Bilocal Bell Inequalities Violated by the Quantum Elegant Joint Measurement

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(Received 7 July 2020; accepted 22 April 2021; published 1 June 2021)

Network Bell experiments give rise to a form of quantum nonlocality that conceptually goes beyond Bell's theorem. We investigate here the simplest network, known as the bilocality scenario. We depart from the typical use of the Bell state measurement in the network central node and instead introduce a family of symmetric isoentangled measurement bases that generalize the so-called "elegant joint measurement." This leads us to report noise-tolerant quantum correlations that elude bilocal variable models. Inspired by these quantum correlations, we introduce network Bell inequalities for the bilocality scenario and show that they admit noise-tolerant quantum violations. In contrast to many previous studies of network Bell inequalities, neither our inequalities nor their quantum violations are based on standard Bell inequalities and standard quantum nonlocality. Moreover, we pave the way for an experimental realization by presenting a simple two-qubit quantum circuit for the implementation of the elegant joint measurement and our generalization.

DOI: 10.1103/PhysRevLett.126.220401

Introduction.—The violation of Bell inequalities is a hallmark property of quantum theory. It asserts that the predictions of quantum theory cannot be accounted for by any physical model based only on local variables [1]. Such violations, referred to as quantum nonlocality, not only provide insights into the foundations of quantum theory but also constitute a powerhouse for a broad scope of applications in quantum information science [2].

A standard Bell experiment features a source that emits a pair of particles shared between two spacelike separated observers who perform local and independent measurements. In quantum theory, the particles can be entangled, thus enabling global randomness [3]. In contrast, in local variable models aiming to simulate the quantum predictions, the particles are endowed with classically correlated stochastic properties that locally determine the outcome of a given measurement. Many decades of research on Bell inequalities have brought a relatively deep understanding of quantum nonlocality and have established standard methods for characterizing correlations in both quantum models and local variable models [2].

The last decade witnessed a significant conceptual advance: much attention was directed at going beyond correlations in standard Bell experiments in favor of investigating correlations in networks featuring many observers and several independent sources of particles [4,5]. While a standard Bell experiment may be viewed as a trivial network (with a single source), the introduction of multiple independent sources is conceptually interesting since it brings into play new physical ingredients and corresponds to the topology of future quantum networks. In contrast to standard Bell experiments, network Bell experiments feature some observers who hold independent particles (from different sources) and therefore *a priori* share no correlations. Moreover, entanglement can be distributed in the network, in particular to initially independent observers, through the process of entanglement swapping [6]. Recent years have seen much attention being directed at characterizing classical, quantum, and postquantum correlations in networks, many times through the construction of network Bell inequalities and the exploration of their quantum violations [7–26]. In general, this is challenging due to the fact that the introduction of multiple independent sources makes the set of local variable correlations nonconvex [4].

Here, we focus on the simplest nontrivial network Bell experiment, known as the "bilocality scenario." It features two independent sources that each produce a pair of particles. The first pair is shared between observers Alice and Bob, while the second pair is shared between Bob and

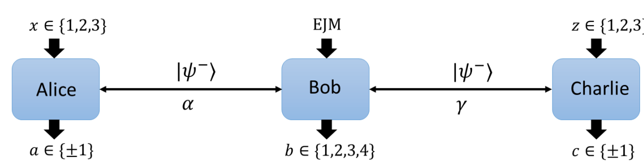

FIG. 1. Bilocality scenario: Bob independently shares a "state" with Alice and Charlie, respectively. In a quantum experiment, these are independent, typically entangled quantum states ($|\psi^-\rangle$), while in a bilocal model these are associated with independent local variables (α and γ).

