble of $N_{\rm sp}$ spins coupled to M different bosonic modes, described by annihilation and creation operators a_μ, $a_\mu^\dagger$, obeying canonical commutation relations [45]. Spins are two-level systems with an excited state $|\bullet\rangle$ and a ground state $|\circ\rangle$. Transitions between states in the kth spin are implemented by the Pauli operator $\sigma_x^{(k)}$, where $\sigma_x|\bullet/\circ\rangle = |\circ/\bullet\rangle$. The operator $\sigma_z^{(k)}$, with $\sigma_z|\bullet\rangle = |\bullet\rangle$ and $\sigma_z|\circ\rangle = -|\circ\rangle$, indicates the presence of an excitation. We also define $\sigma_y^{(k)} = -i\sigma_z^{(k)}\sigma_x^{(k)}$.

The (Markovian) nonequilibrium dynamics of the spin-boson model is implemented by the Lindblad generator $\dot{X} = \mathcal{L}[X]$ [46–48], providing the time evolution of a generic operator X. Defining $n_\mu = a_\mu^\dagger a_\mu$, we consider

$$\mathcal{L}[X] \coloneqq i[H, X] + \sum_{\mu=1}^{M} \kappa_\mu \left(a_\mu^\dagger X a_\mu - \frac{1}{2}\{n_\mu, X\} \right). \quad (1)$$

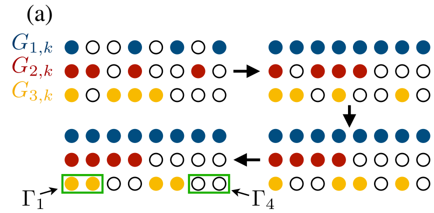
The second term appearing on the right-hand side describes boson losses at rate κ_μ for the different modes, while H is the system Hamiltonian. This operator consists of a free contribution for both spins and bosons,

$$H_F = \Omega \sum_{k=1}^{N_{\rm sp}} \sigma_x^{(k)} + \sum_{\mu=1}^{M} \Omega_\mu n_\mu,$$

and of an interaction term,

$$H_{\rm int} = \frac{g_0}{\sqrt{N_{\rm sp}}} \sum_{\mu=1}^{M} \sum_{k=1}^{N_{\rm sp}} G_{\mu,k}(a_\mu + a_\mu^\dagger)\sigma_z^{(k)}. \quad (2)$$

The coefficients $G_{\mu,k}$ specify the spin-boson interaction. We consider these to be independent identically distributed random variables assuming the values $+1$ or -1 with equal probability, as sketched in Fig. 1(a). The scaling $1/\sqrt{N_{\rm sp}}$, which is typical for these models, is important for establishing a well-defined thermodynamic limit [15] (see also [49] for an application to open systems). For each μ, the string $G_{\mu,k}$ forms a pattern that is encoded in the Hamiltonian. A key result of this work consists in showing that the system can recognize and protect an initially stored pattern for strong enough spin-boson coupling $|g_0|$ (see Fig. 1).



(b)

$f^M_{\mu,k}$	k = 1	2	3	4
μ = 1	+1	+1	+1	+1
2	+1	+1	−1	−1
3	+1	−1	+1	−1

FIG. 2. Mapping to large spins. (a) Example of the mapping for $M = 3$ patterns and $N_{\rm sp} = 8$ spins. The original coupling between the μth mode and the kth spin is encoded in $G_{\mu,k}$. To perform the mapping, we first apply a gauge transformation, making $G_{1,k} = 1$, $\forall\, k$. Then, we reorder $G_{2,k}$ to put all $+1$ first. Finally, the last pattern is reordered by moving the $+1$ toward the right and the -1 toward the left in each subblock identified by the new $G_{2,k}$. In this way, 2^{M-1} subsets of spins Γ_k, equally coupled with each mode, are identified (Γ_1 and Γ_4 are highlighted in the figure for clarity). (b) These subsets of spins are described by "large-spin" operators and couple to bosons as specified by the matrix $f^M_{\mu,k}$.

