Set Coherence: Basis-Independent Quantification of Quantum Coherence

Sébastien Designolle,[1] Roope Uola,[1] Kimmo Luoma,[2] and Nicolas Brunner[1]
[1]Department of Applied Physics, University of Geneva, 1211 Geneva, Switzerland
[2]Institut für Theoretische Physik, Technische Universität Dresden, D-01062 Dresden, Germany

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The coherence of an individual quantum state can be meaningfully discussed only when referring to a preferred basis. This arbitrariness can, however, be lifted when considering sets of quantum states. Here we introduce the concept of set coherence for characterizing the coherence of a set of quantum systems in a basis-independent way. We construct measures for quantifying set coherence of sets of quantum states as well as quantum measurements. These measures feature an operational meaning in terms of discrimination games and capture precisely the advantage offered by a given set over incoherent ones. Along the way, we also connect the notion of set coherence to various resource-theoretic approaches recently developed for quantum systems.

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Introduction.—The superposition principle is a direct consequence of the linearity of quantum mechanics. Given a set of orthogonal quantum states, their coherent superposition also represents a possible state. The coherence stems from the fact that the phase relation between the various orthogonal states in the superposition is well defined. This key concept of quantum theory has broad implications. It is a central element for the existence of genuine randomness in quantum measurements and is also the basis of the phenomenon of entanglement. Consequently, these ideas play a fundamental role in quantum information processing, quantum metrology, quantum transport, and many more important research directions.

A natural question is therefore to characterize the coherence of quantum systems, in particular for quantum states. An intense research effort has been devoted to these questions in recent years, leading notably to the development of a general resource theory of quantum coherence; see, e.g., Refs. [1–8]. There the coherence of a quantum state can be quantified via specific measures. Interestingly, these measures have been shown to have an operational meaning, capturing precisely the advantage offered by a given quantum state (with coherence) for a certain task, compared to any possible incoherent quantum state [9,10]. The case of quantum measurements has been investigated as well [11,12].

However, as intuition suggests, the above ideas can be meaningfully formalized only with respect to a preferred basis (or preferred reference frame). Indeed, a single quantum state has intrinsically no coherence; for instance, all pure quantum states are equivalent to each other if no basis is specified. That is, the notion of quantum coherence for a single state is necessarily a relative property; it can be defined only with regard to a given reference. While the choice of a preferred basis can be motivated in certain cases (for instance, choosing the energy basis in a thermodynamic setting), this basis dependence arguably limits the general scope and applicability of these ideas.

In this work, we follow a different approach for defining and quantifying the coherence of quantum systems in an absolute way, i.e., without referring to any preferred basis. This provides a basis-independent (or reference frame–independent) quantification of coherence. The main idea is to consider a set of quantum states, instead of a single state. Consider for instance a pair of nonorthogonal pure states. Clearly there exists no unitary that can map this pair of states to an orthogonal pair. Hence, for any possible basis choice, the pair will necessarily feature some level of coherence. More generally, given a set of states, a meaningful quantity can be defined by minimizing the coherence of the states in the set over all possible basis choices. We term this quantity set coherence. We note that a related approach was proposed in Ref. [13], where the authors developed an entropic measure for the quantumness of an ensemble of states, reflecting the entropy production in the ensemble. In contrast, our approach is motivated by the more recent resource-theoretic perspective and thus tailored to the problem of characterizing the intrinsic coherence of a set of states.

Below, we start by introducing formally the concept of set coherence. Then, we present two measures for set coherence of quantum states. We derive explicit expressions for the set coherence of sets of qubit states and discuss sets featuring the highest set coherence. This reveals a direct connection to the notion of commutativity. Next, we investigate the notion of set coherence for quantum measurements. Then, we demonstrate the operational meaning of our measures via quantum games. In particular, the measure of set coherence of a set of states quantifies the advantage offered by this set over any incoherent set.

Finally, we conclude with a discussion on possible future directions.

Set coherence.—Consider a set of n quantum states

$$\vec{\varrho} = \{\varrho_j\}_{j=1}^n, \tag{1}$$

where all states ϱ_j are defined on a Hilbert space of dimension d. While each state in the set has *per se* no absolute coherence (one can always express ϱ_j in the basis in which it is diagonal), the situation can be different when considering the entire set of states $\vec{\varrho}$. This motivates us to introduce a notion of set coherence characterizing the coherence of the set $\vec{\varrho}$ in an absolute manner, i.e., without referring to any specific basis or reference frame.