another observer, Charlie (see Fig. 1). Interestingly, there are known Bell inequalities for the bilocality scenario (bilocal inequalities), i.e., inequalities for the observed correlations that are satisfied by all local variable models respecting the independence of the two sources (for simplicity, bilocal models). Importantly, these inequalities are also known to admit quantum violations. The quantum violations typically arise from Bob implementing a Bell state measurement (BSM; encountered in quantum teleportation [27] and entanglement swapping [6]). Conspicuously, both the inequalities and their reported violations strongly resemble those encountered in the standard Bell experiments (see, e.g., [7,18,19,28]). For instance, the standard bilocal inequality, first presented in Ref. [7], is essentially built on the Clauser-Horne-Shimony-Holt (CHSH) inequality [29], and its quantum violations through the BSM turn out to effectively correspond to Bob in a coordinated manner separately testing the CHSH inequality with Alice and Charlie, respectively. Indeed, the BSM measurement amounts to performing simultaneously the two commuting measurements of $\sigma_1 \otimes \sigma_1$ and $\sigma_3 \otimes \sigma_3$ [where $(\sigma_1, \sigma_2, \sigma_3)$ are the three Pauli observables] on Bob's two independent qubits, and ample numerical evidence shows that the optimal measurement settings for Alice and Charlie are at $\pm 45°$ on the Bloch sphere, i.e., exactly those settings tailored for the CHSH inequality. Given this close resemblance to the CHSH inequality, it is perhaps unsurprising that the critical singlet visibility, required for two identical noisy singlet states to enable a violation, is the same as that encountered in the CHSH inequality, namely $1/\sqrt{2}$ for each state.

Here we investigate quantum nonlocality in the bilocality scenario that is not based on the BSM and does not directly trace back to standard quantum nonlocality as in the previous cases. To this end, we present a family of two-qubit entangled measurements generalizing the so-called elegant joint measurement (EJM) [30]. These measurements allow Bob to effectively distribute (in an entanglement swapping scenario) entangled states to Alice and Charlie that are different from those obtained through a BSM. We investigate the possibility of simulating the resulting correlations in bilocal models and find the critical visibility per singlet at which quantum theory eludes such models. Subsequently, we introduce new bilocal inequalities tailored to our quantum correlations and show that they can detect quantum nonlocality in the network at reasonable singlet visibilities. Furthermore, paving the way towards experimental demonstrations of quantum violations of network Bell inequalities that are not based on standard Bell inequalities, we explore the implementation of our generalized EJM. We prove that it cannot be implemented in linear optical schemes without auxiliary photons but it can be implemented with a simple two-qubit quantum circuit.

Entangled measurements with tetrahedral symmetry.— We consider symmetric entangled measurements on two qubits that, most naturally, have four outcomes. Specifically, we present a family of bases $\{|\Phi_b^\theta\rangle\}_{b=1}^4$ of the two-qubit Hilbert space, parameterized by $\theta \in [0, \pi/2]$, such that all elements are equally entangled, and, moreover, the four local states, corresponding to either qubit being traced out, form a shrunk regular tetrahedron inside the Bloch sphere.

To construct such bases, let us first introduce the pure qubit states $|\vec{m}_b\rangle$ that point (on the Bloch sphere) toward the four vertices,

$$\vec{m}_1 = (+1, +1, +1), \qquad \vec{m}_2 = (+1, -1, -1),$$
$$\vec{m}_3 = (-1, +1, -1), \qquad \vec{m}_4 = (-1, -1, +1), \qquad (1)$$

of a regular tetrahedron, as well as the states $|-\vec{m}_b\rangle$ with the antipodal direction. Specifically, we write these tetrahedron vertices in cylindrical coordinates as $\vec{m}_b = \sqrt{3}(\sqrt{1-\eta_b^2}\cos\varphi_b, \sqrt{1-\eta_b^2}\sin\varphi_b, \eta_b)$ and define

$$|\pm \vec{m}_b\rangle = \sqrt{\frac{1 \pm \eta_b}{2}} e^{-i\varphi_b/2}|0\rangle \pm \sqrt{\frac{1 \mp \eta_b}{2}} e^{i\varphi_b/2}|1\rangle. \qquad (2)$$