Before showing this, we make some considerations that bring the model into a convenient form (see Fig. 2). First, without loss of generality, the first pattern, $G_{1,k}$, which is made of ± 1, can be brought into a pattern with all $+1$ by means of the gauge transformation $\sigma_z \to -\sigma_z$ applied to those spins h for which, originally, $G_{1,h} = -1$. Then, we reorder the remaining $M-1$ rows of $G_{\mu,k}$. We look at $G_{2,k}$: this has ± 1 at random positions. We now relabel the spins. We take those with $G_{2,h} = +1$ to the left and those with $G_{2,h} = -1$ to the right. This reshaping does not affect the first pattern. In addition, there is a $\tilde{k}$ such that for $k \le \tilde{k}$, $G_{2,k} = 1$ while $G_{2,k} = -1$ otherwise. We then move to $G_{3,k}$, and we relabel spins as follows. In the subset of spins for which $G_{2,k} = 1$, we have values of $G_{3,k}$ that can be positive and negative. We thus reorder this subsequence in such a way that all $+1$ are moved on the left and -1 on the right. The same can be done for the subset of the sequence $G_{3,k}$ corresponding to values $G_{2,k} = -1$. This procedure, sketched in Fig. 2, is then iterated up to the last pattern.

This mapping generates 2^{M-1} subsets of spins, described by "large-spin" operators and interacting with the bosonic modes. For $N_{\rm sp} \gg 1$, these subsets are expected to have the same number of spins. This is due to the fact that, given the statistical properties of the $G_{\mu,k}$, in a large enough set of randomly chosen spins there is, at leading order in extensivity of the set, an equal number of $+1$ and of -1, in their $G_{\mu,k}$. We can thus consider subsets to contain $N = N_{\rm sp}/2^{M-1}$ spins. In this representation, the interaction Hamiltonian reads

$$H_{\rm int}^N = \frac{g}{\sqrt{N}}\sum_{\mu=1}^{M}\sum_{k=1}^{2^{M-1}} f_{\mu,k}^M (a_\mu + a_\mu^\dagger) S_{z,k}, \quad (3)$$

where $S_{a,k} = \sum_{h\in\Gamma_k}\sigma_a^{(h)}$ is the sum of the σ_a-spin operators, which belong to the kth subset, denoted as Γ_k [see Fig. 2(a)]. In addition, we have defined $g = g_0/\sqrt{2^{M-1}}$. The coefficients $f_{\mu,k}^M = \pm 1$ specify the interaction between spins in Γ_k and the μth boson. This representation provides a more compact formulation of the model. The mapping can be extended to consider models whose spin-only part of the dynamical generator is not invariant under the gauge transformation or also to consider generic distributions for $G_{\mu,k}$ [50].

Mean-field dynamics.—As a consequence of the previous mapping, it is sufficient for understanding the behavior of our nonequilibrium Dicke model to focus on the dynamics of the large-spin operators. In this representation, the generator is $\mathcal{L}_N$, the same as the one in Eq. (1) with the Hamiltonian rewritten as $H_N = H_F + H_{\rm int}^N$. The expectation of time-evolved operators $X_t = e^{t\mathcal{L}_N}[X]$ is given by $\langle X\rangle_t = \omega_t(X) \coloneqq \omega(e^{t\mathcal{L}_N}[X])$, where the functional ω represents the initial state and ω_t the time-evolved one. As a consequence, we have

$$\dot\omega_t(X) = \omega_t(\mathcal{L}_N[X]). \quad (4)$$

We are interested in the "macroscopic" operators [51–54]

$$m_{a,k}^N \coloneqq \frac{1}{N} S_{a,k}, \quad \text{for } a = x, y, z, \qquad \alpha_{\mu,N} \coloneqq \frac{a_\mu}{\sqrt{N}}. \quad (5)$$

The first ones are the usual average magnetization operators of the spin ensembles, while the rescaled bosonic operators appear typically in superradiant transitions. Indeed, a nonvanishing expectation of these operators implies a macroscopic ($\propto N$) bosonic occupation.