To proceed, we follow a resource-theoretic approach. Namely, we first define free sets of states, that is, those featuring zero set coherence. Intuitively, the latter consist of sets of states $\vec{\varrho}$, such that there exists a choice of basis (a unitary U) for which all states in the set $U\varrho_j U^\dagger$ become diagonal. Formally, the free set is given by

$$\mathcal{F}_n = \left\{\vec{\varrho} | \exists\, U,\ \forall\, j, U\varrho_j U^\dagger = \sum_{i=1}^d p(i|j)|i\rangle\langle i|\right\}, \tag{2}$$

where $p(\cdot|j)$ is a probability distribution for $j = 1\ldots n$ and $\{|i\rangle\}_{i=1}^d$ denotes the computational basis. If no such unitary can be found, then the set of states features nonzero set coherence. Our goal is now to construct a measure for this effect, which is not straightforward due to the nonconvexity of the free set $\mathcal{F}_n$ (see the Supplemental Material [14], Sec. I).

We first consider the free set defined in Eq. (2) but restricting for the moment to a fixed unitary, taken for simplicity to be $U = \mathbb{1}$. This corresponds to having a fixed reference basis, namely, the computational one; in due course, reference to this arbitrary choice of basis will disappear. The free set is given by [8]

$$F_n = \left\{\vec{\varrho} | \forall\, j, \varrho_j = \sum_{i=1}^d p(i|j)|i\rangle\langle i|\right\} \tag{3}$$

and is now convex, so that one can define the so-called generalized robustness of a given set of states $\vec{\varrho}$ with respect to F_n, i.e.,

$$\mathcal{R}_{F_n}(\vec{\varrho}) = \min\left\{t \geq 0 | \frac{\vec{\varrho} + t\vec{\tau}}{1+t} \in F_n\right\}, \tag{4}$$

where the optimization is performed over all sets $\vec{\tau}$ with the same number of states and dimension as $\vec{\varrho}$.

To remove the dependency on a reference basis, we now minimize the above measure with respect to any possible basis choice. Formally, we define the max robustness of set coherence of $\vec{\varrho}$ as

$$\mathcal{R}(\vec{\varrho}) = \min_U \mathcal{R}_{F_n}(U\vec{\varrho}U^\dagger). \tag{5}$$

where the minimization is performed over all unitaries U acting on $\mathbb{C}^d$. Clearly, the above quantity is basis independent and corresponds to the intrinsic (or minimal) amount of coherence present in the set.

While this measure captures a general property of the set $\vec{\varrho}$, one can nevertheless express it in terms of the robustness of the individual states ϱ_j in the set. More precisely, we show in [14], Sec. I that

$$\mathcal{R}(\vec{\varrho}) = \min_U \max_j \mathcal{R}_{F_1}(U\varrho_j U^\dagger), \tag{6}$$

hence the name max robustness. This naturally suggests another possible measure for set coherence, replacing the maximum by the average over the states. We thus define the mean robustness of set coherence as

$$\mathcal{R}_1(\vec{\varrho}) = \min_U \frac{1}{n}\sum_{j=1}^n \mathcal{R}_{F_1}(U\varrho_j U^\dagger). \tag{7}$$

Importantly, both $\mathcal{R}$ and $\mathcal{R}_1$ are faithful measures, in the sense that $\mathcal{R}(\vec{\varrho}) = 0$ if and only if the set $\vec{\varrho}$ is incoherent, i.e., belongs to $\mathcal{F}_n$. While these measures are not convex (see [14], Sec. I), similarly to the free set $\mathcal{F}_n$, we will see below that they nevertheless have a clear operational meaning.

In the following we will investigate these measures in various scenarios. The max robustness $\mathcal{R}$ will be useful for discussing the set coherence of quantum measurements. For sets of states, we will focus our attention mostly on the second measure $\mathcal{R}_1$. It turns out that $\mathcal{R}_1$ is more convenient to calculate, as we will see below, and it provides a lower bound on $\mathcal{R}$, as clearly $\mathcal{R}_1(\vec{\varrho}) \leq \mathcal{R}(\vec{\varrho})$.