Our family of generalized EJM bases, with the above properties, is then given by

$$|\Phi_b^\theta\rangle = \frac{\sqrt{3} + e^{i\theta}}{2\sqrt{2}} |\vec{m}_b, -\vec{m}_b\rangle + \frac{\sqrt{3} - e^{i\theta}}{2\sqrt{2}} |-\vec{m}_b, \vec{m}_b\rangle. \qquad (3)$$

Notice that for $\theta = 0$, we obtain the EJM introduced in Ref. [30] (the largest local tetrahedron in our family, of radius $\sqrt{3}/2$), while for $\theta = \pi/2$, we obtain the BSM (the smallest local tetrahedron, of radius zero) up to local unitaries (which can, for instance, be chosen as $U_1 \otimes U_2 = \mathbb{1} \otimes e^{(2\pi i/3)[(\sigma_1+\sigma_2+\sigma_3)/\sqrt{3}]}$ to recover the standard BSM). By varying θ, we thus continuously interpolate between the EJM and the BSM.

Quantum correlations.—While we will later derive bilocal inequalities that apply to fully general tripartite correlations, we begin with considering a specific quantum implementation of the bilocality experiment illustrated in Fig. 1. We let Bob apply the generalized EJM and consider that both sources emit pairs of qubits corresponding to noisy singlets (so-called Werner states [31]),

$$\rho_i = V_i |\psi^-\rangle\langle\psi^-| + \frac{1 - V_i}{4} \mathbb{1}, \qquad (4)$$

for $i \in \{1, 2\}$ where $V_i \in [0, 1]$ denotes the visibility of each singlet $|\psi^-\rangle = 1/\sqrt{2}(|0,1\rangle - |1,0\rangle)$. By applying his measurement onto distributed (pure) singlets, Bob effectively prepares Alice's and Charlie's joint state in an entangled state similar to that of Eq. (3), up to a change in signs for $\vec{m}_b$ and θ. Because of the tetrahedral structure of the distributed states, we expect to find strong correlations between Alice and Charlie when they perform measurements of the three Pauli observables [32]. We therefore let each of them have three possible measurement settings, $x, z \in \{1, 2, 3\}$ [corresponding to the observables (σ_x, σ_z)], with binary outcomes denoted $a, c \in \{+1, -1\}$.

To reflect the symmetry of our scenario, it is convenient to identify Bob's outcome b with the corresponding vector $\vec{m}_b$ from Eq. (1), i.e., to write b as ± 1-valued 3-vector $b = (b^1, b^2, b^3)$. The conditional probability distribution $p(a, b, c|x, z)$ obtained in the experiment can then be characterized in terms of the single-, two-, and three-party correlators $\langle A_x \rangle$, $\langle B^y \rangle$, $\langle C_z \rangle$, $\langle A_x B^y \rangle$, $\langle B^y C_z \rangle$, $\langle A_x C_z \rangle$ $(= \langle A_x \rangle \langle C_z \rangle$ in the bilocality scenario) and $\langle A_x B^y C_z \rangle$ for all $x, y, z \in \{1, 2, 3\}$, with e.g., $\langle A_x B^y C_z \rangle = \sum_{a,b^1,b^2,b^3,c=\pm 1} a b^y c\, p(a, b, c|x, z)$ and similarly for the other correlators [33]. For the quantum correlation p_{Q}^{θ} obtained from our above choice of states and measurements, these correlators become

$$\langle A_x \rangle = \langle B^y \rangle = \langle C_z \rangle = \langle A_x C_z \rangle = 0,$$
$$\langle A_x B^y \rangle = -\frac{V_1}{2} \cos\theta \delta_{x,y}, \qquad \langle B^y C_z \rangle = \frac{V_2}{2} \cos\theta \delta_{y,z},$$
$$\langle A_x B^y C_z \rangle = \begin{cases} -\frac{V_1 V_2}{2}(1 + \sin\theta) & \text{if } xyz \in \{123, 231, 312\} \\ -\frac{V_1 V_2}{2}(1 - \sin\theta) & \text{if } xyz \in \{132, 213, 321\}, \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

where δ is the Kronecker symbol.