We want to derive the dynamics of these quantum operators in the thermodynamic limit N, $N_{\rm sp}\to\infty$. We thus compute the action of the generator $\mathcal{L}_N$ on the operators in Eq. (5) and get [50]

$$\mathcal{L}_N[m_{a,k}^N] = \sum_b\left(-2\Omega\epsilon_{xab} - 2g\sum_\mu \epsilon_{zab} f_{\mu,k}^M(\alpha_{\mu,N}^\dagger + \alpha_{\mu,N})\right) m_{b,k}^N$$

$$\mathcal{L}_N[\alpha_{\mu,N}] = -\left(i\Omega_\mu + \frac{\kappa_\mu}{2}\right)\alpha_{\mu,N} - ig\sum_{k=1}^{2^{M-1}} f_{\mu,k}^M m_{z,k}^N, \quad (6)$$

where ϵ_{abc} is the fully antisymmetric tensor. To make progress, one typically assumes that the dynamics does not generate correlations among the different constituents in the thermodynamic limit, so that expectation values factorize. This leads to the mean-field equations

$$\dot m_{a,k} = -2\Omega\sum_b \epsilon_{xab} m_{b,k} - 2g\sum_{b,\mu}\epsilon_{zab} f_{\mu,k}^M(\alpha_\mu^\dagger + \alpha_\mu) m_{b,k},$$

$$\dot\alpha_\mu = -\left(i\Omega_\mu + \frac{\kappa_\mu}{2}\right)\alpha_\mu - ig\sum_{k=1}^{2^{M-1}} f_{\mu,k}^M m_{z,k}. \quad (7)$$

In order to show that they are exact in the thermodynamic limit, we need to prove that

$$\lim_{N\to\infty}\omega_t(m_{a,k}^N) - m_{a,k}(t) = 0 = \lim_{N\to\infty}\omega_t(\alpha_{\mu,N}) - \alpha_\mu(t), \quad (8)$$

meaning that the expectation of the operators of Eq. (5) behaves, for large N, as the time-dependent scalar functions $m_{a,k}(t)$, $\alpha_\mu(t)$ obeying Eq. (7). To obtain this result, a proper strategy must be identified. In particular, an appropriate "cost function" controlling the above limits is needed. Defining $E_{a,k} = m_{a,k}^N - m_{a,k}(t)$ and $A_\mu = \alpha_{\mu,N} - \alpha_\mu(t)$, we consider

$$\mathcal{E}_N(t) \coloneqq \sum_{k=1,a=x,y,z}^{2^{M-1}}\omega_t(E_{a,k}^2) + \sum_{\mu=1}^{M}\omega_t(A_\mu^\dagger A_\mu + A_\mu A_\mu^\dagger). \quad (9)$$

This quantity is a sum of positive contributions consisting of the expectation of the square of the distance of the operators from their mean-field counterpart. Namely, $\mathcal{E}_N(t)$ measures the fraction of spins or bosons not behaving as dictated by Eq. (7). In addition, via Cauchy-Schwarz inequality, one can show that

$$|\omega_t(E_{a,k})| \le \sqrt{\omega_t(E_{a,k}^2)} \le \sqrt{\mathcal{E}_N(t)}, \quad (10)$$

and thus $\lim_{N\to\infty}\mathcal{E}_N(t)$ controls the limits in Eq. (8), as desired. For physical initial states [52–54], with short-range correlations, one has $\lim_{N\to\infty}\mathcal{E}_N(0) = 0$. As we now show, for these states, $\mathcal{E}_N(t)$ vanishes for large N, implying the exactness of the mean-field assumption for these nonequilibrium multimode Dicke models.

Theorem.—With the above definitions, if the initial state of the system is such that $\lim_{N\to\infty}\mathcal{E}_N(0) = 0$, then, for all finite t, we have that $\lim_{N\to\infty}\mathcal{E}_N(t) = 0$.