Sets of qubit states.—We now illustrate the above ideas considering sets of qubit states ($d = 2$). We take advantage of the Bloch sphere representation: $\vec{q}_j$ denotes the Bloch vector of the state ϱ_j, i.e., $\varrho_j = (\mathbb{1} + \vec{q}_j\cdot\vec{\sigma})/2$, where $\vec{\sigma}$ contains the Pauli matrices $(\sigma_x, \sigma_y, \sigma_z)$.

To compute the set coherence measure $\mathcal{R}_1(\vec{\varrho})$ for qubit sets, we proceed as follows. Note first that the basis-dependent free set F_n, as defined in Eq. (3), now corresponds to sets of states all aligned with the vertical axis of the sphere (i.e., diagonal in the σ_z basis). As shown in Ref. [10] the robustness of each individual state reduces to the norm of its off-diagonal elements, i.e.,

$$\mathcal{R}_{F_1}(\varrho_j) = 2\left|\langle 0\left|\frac{\mathbb{1} + \vec{q}_j\cdot\vec{\sigma}}{2}\right|1\rangle\right| = \|\vec{q}_j\||\sin(\vec{e}_z, \vec{q}_j)|, \tag{8}$$

where $(\vec{e}_z, \vec{q}_j)$ is the angle between $\vec{q}_j$ and the z axis. With this, Eq. (7) simplifies to

TABLE I. Maximal values $\mathcal{R}_1^*$ of the set coherence measure $\mathcal{R}_1$ for sets of n qubit states ($d = 2$). The sets achieving these optimal values are also described by means of the geometry of their Bloch representation (see the main text for details).

n	$\mathcal{R}_1^*$	Optimal sets
2	$\frac{1}{2}$	Orthonormal pair (e.g., X, Z)
3	$\frac{2}{3}$	Orthonormal basis (e.g., X, Y, Z)
4	$1/\sqrt{2} \approx 0.7071$	Regular tetrahedron
6	$\sqrt{5}/3 \approx 0.7454$	Half of an icosahedron
∞	$\pi/4 \approx 0.7854$	Uniform distribution

$$\mathcal{R}_1(\vec{\varrho}) = \frac{1}{n} \min_{\vec{p} \in S^2} \sum_{j=1}^{n} \|\vec{q}_j\| |\sin(\vec{p}, \vec{q}_j)|, \tag{9}$$

where the minimization is now performed over unit-length vectors $\vec{p}$ on the Bloch sphere S^2. The optimal vector $\vec{p}$ indicates the basis choice where the coherence of the set $\vec{\varrho}$ is minimized. Note also that by using the relation $\|[\rho, \eta]\| = \frac{1}{2}\|\vec{r}\|\|\vec{v}\||\sin(\vec{r}, \vec{v})|$, where $\vec{r}$ and $\vec{v}$ are the Bloch vectors of the states ρ and η, respectively, one can rewrite

$$\mathcal{R}_1(\vec{\varrho}) = \frac{2}{n} \min_{|\psi\rangle} \sum_{j=1}^{n} \|[|\psi\rangle\langle\psi|, \varrho_j]\|, \tag{10}$$

where the optimization is over pure qubit states. Hence there is here a direct connection between set coherence and commutativity in the qubit case.

We start our analysis with sets of $n = 2$ qubit states, the case $n = 1$ trivially giving $\mathcal{R}(\varrho_1) = 0$ by aligning $\vec{p}$ with $\vec{q}_1$. For pairs of pure qubit states the minimum in Eq. (9) is reached when $\vec{p}$ is either $\vec{q}_1$ or $\vec{q}_2$; hence

$$\mathcal{R}_1(\vec{\varrho}) = \frac{1}{2}|\sin(\vec{q}_1, \vec{q}_2)| = \sqrt{\mathrm{tr}(\varrho_1\varrho_2)[1 - \mathrm{tr}(\varrho_1\varrho_2)]}. \tag{11}$$

Note that for pairs of mixed states, the optimal $\vec{p}$ is aligned with the Bloch vector of the purest ϱ_j (i.e., the longest Bloch vector). From the above equation, we see that, to maximize the set coherence, one should choose a pair of pure qubit states with Bloch vectors that are orthogonal, i.e., $|(\vec{q}_1, \vec{q}_2)| = \pi/2$. For $n = 2$ we thus get $\mathcal{R}_1 \leq \mathcal{R}_1^* \coloneqq \max_{\vec{\varrho}} \mathcal{R}_1(\vec{\varrho}) = 1/2$.