Simulating quantum correlations in bilocal models.— Let us first investigate whether the quantum probability distribution p_{Q}^{θ} admits a bilocal model. In such a model, each pair of particles is associated with a local variable denoted as α and γ, respectively (see Fig. 1). Alice's (Charlie's) outcome is determined by her (his) setting and α (γ). Since they each have three possible settings, we can without loss of generality represent the local variables as triples $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ and $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ with entries ± 1, with each α_x, γ_z denoting Alice's or Charlie's deterministic outcome for the setting x or z. A bilocal model can thus be written as

$$p_{\mathrm{biloc}}(a, b, c|x, z) = \sum_{\alpha,\gamma} q_{\alpha}^{(1)} q_{\gamma}^{(2)} \delta_{a,\alpha_x} \delta_{c,\gamma_z} p(b|\alpha, \gamma), \tag{6}$$

where $\{q_{\alpha}^{(1)}\}_{\alpha}$ and $\{q_{\gamma}^{(2)}\}_{\gamma}$ are probability distributions representing the stochastic nature of the local variables α and γ, respectively, and $p(b|\alpha, \gamma)$ are probability distributions representing the stochastic response of Bob upon receiving (α, γ).

The central question is whether the quantum correlations characterized by Eq. (5) can be simulated in a bilocal model. We investigate the matter with three different approaches. First, we set $V \equiv V_1 = V_2$ (equal noise on both sources), and $\theta = 0$ (as in the original EJM [30]). By employing semidefinite relaxations of the set of bilocal correlations, one can obtain a necessary condition for the existence of a bilocal model [34]. An evaluation of the relevant semidefinite program guarantees that a violation of bilocality is obtained whenever $V \gtrsim 83\%$ [35]. However, this bound is not expected to be tight due to the nonconvex nature of the set of quantum correlations with independent sources.

Second, we provide a better characterization of the power of bilocal models by explicitly considering their ability to simulate the quantum correlations. To this end, we have used an efficient search method that exploits the fact that the numerical difficulties associated with the bilocality assumption are significantly reduced if the bilocal model first undergoes a Fourier transformation [7]. For the case of $V \equiv V_1 = V_2$ and $\theta = 0$ considered above, we look for the largest V for which $p_{\mathrm{Q}}^{\theta=0}$ admits a bilocal model and find the critical visibility

$$V_{\mathrm{crit}} \approx 79.1\%. \tag{7}$$

This contrasts with conventional quantum nonlocality in entanglement swapping experiments in which the swapped state, postselected on b, is tested in a standard Bell experiment. For any outcome b, our swapped (noisy, partially entangled) state can only violate the CHSH inequality for $V > 5^{-1/4}\sqrt{2} \approx 94.6\%$, as can be verified using the Horodecki criterion [36]. The advantage persists even if we compare the per source visibility [Eq. (7)] to the total visibility in the sources required for the swapped state to violate the CHSH inequality $[V^2 > (5^{-1/4}\sqrt{2})^2 \approx 89.4\%]$, which also contrasts with established bilocal Bell inequalities [4,7].

Next, we consider, for a given V_1, the largest V_2 for which a bilocal model exists. Figure 2 shows the region in

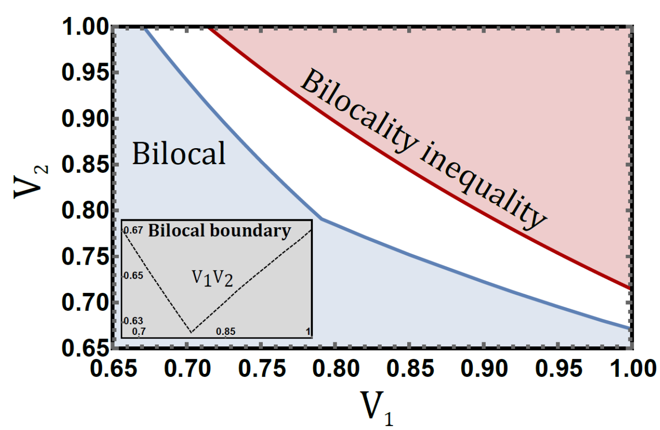

FIG. 2. The blue region represents the set of bilocal quantum correlations $p_{\mathrm{Q}}^{\theta=0}$ in the plane of visibilities (V_1, V_2), with the dashed line in the figure inset showing the product of the visibilities on the boundary of this bilocal region. The red area is the part of the quantum region that can be detected as nonbilocal through the violation of our bilocal inequality [Eq. (9)].