Proof of theorem.—The full proof is reported in Ref. [50]. Here we provide the main steps. The idea is to use Gronwall's lemma [55,56], which states that if a positive, bounded, and N-independent constant C such that $\dot{\mathcal{E}}_N(t) \le C\mathcal{E}_N(t)$ exists, then

$$\mathcal{E}_N(t) \le e^{Ct}\mathcal{E}_N(0). \quad (11)$$

With the assumption $\lim_{N\to\infty}\mathcal{E}_N(0) = 0$, letting $N\to\infty$ in the above relation would prove the theorem. What is missing is to show that such constant C indeed exists. This can be achieved by directly inspecting the time derivative of all terms forming $\mathcal{E}_N(t)$. They are given by sums of contributions having, for instance, the form $\omega_t(E_{b,k}BA_\mu)$, where B can either be an operator or a scalar from Eq. (7). In addition, it can be shown that

$$|\omega_t(E_{b,k}BA_\mu)| \leq \|B\|\mathcal{E}_N(t),$$

and this gives a way to estimate a suitable constant C. We thus obtain

$$\frac{d}{dt}\mathcal{E}_N(t) \leq \left|\frac{d}{dt}\mathcal{E}_N(t)\right| \leq C\mathcal{E}_N(t),$$

and we can exploit Gronwall's lemma to finish the proof of the theorem as already discussed. □

Pattern-recognition phase transition.—With the above result, we establish that the semiclassical mean-field equations [Eq. (7)] correctly capture the behavior of our system in the thermodynamic limit. As such, we can now use these equations to unveil the presence of a nonequilibrium pattern-recognition phase transition.

In the original formulation of the problem [see Eq. (2) and Fig. 1(a)], we can define the overlap of the quantum state of the spins with the pattern μ as

$$\xi_\mu \coloneqq \lim_{N_{\rm sp}\to\infty}\frac{1}{N_{\rm sp}}\sum_{k=1}^{N_{\rm sp}} G_{\mu,k}\langle\sigma_z^{(k)}\rangle_t.$$

This equation shows that, if the expectation value of the operator σ_z is, for each spin, aligned with the corresponding value of $G_{\mu,k}$, then the overlap $|\xi_\mu|$ is different from zero (pattern retrieval). Otherwise, ξ_μ tends to vanish for $N_{\rm sp}\to\infty$ (pattern not retrieved). In the large-spin representation, the overlaps can be expressed in terms of the coefficients $f^M_{\mu,k}$ and of the macroscopic operators $m^N_{z,k}$, [cf. Eq. (3) and Fig. 2]. In particular,

$$\xi_\mu = \frac{1}{2^{M-1}}\sum_{k=1}^{2^{M-1}} f^M_{\mu,k}\lim_{N\to\infty}\omega_t(m^N_{z,k}).$$

Invoking our theorem, we can thus study the dynamics and the stationary properties of these overlaps through the scalars $m_{z,k}$, obeying the mean-field equations [Eq. (7)].

To prove the existence of the phase transition, we first show the presence of different stationary solutions to Eq. (7), featuring a finite overlap with one of the patterns. Without loss of generality, we consider all rates of the dynamical generator to be positive and, further, that the constant of motion $m^2_{T,k} = \sum_a m^2_{a,k} = 1,\ \forall\, k$. Then, we take the ansatz solution $m_{z,k} = f^M_{\nu,k}|z|$, aligned with pattern ν, and look for conditions ensuring its existence as a stationary solution for Eq. (7). Note that such ansatz has indeed a finite overlap with pattern ν, since $\xi_\nu = |z|$ while $\xi_\mu = 0\ \forall\, \mu \neq \nu$, and that also $m_{z,k} = -f^M_{\nu,k}|z|$ would be valid, with $\xi_\nu = -|z|$.