More generally, we can characterize the sets of n qubit states featuring the largest set coherence; see [14], Sec. II for details. For triplets, i.e., $n = 3$, $\mathcal{R}_1 \leq \mathcal{R}_1^* = 2/3$, the upper bound being attained when the three Bloch vectors form an orthonormal basis of $\mathbb{R}^3$. For some values of n, we can go further by using known results on optimization over the sphere [19,20]. For $n = 4$ one has $\mathcal{R}_1 \leq \mathcal{R}_1^* = 1/\sqrt{2}$, reached by states whose Bloch vectors form a regular tetrahedron, while for $n = 6$ one has $\mathcal{R}_1 \leq \mathcal{R}_1^* = \sqrt{5}/3$, obtained from half of an icosahedron. The case of $n = 5$ does not have any general answer in the literature and is notoriously hard [21]. Note that when $n \to \infty$ the optimal distribution of pure states tends to be uniform over the Bloch sphere and it follows that $\mathcal{R}_1 \leq \mathcal{R}_1^* = \pi/4$. All these results are summarized in Table I.

Let us also discuss our alternative measure, namely, the max robustness of set coherence. As mentioned above, we have that $\mathcal{R}_1(\vec{\varrho}) \leq \mathcal{R}(\vec{\varrho})$, the inequality being tight only for incoherent sets of states. For $n = 2$, similarly to Eq. (11) one can show that $\mathcal{R}(\varrho_1, \varrho_2) = \sqrt{1 - \mathrm{tr}(\varrho_1\varrho_2)}$ so that the maximal value is $\mathcal{R}^* \coloneqq \max_{\vec{\varrho}} \mathcal{R}(\vec{\varrho}) = 1/\sqrt{2}$, also obtained for an orthogonal pair. For $n = 3$, it seems numerically that $\mathcal{R}^* = \sqrt{3}/2$, obtained when the three vectors form a trine on an equator of the Bloch sphere.

Qudits.—Going beyond qubits using our methods turns out to be challenging, as the Bloch representation becomes more complex. We can nonetheless still make a few statements.

First, observe that the case of a pair of pure states of arbitrary dimension corresponds to the qubit case, as the states span only a qubit subspace. Therefore the rightmost expression in Eq. (11) is applicable to any pair of pure states. For $n > 2$ and $d > 2$ the situation is more complicated. One can nevertheless prove from the inequality $\mathcal{R}_{F_1}(\varrho) \leq d - 1$ of Ref. [10] that

$$\mathcal{R}^* \leq d - 1 \quad \text{and} \quad \mathcal{R}_1^* \leq \frac{(n-1)(d-1)}{n}. \tag{12}$$

Although for $d = 2$ the bound on $\mathcal{R}$ is tight for $n \to \infty$ and the one on $\mathcal{R}_1$ for $n = 2$, 3, we observe numerically that this is not the case in general. Moreover, it seems that constructions based on mutually unbiased bases do not lead to the largest values of $\mathcal{R}_1$. For the case of sets of pure states, an interesting open question is whether the set coherence relates to properties of the Gram matrix (a matrix with entries given by the inner products of each pair of states in the set), which is known to identify uniquely its set of states, up to a unitary.

Finally, we note that lower bounds on the set coherence could be obtained by adapting the method developed in Ref. [22], where optimization problems over unitaries can be relaxed to a hierarchy of semidefinite programs [23].

Quantum measurements.—The notion of set coherence naturally applies to the case of quantum measurements (or positive operator-valued measures, POVMs). Indeed the latter are represented by a set of Hermitian operators: $\mathcal{A} = \{A_a\}_{a=1}^n$ with the properties $A_a \geq 0$ for all a and $\sum_a A_a = \mathbb{1}$. Note that the operators must sum up to the identity but do not need to have unit trace.

To quantify the set coherence of a POVM $\mathcal{A}$ we use the robustness $\mathcal{R}(\mathcal{A})$ defined as in Eq. (5) by replacing the set of density matrices $\{\varrho_j\}_{j=1}^n$ by an n-outcome POVM

$\{A_a\}_{a=1}^n$. The free set and basis-dependent robustness are constructed as in Eqs. (3) and (4), with the only difference that all sets of operators considered, such as $\vec{\tau}$, should form a POVM. The detailed construction of this measure, as well as its related mean robustness, can be found in [14], Sec. III. In what follows, we concentrate on the measure constructed in line with Eq. (5).