the (V_1, V_2) plane for which we find a bilocal simulation of $p_Q^{\theta=0}$ [see the Supplemental Material (SM) [37], Sec. I, for the analysis of $\theta > 0$]. It also displays the product $V_1 V_2$ associated with the boundary of the bilocal region (the critical pairs). In previously studied quantum correlations that arise from the BSM [7], this product of visibilities determines the existence of a bilocal model. Similarly, the violations of many bilocal inequalities that are based on coordinated tests of standard Bell inequalities [7,9,18,19,28] are determined by such products. We find that such is not the case in our scenario.

Third, we employ an intuitive ansatz for analytically constructing bilocal models that mimic the symmetry of $p_Q^{\theta=0}$. Namely, we stipulate that the (unobserved) probability distribution $p_{\text{biloc}}(\alpha, b, \gamma) = q_\alpha^{(1)} q_\gamma^{(2)} p(b|\alpha, \gamma)$ of the bilocal model should have the same tetrahedral symmetry: for every permutation π of the tetrahedron vertices $\{\vec{m}_b\}$ in Eq. (1), extended to the opposite vertices via $\pi(-\vec{m}_b) = -\pi(\vec{m}_b)$, and applied to the 3-vector variables α, b, γ, one should have $p_{\text{biloc}}[\pi(\alpha), \pi(b), \pi(\gamma)] = p_{\text{biloc}}(\alpha, b, \gamma)$. Under this symmetry ansatz, we are able to analytically construct efficient bilocal simulations of $p_Q^{\theta=0}$. Interestingly, along the entire boundary of the bilocal region, the obtained results match those presented in Fig. 2 up to the fifth decimal digit. This shows that simple and highly symmetric bilocal models are very nearly optimal for simulating $p_Q^{\theta=0}$. These bilocal models and the critical visibilities are detailed in the SM (Sec. II).

Bilocal Bell inequalities.—We now draw inspiration from the structure of the nonbilocal quantum correlations obtained from the EJM to construct a bilocal inequality that can be applied to general quantum states and measurements in the considered scenario. Hence, in contrast to several previous bilocal inequalities, the present one is neither based on nor apparently resembles a standard Bell inequality. To build the Bell expression, we introduce the following quantities:

$$S = \sum_{y=z} \langle B^y C_z \rangle - \sum_{x=y} \langle A_x B^y \rangle,$$
$$T = \sum_{x \neq y \neq z \neq x} \langle A_x B^y C_z \rangle, \;\; Z = \max(\mathcal{C}_{\text{other}}), \tag{8}$$

where $\mathcal{C}_{\text{other}} = \{|\langle A_x \rangle|, |\langle A_x B^y \rangle|, \ldots, |\langle A_x B^y C_z \rangle|\}$ is the set of the absolute values of all one-, two-, and three-party correlators other than those appearing in the expressions of S and T. This leads us to the following bilocal inequality:

$$\mathcal{B} \equiv \frac{S}{3} - T \overset{\text{biloc}}{\leq} 3 + 5Z. \tag{9}$$