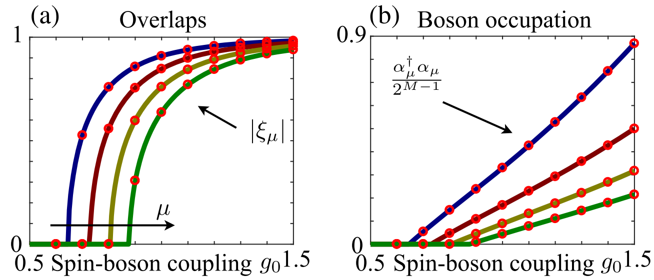


FIG. 3. Pattern-recognition phase transition. Comparison between theoretical prediction (solid lines) and numerical simulations of the mean-field equations (circles). We consider $M = 4$. (a) Each curve corresponds to the stationary overlap $|\xi_\mu|$ computed from the initial condition $\xi_\mu = 1$ as a function of g_0, for $\Omega = 0.5$, $\Omega_\mu = \Omega(\mu + 2)$. Rates are in units of κ. Different colors correspond to values of μ growing as indicated by the arrow. Both theoretical and numerical results display a nonequilibrium phase transition, as shown by the behavior of the overlap. (b) Same parameters and same order for the curves as in (a). The occupation of the μth bosonic mode becomes macroscopically occupied when the corresponding pattern is stored in the stationary state.

By substituting the ansatz for $m_{z,k}$ in Eq. (7), taking $m_{y,k} = 0$ and appropriately fixing the values of $m_{x,k}$ (see Ref. [50] for details), we find that the relation

$$|z| = \sqrt{1 - \frac{1}{4g_0^4}\left(\frac{\Omega}{\Omega_\nu}\right)^2\left[\Omega_\nu^2 + \left(\frac{\kappa_\nu}{2}\right)^2\right]^2} \tag{12}$$

must be satisfied in order for the assumed stationary solution to exist. This is not always the case; indeed, $|z|$ must be a positive real number, $|z| \in [0; 1]$, and this only happens if the argument of the square root is positive. This observation yields a critical value,

$$g_{\rm crit} = \sqrt{\frac{1}{2}\left(\frac{\Omega}{\Omega_\nu}\right)\left[\Omega_\nu^2 + \left(\frac{\kappa_\nu}{2}\right)^2\right]},$$

such that for $g_0 \geq g_{\rm crit}$ the ansatz solution exists, with $|z|$ given by Eq. (12). On the other hand, if $g_0 < g_{\rm crit}$, we can only have $|z| = 0$, and we are outside the pattern-recognition phase. The critical g depends on the pattern through the parameters Ω_ν, κ_ν [see also Figs. 3(a),(b)]. Further, note that a finite stationary overlap corresponds to a macroscopic occupation of the associated bosonic mode. Our theorem indeed implies $N^{-1}\langle a^\dagger_\mu a_\mu\rangle \to |\alpha_\mu|^2$ for $N \to \infty$, and we have $|\alpha_\mu| \propto |\xi_\mu|$ [50]. This feature, shown in Fig. 3(b), establishes a connection between pattern recognition and the superradiant phase transitions in open multimode Dicke models.

Discussion.—We have derived two key results for multimode Dicke models. First, we have shown that the mean-field assumption, typically exploited to consider the large-scale behavior of these systems, actually provides an exact description in the thermodynamic limit. Second, we have used this new insight to reveal the presence of a nonequilibrium phase transition from a disordered phase to a pattern-recognition phase in open multimode Dicke models. The stability of stationary solutions, such as the one of Eq. (12), for open Dicke models has been shown, for instance, in Refs. [15,19]. For the multimode settings investigated here, the agreement of our numerical results with analytical ones (cf. Fig. 3) suggests that the proposed stationary states, having finite overlap with the patterns, possess stable basins of attraction in the pattern-recognition phase. Interestingly, the critical spin-boson coupling strength depends on the specific pattern through the corresponding bosonic mode parameters. This may allow for intermediate regimes of pattern recognition, where only certain patterns can be stored and retrieved.