We first discuss the set coherence for the case of a single POVM. Clearly, $\mathcal{R}(\mathcal{A}) = 0$ for all projective measurements, i.e., measurements with $A_a^2 = A_a$ for all a, as the POVM elements commute with on another and can hence be simultaneously diagonalized. Note that this is also the case for binary measurements. There are, however, nonprojective POVMs that feature nonzero set coherence. For qubit POVMs, we find numerically (similarly to the case of states; see [14], Sec. III) that the most set coherent ternary ($n = 3$) POVM is the trine ($\mathcal{R} = 1/\sqrt{3}$), while for $n = 4$ we get $\mathcal{R} = 1/\sqrt{2}$ for the symmetric informationally complete POVM (with Bloch vectors forming a regular tetrahedron).

It is relevant to comment on the relation between the set coherence of a POVM and some recently developed quantifiers of the usefulness of a POVM. First, in Ref. [24] the authors quantify the informativeness of a POVM through its robustness with respect to those POVMs that have elements proportional to the identity operator. The latter correspond to generalized coin-flip measurements, i.e., measurements with state-independent outcome distributions. Clearly these POVMs have zero set coherence, and hence they belong to our free set. Thus any feasible point of the robustness measure for informativeness is also a feasible point in our optimization. This further implies that the robustness of informativeness upper bounds the set coherence. There exist, however, POVMs that are informative but have no set coherence (e.g., projective measurements). Second, one can consider POVMs that are simulable with projective measurements [25,26], i.e., measurements in the convex hull of projective measurements. Any POVM with no set coherence is in this convex hull, as we show in [14], Sec. IV, based on a generalization of the Birkhoff–von Neumann theorem [27]. Hence, the robustness of a POVM with respect to the set of projective simulable measurements lower bounds the set coherence. Finally, note that both the informativeness and nonprojective simulability have an operational interpretation in terms of a performance in a state discrimination task, as their corresponding free sets are convex [24,25,28,29]. For set coherence, the free set is not convex, although it is somewhere in between the two mentioned convex free sets. In the next section we show that an interpretation through discrimination tasks is nevertheless possible.

We now move to the case of a set of POVMs. In this case, one could expect a connection between the set coherence and the incompatibility of sets of POVMs as captured, e.g., via nonjoint measurability. Joint measurability asks whether for a set of POVMs there exists a common POVM that functions as their common readout. Clearly, zero set coherence guarantees mutual commutativity and hence joint measurability by using the product POVM $G_{\vec{a}} = \prod_x A_{a_x|x}$. This is indeed a POVM because $[A_{a_x|x}, A_{a_y|y}] = 0$ and it has the property $A_{a|x} = \sum_{\vec{a}} \delta_{a_x,a} G_{\vec{a}}$; i.e., neglecting all but the outcome a_x gives the POVM $\{A_{a_x|x}\}_{a_x}$. Hence, the set coherence is an upper bound on the incompatibility robustness [30]. However, there exist compatible sets of POVMs featuring nonzero set coherence, such as noisy X and Z measurements, and one may expect that a high enough set coherence ensures incompatibility. These are interesting questions for future work.

Set coherence as a quantum game.—We now discuss the operational meaning of set coherence in terms of a quantum discrimination game. We consider the case of a set of states $\vec{\varrho}$ and show that if $\mathcal{R}_1(\vec{\varrho}) > 0$, then there exists a specific discrimination game for which $\vec{\varrho}$ provides an advantage over any incoherent set (i.e., sets of states that have zero set coherence). Moreover, the value of $\mathcal{R}_1(\vec{\varrho})$ quantifies precisely the relative advantage provided by $\vec{\varrho}$ over any incoherent set.