Notice that the Z quantity makes this general inequality nonlinear. The most interesting case is when $Z = 0$, as satisfied by the quantum correlation of Eq. (5). For this case, we have proved the bilocal bound under the previously considered symmetry ansatz (which enforces $Z = 0$; see the SM, Sec. III). Then, we have confirmed the bilocal bound using two different numerical methods applied to general bilocal models [38]. We find that the bilocal inequality above, for $Z = 0$, is tight in the sense that it constitutes one of the facets of the projection of the "$Z = 0$ slice" of the bilocal set of correlations onto the (S, T) plane. Remarkably, this projection of the $Z = 0$ slice is delimited by linear inequalities (see the SM, Sec. IV); this stands in contrast to previous bilocal inequalities that use nonlinear Bell expressions. Finally, for $Z > 0$, we have again applied the same numerical search methods to justify the correction term $5Z$ in the bilocal bound of $\mathcal{B}$. Notably, more accurate corrections are also possible (see the SM, Sec. V).

For our quantum correlation of Eq. (5), we straightforwardly obtain $(S,T,Z)=\{3[(V_1+V_2)/2]\cos\theta, -3V_1V_2, 0\}$, and $\mathcal{B} = 3V_1V_2 + [(V_1 + V_2)/2]\cos\theta$. In the noiseless case ($V_1 = V_2 = 1$), we thus get $\mathcal{B} = 3 + \cos\theta$, which gives a violation of our bilocal inequality [Eq. (9)] for our whole family of generalized EJMs (i.e., the whole range of θ) except for the special case of a BSM ($\theta = \pi/2$, for which our quantum correlation turns out to be bilocal; see the SM, Sec. I). In contrast, when white noise is present and both sources are equally noisy ($V \equiv V_1 = V_2$), we get a violation of our inequality whenever $3V^2 + V\cos\theta > 3$. For $\theta = 0$, the critical visibility per singlet required for a violation is

$$V_{\text{crit}} = \frac{\sqrt{37} - 1}{6} \approx 84.7\%. \tag{10}$$

This shows that the quantum violation is robust to white noise on the singlet states but not optimally robust as no violation is found here for $V \in [0.791, 0.847]$. More generally, the bilocal inequality enables the detection of quantum correlations in a sizable segment of the (V_1, V_2) plane (see Fig. 2).

Finally, we note that several different bilocal inequalities can be constructed based on the correlations from the EJM. As another example, in the SM (Sec. VI), we consider the following Bell expression:

$$\mathcal{B}' \equiv \sum_{x,b} \sqrt{p(b)[1 - b^x E_b^{\mathrm{A}}(x)]} + \sum_{z,b} \sqrt{p(b)[1 + b^z E_b^{\mathrm{C}}(z)]}$$
$$+ \sum_{x \neq z, b} \sqrt{p(b)[1 - b^x b^z E_b^{\mathrm{AC}}(x, z)]}, \tag{11}$$

where $E_b^{\mathrm{A}}(x)$, $E_b^{\mathrm{C}}(z)$, and $E_b^{\mathrm{AC}}(x, z)$ denote one- and two-party expectation values for Alice and Charlie, conditioned on Bob's output $b = (b^1, b^2, b^3)$ (see the SM). Numerical methods similar to the previous ones are employed to evidence that $\mathcal{B}' \leq 12\sqrt{3} + 2\sqrt{15}$ holds for bilocal models.

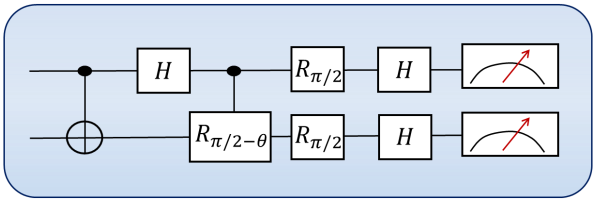

FIG. 3. Quantum circuit for implementing our family of generalized elegant joint measurements parameterized by θ. A controlled-NOT gate is followed by a Hadamard rotation $[H = (\sigma_1 + \sigma_3)/\sqrt{2}]$ on the control qubit, a controlled phase shift gate $R_{\pi/2-\theta}$, and a separate rotation of each qubit composed of $R_{\pi/2}$ and H. Finally, a measurement is performed in the basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$.