Following Ref. [37], we remark that the validity of the semiclassical Eq. (5) provides a necessary ingredient to obtain mathematically rigorous results on quantum fluctuations. It would be interesting to exploit it to reobtain bosonic descriptions [57,58] employed for the investigation of quantum fluctuations in closed Dicke models and to extend these to open systems, via quantum central limit theorems [37,53,59]. Contrary to Holstein-Primakoff approximations, these procedures do not assume a conserved total spin operator and are thus more general [15].

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[1] R. H. Dicke, Coherence in spontaneous radiation processes, Phys. Rev. **93**, 99 (1954).

[2] Y. K. Wang and F. T. Hioe, Phase transition in the Dicke model of superradiance, Phys. Rev. A **7**, 831 (1973).

[3] F. T. Hioe, Phase transitions in some generalized Dicke models of superradiance, Phys. Rev. A **8**, 1440 (1973).

[4] H. J. Carmichael, C. W. Gardiner, and D. F. Walls, Higher order corrections to the Dicke superradiant phase transition, Phys. Lett. **46A**, 47 (1973).

[5] K. Hepp and E. H. Lieb, Equilibrium statistical mechanics of matter interacting with the quantized radiation field, Phys. Rev. A **8**, 2517 (1973).

[6] G. Comer Duncan, Effect of antiresonant atom-field interactions on phase transitions in the Dicke model, Phys. Rev. A **9**, 418 (1974).

[7] E. B. Davies, Exact dynamics of an infinite-atom Dicke maser model, Commun. Math. Phys. **33**, 187 (1973).

[8] F. Dimer, B. Estienne, A. S. Parkins, and H. J. Carmichael, Proposed realization of the Dicke-model quantum phase transition in an optical cavity QED system, Phys. Rev. A **75**, 013804 (2007).

[9] Z. Zhang, C. H. Lee, R. Kumar, K. J. Arnold, S. J. Masson, A. S. Parkins, and M. D. Barrett, Nonequilibrium phase transition in a spin-1 Dicke model, Optica **4**, 424 (2017).

[10] P. Domokos and H. Ritsch, Collective Cooling and Self-Organization of Atoms in a Cavity, Phys. Rev. Lett. **89**, 253003 (2002).

[11] A. T. Black, H. W. Chan, and V. Vuletić, Observation of Collective Friction Forces due to Spatial Self-Organization of Atoms: From Rayleigh to Bragg Scattering, Phys. Rev. Lett. **91**, 203001 (2003).

[12] D. Nagy, G. Kónya, G. Szirmai, and P. Domokos, Dicke-Model Phase Transition in the Quantum Motion of a Bose-Einstein Condensate in an Optical Cavity, Phys. Rev. Lett. **104**, 130401 (2010).

[13] K. Baumann, C. Guerlin, F. Brennecke, and T. Esslinger, Dicke quantum phase transition with a superfluid gas in an optical cavity, Nature (London) **464**, 1301 (2010).

[14] K. Baumann, R. Mottl, F. Brennecke, and T. Esslinger, Exploring Symmetry Breaking at the Dicke Quantum Phase Transition, Phys. Rev. Lett. **107**, 140402 (2011).

[15] P. Kirton, M. M. Roses, J. Keeling, and E. G. Dalla Torre, Introduction to the Dicke model: From equilibrium to nonequilibrium, and vice versa, Adv. Quantum Tech. **2**, 1800043 (2019).

[16] M. M. Roses and E. G. Dalla Torre, Dicke model, PLoS One **15**, e0235197 (2020).

[17] C.-M. Halati, A. Sheikhan, H. Ritsch, and C. Kollath, Numerically Exact Treatment of Many-Body Self-Organization in a Cavity, Phys. Rev. Lett. **125**, 093604 (2020).

[18] A. V. Bezvershenko, C.-M. Halati, A. Sheikhan, C. Kollath, and A. Rosch, Dicke transition in open many-body systems determined by fluctuation effects, arXiv:2012.11823.