To construct such a game we consider a task of subchannel discrimination, i.e., distinguishing between different branches of a time evolution. The branches are modeled as sets of completely positive maps $\mathcal{C} = \{\mathcal{I}_a\}_a$ with the property that $\sum_a \mathcal{I}_a$ is trace preserving. Given $\mathcal{C}$ and a final measurement $\mathcal{A}$, the goal is to identify which subchannel $\mathcal{I}_a$ has been applied. The resource is the initial state ϱ. The success probability is given by

$$p_{\mathrm{succ}}(\varrho, \mathcal{C}, \mathcal{A}) \coloneqq \sum_a \mathrm{tr}(\mathcal{I}_a(\varrho) A_a). \tag{13}$$

In the case in which one considers a fixed reference basis, the robustness measure $\mathcal{R}_{F_1}$ quantifies the relative advantage offered by ϱ over any state τ in the free set [9,10], i.e.,

$$\frac{p_{\mathrm{succ}}(\varrho, \mathcal{C}, \mathcal{A})}{\max_{\tau \in U^\dagger F_1 U} p_{\mathrm{succ}}(\tau, \mathcal{C}, \mathcal{A})} \leq 1 + \mathcal{R}_{U^\dagger F_1 U}(\varrho). \tag{14}$$

This relation can be derived from Eq. (4) (with $n = 1$) and holds for any unitary U (specifying the reference frame), any set of subchannels $\mathcal{C}$, and any POVM $\mathcal{A}$.

Moving now to the case of a set of states $\vec{\varrho}$ and making the construction basis independent by minimizing over unitaries, we show in [14], Sec. V that

$$\frac{1}{n} \min_U \sum_j \sup_{\mathcal{C}_j, \mathcal{A}_j} \frac{p_{\mathrm{succ}}(\varrho_j, \mathcal{C}_j, \mathcal{A}_j)}{\max_{\tau_j \in U^\dagger F_1 U} p(\tau_j, \mathcal{C}_j, \mathcal{A}_j)} = 1 + \mathcal{R}_1(\vec{\varrho}). \tag{15}$$

Note that here the discrimination procedure $(\mathcal{C}_j, \mathcal{A}_j)$ depends on U. In other words, for any reference basis, i.e., for any given U, there exists a set of subchannel discrimination tasks $(\vec{\mathcal{C}}, \vec{\mathcal{A}})$, in which the set of states $\vec{\varrho}$ outperforms any incoherent set in this reference frame, with relative advantage given by $\mathcal{R}_{U^\dagger F_1 U}(\vec{\varrho})$. When minimizing this advantage over reference frames, one gets the mean robustness of set coherence $\mathcal{R}_1(\vec{\varrho})$.

A similar construction can be made for our other measure of set coherence $\mathcal{R}$. In particular, this can also be adapted to the case of quantum measurements, where the set-coherence robustness of a POVM quantifies the relative advantage in a state discrimination task. We have sketched the proofs for these scenarios in [14], Sec. V.

We note that, in the case of measurements, the task-based interpretation sheds light on a natural question in quantum measurement theory. Namely, the notion of commutativity of POVMs, i.e., the requirement that $[A_{a|1}, A_{b|2}] = 0$ for all a, b, is a type of measurement compatibility that lacks an operational interpretation. Commutativity implies all known types of compatibility such as joint measurability [31], unavoidable measurement disturbance [32], and coexistence [33], all of which can be given a task-oriented interpretation [28,29,34–41]. It is clear that our notion of set coherence of measurements does not exhaust commutativity of POVMs, as one can easily construct a POVM that does not commute with itself but commutes with a trivial POVM. However, for binary measurements our notion coincides with commutativity, and hence in this scenario we get an operational interpretation of commutativity in the spirit of Eq. (15).

Conclusion.—We developed a notion of set coherence for characterizing the coherence of a set of quantum systems. This provides an approach for quantifying quantum coherence in a basis-independent manner. This is appealing from the physical standpoint but becomes formally more challenging due to the nonconvexity of the resource theory. Nevertheless, we showed that meaningful measures can be constructed for set coherence. Some of these ideas could be useful for building resource theories for other quantum resources, such as non-Markovianity, as the set of Markovian channels is also known to be nonconvex [42]. In parallel, it would also be interesting to see if the present resource theory of set coherence can be "convexified," for instance by taking as the free set the convex hull of $\mathcal{F}_n$.

Finally, it would be interesting to investigate the relevance of set coherence in settings where sets of quantum states (or measurements) naturally appear, for instance in quantum key distribution (QKD) or quantum computation. Could one design a secure QKD protocol based on any set of states featuring nonzero set coherence? One may also consider quantifying the coherence of a quantum dynamical evolution, where a continuous set of states is explored over time.

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