In the SM, we prove that there are quantum distributions whose nonbilocality is detected with this bilocal inequality but not with the inequality [Eq. (9)]. Furthermore, we also prove that if Bob has uniformly distributed outcomes ($p(b) = 1/4$), then $\mathcal{B}' \lesssim 30.70$ is respected by all quantum models with independent sources, and hence it constitutes a quantum Bell inequality for the network [35].

Implementation of the elegant joint measurement.—It is both interesting and practically relevant to address the question of how one may implement experimentally the EJM [39] and its generalization. In general, the implementation of joint (two-qubit) measurements requires the interaction of different signals. Optical implementations are of particular interest since they are common and convenient for Bell-type experiments. However, many such measurements, including the BSM, cannot be implemented with the basic tools applied in linear optics schemes (phase shifters and beam splitters) when no auxiliary photons are present [40]. It turns out that our family of generalized EJMs as defined by Eq. (3) can also not be implemented with two-photon linear optics, as can be shown by evaluating the criterion provided in Ref. [41].

Our measurement family can in fact be implemented by the two-qubit circuit presented in Fig. 3. This circuit maps the four measurement basis states $\{|\Phi_b^\theta\rangle\}_b$ onto the computational basis product states $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ (up to global phases). The proposed implementation involves (in addition to single-qubit gates) two different controlled unitary operations, namely a standard controlled-NOT gate and a controlled implementation of the phase shift gate

$$R_\phi = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\phi} \end{pmatrix}. \quad (12)$$

We remark that this controlled phase gate itself can be implemented using two controlled-NOT gates and unitaries acting on the target qubit as described in Ref. [42]. Finally, notice that when $\theta = \pi/2$, we have $R_{\pi/2-\theta} = \mathbb{1}$, and thus the circuit only involves a single two-qubit gate, just like the standard scheme for a BSM [43].

Discussion and open questions.—We have investigated quantum violations of bilocality based on the elegant joint measurement and a new generalization thereof. In contrast to several previous works in which quantum correlations were generated through a Bell state measurement, our setup does not effectively reduce to separate implementations of the standard CHSH scenario. We nevertheless constructed new bilocal inequalities and exhibited violations that we could not directly trace back to violations of a standard Bell inequality. Finally, we paved the way toward a bilocality experiment based on the EJM by constructing a quantum circuit for its implementation.

Several intriguing questions are left open: (1) What is the largest possible quantum violation of the bilocal inequalities? (2) Can the inequalities be proven in full generality? (3) How can one formalize the intuitive idea that some bilocal inequalities may or may not trace back to standard Bell inequalities? (4) Can our EJM family be further generalized for two higher-dimensional systems or for more than two qubits such that it preserves its elegant properties?

We thank Alejandro Pozas-Kerstjens for sharing with us both his codes and results for semidefinite programs based on Ref. [34], and Norbert Lütkenhaus for directing us to Ref. [41]. This work was supported by the Swiss National Science Foundation via the National Centres of Competence in Research (NCCR)-SwissMap.

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[32] Another natural scenario is to let Alice and Charlie perform four measurements with Bloch vectors pointing to the vertices of a tetrahedron. It turns out, however, that the resulting correlations are less robust to noise than those obtained from measuring the three Pauli observables.

[33] Specifically, one has $p(a,b^1,b^2,b^3,c|x,z)=\frac{1}{16}[1+a\langle A_x\rangle+\sum_y b^y\langle B^y\rangle+c\langle C_z\rangle+\sum_y ab^y\langle A_xB^y\rangle+\sum_y b^yc\langle B^yC_z\rangle+ac\langle A_xC_z\rangle+\sum_y ab^yc\langle A_xB^yC_z\rangle]$. Notice the (convenient) redundancy in our encoding of Bob's outcome, as the product of its three ±1-valued bits $b^1b^2b^3$ is always $+1$. As in Ref. [7], we write y as superscripts in b^y, B^y to distinguish the case where the outputs (b^y) are all observed together from the case of outputs obtained for different inputs (as in A_x, C_z).

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