[19] P. Kirton and J. Keeling, Suppressing and Restoring the Dicke Superradiance Transition by Dephasing and Decay, Phys. Rev. Lett. **118**, 123602 (2017).

[20] J. Klinder, H. Keßler, M. Wolke, L. Mathey, and A. Hemmerich, Dynamical phase transition in the open Dicke model, Proc. Natl. Acad. Sci. U.S.A. **112**, 3290 (2015).

[21] J. J. Hopfield, Neural network and physical systems with emergent collective computational abilities, Proc. Natl. Acad. Sci. U.S.A. **79**, 2554 (1982).

[22] A. Fuchs and H. Haken, Pattern recognition and associative memory as dynamical processes in a synergetic system, Biol. Cybern. **60**, 17 (1988).

[23] S. Gopalakrishnan, B. L. Lev, and P. M. Goldbart, Frustration and Glassiness in Spin Models with Cavity-Mediated Interactions, Phys. Rev. Lett. **107**, 277201 (2011).

[24] S. Gopalakrishnan, B. L. Lev, and P. M. Goldbart, Exploring models of associative memory via cavity quantum electrodynamics, Philos. Mag. **92**, 353 (2012).

[25] P. Rotondo, M. Cosentino Lagomarsino, and G. Viola, Dicke Simulators with Emergent Collective Quantum Computational Abilities, Phys. Rev. Lett. **114**, 143601 (2015).

[26] V. Torggler, S. Krämer, and H. Ritsch, Quantum annealing with ultracold atoms in a multimode optical resonator, Phys. Rev. A **95**, 032310 (2017).

[27] P. Rotondo, M. Marcuzzi, J. P. Garrahan, I. Lesanovsky, and M. Müller, Open quantum generalization of Hopfield neural networks, J. Phys. A **51**, 115301 (2018).

[28] E. Fiorelli, M. Marcuzzi, P. Rotondo, F. Carollo, and I. Lesanovsky, Signatures of Associative Memory Behavior in a Multimode Dicke Model, Phys. Rev. Lett. **125**, 070604 (2020).

[29] B. P. Marsh, Y. Guo, R. M. Kroeze, S. Gopalakrishnan, S. Ganguli, J. Keeling, and B. L. Lev, Enhancing associative memory recall and storage capacity using confocal cavity QED, arXiv:2009.01227 [Phys. Rev. X (to be published)].

[30] E. Fiorelli, P. Rotondo, F. Carollo, M. Marcuzzi, and I. Lesanovsky, Dynamics of strongly coupled disordered dissipative spin-boson systems, Phys. Rev. Research **2**, 013198 (2020).

[31] K. C. Stitely, A. Giraldo, B. Krauskopf, and S. Parkins, Nonlinear semiclassical dynamics of the unbalanced, open dicke model, Phys. Rev. Research **2**, 033131 (2020).

[32] P. Pickl, A simple derivation of mean field limits for quantum systems, Lett. Math. Phys. **97**, 151 (2011).

[33] W. Niedenzu and G. Kurizki, Cooperative many-body enhancement of quantum thermal machine power, New J. Phys. **20**, 113038 (2018).

[34] F. Carollo, F. M. Gambetta, K. Brandner, J. P. Garrahan, and I. Lesanovsky, Nonequilibrium Quantum Many-Body Rydberg Atom Engine, Phys. Rev. Lett. **124**, 170602 (2020).

[35] F. Carollo, K. Brandner, and I. Lesanovsky, Nonequilibrium Many-Body Quantum Engine Driven by Time-Translation Symmetry Breaking, Phys. Rev. Lett. **125**, 240602 (2020).

[36] F. Bagarello and G. Morchio, Dynamics of mean-field spin models from basic results in abstract differential equations, J. Stat. Phys. **66**, 849 (1992).

[37] F. Benatti, F. Carollo, R. Floreanini, and H. Narnhofer, Quantum spin chain dissipative mean-field dynamics, J. Phys. A **51**, 325001 (2018).

[38] F. Iemini, A. Russomanno, J. Keeling, M. Schirò, M. Dalmonte, and R. Fazio, Boundary Time Crystals, Phys. Rev. Lett. **121**, 035301 (2018).

[39] M. A. Norcia, R. J. Lewis-Swan, J. R. K. Cline, B. Zhu, A. M. Rey, and J. K. Thompson, Cavity-mediated collective spin-exchange interactions in a strontium superradiant laser, Science **361**, 259 (2018).

[40] B. Buča and D. Jaksch, Dissipation Induced Nonstationarity in a Quantum Gas, Phys. Rev. Lett. **123**, 260401 (2019).

[41] D. Huybrechts, F. Minganti, F. Nori, M. Wouters, and N. Shammah, Validity of mean-field theory in a dissipative critical system: Liouvillian gap, $\mathbb{PT}$-symmetric antigap, and permutational symmetry in the XYZ model, Phys. Rev. B **101**, 214302 (2020).

[42] P. Wang and R. Fazio, Dissipative phase transitions in the fully-connected Ising model with p-spin interaction, Phys. Rev. A **103**, 013306 (2021).

[43] A. L. Grimsmo and A. S. Parkins, Dissipative Dicke model with nonlinear atom–photon interaction, J. Phys. B **46**, 224012 (2013).

[44] L. Garbe, P. Wade, F. Minganti, N. Shammah, S. Felicetti, and F. Nori, Dissipation-induced bistability in the two-photon Dicke model, Sci. Rep. **10**, 13408 (2020).

[45] D. Petz, *An invitation to the Algebra of Canonical Commutation Relations.* (Leuven University Press, Leuven, 1990).

[46] G. Lindblad, On the generators of quantum dynamical semigroups, Commun. Math. Phys. **48**, 119 (1976).

[47] V. Gorini, A. Kossakowski, and E. C. G. Sudarshan, Completely positive dynamical semigroups of N-level systems, J. Math. Phys. (N.Y.) **17**, 821 (1976).

[48] H. P. Breuer and F. Petruccione, *The Theory of Open Quantum Systems* (Oxford University Press, Oxford, 2002).

[49] M. Merkli and A. Rafiyi, Mean field dynamics of some open quantum systems, Proc. R. Soc. A **474**, 20170856 (2018).

[50] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.126.230601 for details on the proof of the theorem and on the derivation of the stationary solutions to the mean-field equations with finite overlap.

[51] O. E. Lanford and D. Ruelle, Observables at infinity and states with short range correlations in statistical mechanics, Commun. Math. Phys. **13**, 194 (1969).

[52] F. Strocchi, *Symmetry Breaking* (Springer-Verlag, Berlin Heidelberg, 2005).

[53] A. F. Verbeure, *Many-Body Boson Systems: Half a Century Later* (Spinger-Verlag, London, 2010).

[54] O. Bratteli and D. W. Robinson, *Operator Algebras and Quantum Statistical Mechanics: Volume 1: C*-and W*-Algebras. Symmetry Groups. Decomposition of States* (Springer-Verlag, Berlin Heidelberg, 2012).

[55] T. H. Gronwall, Note on the derivatives with respect to a parameter of the solutions of a system of differential equations, Ann. Math. **20**, 292 (1919).

[56] R. Bellman, The stability of solutions of linear differential equations, Duke Math. J. **10**, 643 (1943).

[57] C. Emary and T. Brandes, Quantum Chaos Triggered by Precursors of a Quantum Phase Transition: The Dicke Model, Phys. Rev. Lett. **90**, 044101 (2003).

[58] C. Emary and T. Brandes, Chaos and the quantum phase transition in the Dicke model, Phys. Rev. E **67**, 066203 (2003).

[59] D. Goderis and P. Vets, Central limit theorem for mixing quantum systems and the CCR-algebra of fluctuations, Commun. Math. Phys. **122**, 249 (1989).